SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

NAYLOR TRANSFORMS OF MELLIN TYPE*

W. J. HARRINGTON AND K. A. PATEL†

Abstract. Some transforms introduced by Naylor (1963) are characterized in terms of Mellin transforms. This facilitates the analysis of transform properties. A problem of steady-state heat in a finite circular sector (or wedge) is considered to illustrate the use of one of the transforms and its properties.

1. Introduction. In [1], Naylor introduced several Mellin-type integral transforms related to boundary value problems involving the Laplacian differential operators in polar and spherical coordinates. The transforms were generated through an integral representation of solutions in terms of suitable Green's functions.

In this paper, it is shown that each of these Naylor transforms can be identified as an ordinary Mellin transform by considering an extended domain of definition of the function involved. This interpretation enables one to obtain a variety of useful properties of the Naylor transforms directly from properties of the Mellin transform. These include the inversion integral, operational properties, and convolution formulas.

Although Naylor defines the transforms on the domains $0 < r < a$ and $a < r < \infty$ it is sufficient, and more convenient for our purposes, to consider scaled radial variables r on $(0, 1)$ or $(1, \infty)$ respectively.

2. Naylor transforms on $0 < r < 1$ for plane polar coordinates.

2.1. Definitions and relationships to Mellin transforms. Consider f to be a real function on $0 < r < 1$ such that:

(i) f is piecewise continuous and of bounded variation in every finite interval $[a, b]$ where $0 < a < b < 1$;

(ii) For some real $\alpha > 0$, $\int_0^1 r^{-\alpha-1}|f(r)|\,dr$ is convergent.

Let the Naylor transforms $\mathcal{N}_1$ and $\mathcal{N}_2$ be defined as follows:

$$\mathcal{N}_1\{f(r); r \to s\} = F_1(s) \triangleq \int_0^1 (r^{s-1} + r^{-s-1}) f(r)\,dr; \tag{1}$$

$$\mathcal{N}_2\{f(r); r \to s\} = F_2(s) \triangleq \int_0^1 (r^{s-1} - r^{-s-1}) f(r)\,dr. \tag{2}$$

We extend the domain of r to $(0, \infty)$ and consider two extensions of the function f, f_1 and f_2, defined by

$$f_{1,2}(r) = \begin{cases} f(r), & 0 < r < 1, \\ \pm f(1/r), & 1 < r < \infty, \end{cases} \tag{3}$$

* Received by the editors September 8, 1971.

† Department of Mathematics, North Carolina State University at Raleigh, Raleigh, North Carolina 27607. This work was supported in part by the U.S. Air Force Office of Scientific Research under Grant AFOSR-69-1779.

with the $+$ being used for f_1 and the $-$ for f_2. It is clear that $\int_1^\infty r^{\alpha-1}|f_{1,2}(r)|\,dr = \int_0^1 r^{-\alpha-1}|f(r)|\,dr$ and is convergent. Thus one recognizes that the Mellin transforms

$$\mathscr{M}\{f_{1,2}(r); r \to s\} = \int_0^\infty r^{s-1} f_{1,2}(r)\,dr \tag{4}$$

are convergent if $-\alpha < \operatorname{Re}(s) < \alpha$ and that (with $\rho = 1/r$)

$$\mathscr{N}_{1,2}\{f\} = \int_0^1 r^{s-1} f(r)\,dr \pm \int_1^\infty \rho^{s-1} f(1/\rho)\,d\rho = \mathscr{M}\{f_{1,2}\}, \tag{5}$$

with the transform functions defined[1] and analytic in the strip $|\operatorname{Re}(s)| < \alpha$.

2.2. Inversion integral. The inversion of (1) and (2), in the light of (5), is given by the standard Mellin inversion integral

$$\mathscr{N}_{1,2}^{-1}\{F_{1,2}(s)\} = f(r) = \frac{1}{2\pi i}\int_L r^{-s} F_{1,2}(s)\,ds, \qquad 0 < r < 1, \tag{6}$$

where L denotes the line, $\operatorname{Re} s = \lambda$, with $|\lambda| < \alpha$.

2.3. Operational properties. Let f be continuous on $[0, 1]$. Then on $[0, \infty)$, f_1 is continuous and f_2 is continuous except for a jump of $[-2f(1)]$ at $r = 1$. Thus, if the Naylor transforms $\mathscr{N}_{1,2}\{rf'(r)\}$ exist, one readily obtains the formulas

$$\mathscr{N}_1\{rf'(r)\} = \mathscr{M}\{rf'_2(r)\} = -sF_2(s) + 2f(1), \tag{7}$$

$$\mathscr{N}_2\{rf'(r)\} = \mathscr{M}\{rf'_1(r)\} = -sF_1(s). \tag{8}$$

If f and f' are continuous on $[0, 1]$ and if $\mathscr{N}_{1,2}\{[r(d/dr)]^2 f\}$ exist, then using (7) and (8) with $g_{1,2}(r) = rf'_{1,2}(r)$, one obtains

$$\mathscr{N}_1\left\{\left(r\frac{d}{dr}\right)^2 f\right\} = \mathscr{M}\{rg'_2(r)\} = -sG_2(s) + 2g(1) = s^2F_1(s) + 2f'(1), \tag{9}$$

$$\mathscr{N}_2\left\{\left(r\frac{d}{dr}\right)^2 f\right\} = \mathscr{M}\{rg'_1(r)\} = -sG_1(s) = s^2F_2(s) - 2sf(1). \tag{10}$$

2.4. Convolution formulas. Consider $\mathscr{N}_{1,2}\{f\} = F_{1,2}(s) = \mathscr{M}\{f_{1,2}\}$ and $\mathscr{N}_{1,2}\{g\} = G_{1,2}(s) = \mathscr{M}\{g_{1,2}\}$. In terms of the extended functions $f_{1,2}$ and $g_{1,2}$ the standard Mellin convolution formula for $\mathscr{M}^{-1}\{F_i(s)G_j(s)\}$ is applicable provided F_i and G_j have a common strip of convergence. Thus

$$\mathscr{M}^{-1}\{F_i(s)G_j(s)\} = \int_0^\infty x^{-1} f_i(r/x) g_j(x)\,dx. \tag{11}$$

Before proceeding further, one should note that the definitions (1) and (2) imply that $F_1(s)$ and $F_2(s)$ are respectively even and odd functions of s. Thus one can

[1] In certain cases one can consider $\alpha = 0$ as a limiting case. Thus for each $\delta > 0$, $\mathscr{N}_2\{r^\delta\} = 2s/(s^2 - \delta^2)$, $|\operatorname{Re} s| < \delta$. In a limiting sense, we shall consider $\mathscr{N}_2\{1\} = 2/s$, defined on $\operatorname{Re} s = 0$ except at $s = 0$. The inversion integral on $\operatorname{Re} s = 0$, interpreted as a Cauchy principal value at $s = 0$, is applicable.

readily obtain the following meaningful formulas in terms of f and g on their basic domain $[0, 1]$:

$$(12)^2 \quad \begin{aligned} \mathcal{N}_1^{-1}\{F_1(s)G_1(s)\} &= \int_0^r x^{-1}[f(x/r) + f(rx)]g(x)\,dx \\ &\quad + \int_r^1 x^{-1}[f(r/x) + f(rx)]g(x)\,dx \\ &= \int_0^1 x^{-1}[f_1(rx) + f_1(x/r)]g(x)\,dx, \end{aligned}$$

$$(13) \quad \begin{aligned} \mathcal{N}_1^{-1}\{F_2(s)G_2(s)\} &= -\int_0^r x^{-1}[f(x/r) + f(rx)]g(x)\,dx \\ &\quad + \int_r^1 x^{-1}[f(r/x) - f(rx)]g(x)\,dx \\ &= -\int_0^1 x^{-1}[f_2(rx) + f_2(x/r)]g(x)\,dx, \end{aligned}$$

$$(14) \quad \begin{aligned} \mathcal{N}_2^{-1}\{F_1(s)G_2(s)\} &= \int_0^r x^{-1}[f(x/r) - f(rx)]g(x)\,dx \\ &\quad + \int_r^1 x^{-1}[f(r/x) - f(rx)]g(x)\,dx \\ &= \int_0^1 x^{-1}[f_1(x/r) - f_1(rx)]g(x)\,dx. \end{aligned}$$

2.5. Extension of another Mellin property. As in the case of Mellin transforms (see Harrington [2]), the application of Naylor transforms to problems involving the Laplacian operator in polar coordinates often leads to transforms, $\hat{f}(s)\cos\theta s$ or $\hat{f}(s)\sin\theta s$, where $\hat{f}(s)$ is a known transform. The extension of the Mellin property [2] to Naylor transforms is not automatic but the analogous results can be established for $\mathcal{N}_1$ and $\mathcal{N}_2$ as set forth in the following two theorems.

THEOREM 1. *Let f be a real continuous function on $(0, \infty)$ such that*[3] $f(1/r) = f(r)$ *with* $\mathcal{M}\{f\} = \mathcal{N}_1\{f\} = \hat{f}(s)$, $|\mathrm{Re}\,(s)| < \alpha$. *If, within some sector,* $-\beta < \arg z < \beta$, *f has an analytic extension in the subregion* $0 < |z| < 1$ *with continuity on the region to include* $|z| = 1$ *and if* $\lim_{z\to 0} z^s f(z) = 0$, $|\mathrm{Re}\,(s)| < \alpha$, *then for* $|\theta| < \beta$,

$$(15) \qquad \hat{f}(s)\cos\theta s = \mathcal{N}_1\{\mathrm{Re}\,f(re^{i\theta})\} \quad \text{and} \quad \hat{f}(s)\sin\theta s = \mathcal{N}_2\{-\mathrm{Im}\,f(re^{i\theta})\}.$$

THEOREM 2. *In Theorem* 1, *interchange* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *and change the restrictive property on f to read*: "$f(1/r) = -f(r)$".

Proof of Theorem 2. In the sector $|\arg z| < \beta$, $|z| < 1$, one has $f(\bar{z}) = \overline{f(z)}$ and $2\,\mathrm{Re}\,f(z) = f(z) + f(\bar{z})$. For each θ, $0 \leqq \theta < \beta$, let L_θ and $L_{-\theta}$ denote, as in

[2] In the last integral form one can observe that the integral defines a function $h(r)$ on $(0, \infty)$ such that $h(1/r) = h(r)$. This is also true in (13); in (14), $h(1/r) = -h(r)$.

[3] Functions arising from convolution have this property. See (12) and (13).

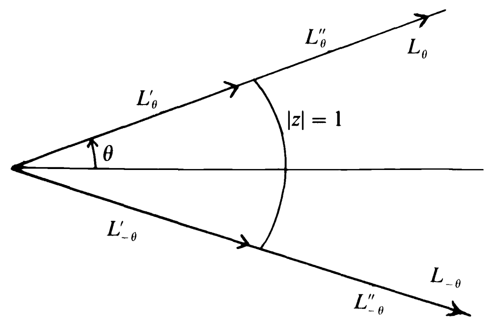

FIG. 1

Fig. 1, the directed radial lines each consisting of segments L'_θ, L''_θ and $L'_{-\theta}$, $L''_{-\theta}$ respectively.

We consider the real-valued function,

$$2g(r) = f(re^{i\theta}) + f(re^{-i\theta}), \qquad 0 < r < 1,$$

and

$$(16) \qquad 2\mathcal{N}_2\{g\} = \int_0^1 [f(re^{i\theta}) + f(re^{-i\theta})](r^{s-1} - r^{-s-1})\,dr = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_0^1 f(re^{i\theta})r^{s-1}\,dr = e^{-is\theta}\int_{L'_\theta} f(z)z^{s-1}\,dz,$$

$$I_2 = \int_0^1 f(re^{-i\theta})r^{s-1}\,dr = e^{is\theta}\int_{L'_{-\theta}} f(z)z^{s-1}\,dz,$$

$$I_3 = -\int_0^1 f(re^{i\theta})r^{-s-1}\,dr = e^{is\theta}\int_{L''_{-\theta}} f(z)z^{s-1}\,dz$$

and

$$I_4 = -\int_0^1 f(re^{-i\theta})r^{-s-1}\,dr = e^{-is\theta}\int_{L''_\theta} f(z)z^{s-1}\,dz.$$

In the consideration of I_3 and I_4, we extend f into the region $|z| > 1, |\arg z| < \beta$, by defining $f(z) = -f(1/z)$. Here also f is analytic with continuity on the region including the boundary $|z| = 1$.

Thus (16) can be written in the form

$$(17) \qquad 2\mathcal{N}_2\{g\} = e^{-is\theta}\int_{L_\theta} f(z)z^{s-1}\,dz + e^{is\theta}\int_{L_{-\theta}} f(z)z^{s-1}\,dz.$$

If each of the complex integrals in (17) is equal to the corresponding integral along $\theta = 0$, then one obtains

$$\mathcal{N}_2\{g\} = (\cos\theta s)\,\mathcal{N}_2\{f\},$$

where

$$g(r) = \operatorname{Re} f(re^{i\theta}).$$

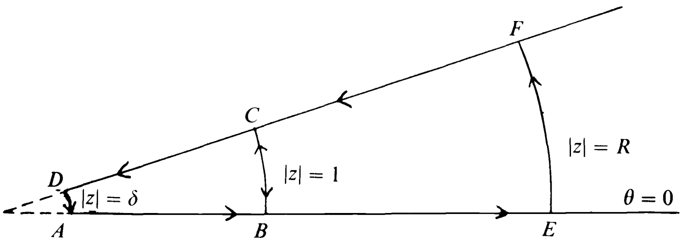

FIG. 2. *Paths of integration*

To establish this, consider the paths of integration $\Gamma_1 = ABCDA$ and $\Gamma_2 = BEFCB$ as shown in Fig. 2.

By the stronger form of the Cauchy integral theorem [3],

$$\int_{\Gamma_1} f(z)z^{s-1}\,dz = 0 = \int_{\Gamma_2} f(z)z^{s-1}\,dz.$$

The conditions $\lim_{z\to 0} z^s f(z) = \lim_{|z|\to\infty} z^s f(z) = 0$,[4] $|\mathrm{Re}\,(s)| < \alpha$, imply that the integrals on the arcs, $|z| = \delta$ and $|z| = R$, tend to zero as $\delta \to 0^+$ and $R \to \infty$ respectively. Also the contributions to the two integrals from the arc BC nullify each other.

Thus, with the analogous argument relating to $L_{-\theta}$, we conclude that

$$\int_{L_\theta} f(z)z^{s-1}\,dz = \int_{L_{-\theta}} f(z)z^{s-1}\,dz = \int_0^\infty r^{s-1}f(r)\,dr = \mathscr{M}\{f\} = \mathscr{N}_2\{f\}.$$

Similar considerations of the combination $f(re^{i\theta}) - f(re^{-i\theta})$ yield the other conclusion of Theorem 2.

The proof of Theorem 1 is analogous.

3. Naylor transforms on $1 < r < \infty$ **for polar coordinates.** Closely related to the transforms $\mathscr{N}_1$ and $\mathscr{N}_2$ of § 2 are two transforms defined by Naylor on the domain $1 < r < \infty$:

(18) $$\mathscr{N}_3\{f(r)\} = F_3(s) \triangleq \int_1^\infty (r^{s-1} + r^{-s-1})f(r)\,dr,$$

(19) $$\mathscr{N}_4\{f(r)\} = F_4(s) \triangleq \int_1^\infty (r^{s-1} - r^{-s-1})f(r)\,dr.$$

We simply note that if $g(r) = f(1/r)$, $0 < r < 1$, then directly from (18) and (19) one obtains

(20) $$\mathscr{N}_3\{f(r)\} = \mathscr{N}_1\{g(r)\},$$

(21) $$\mathscr{N}_4\{f(r)\} = \mathscr{N}_2\{-g(r)\}.$$

The conditions, properties, and theorems of §§ 2.1–2.5 can be readily translated so as to apply equally well to $\mathscr{N}_3$ and $\mathscr{N}_4$ on the domain $(1, \infty)$.

---

[4] The second of these conditions follows from the first because of the property $f(z) = -f(1/z)$, $|z| > 1, |\arg z| < \beta$.

4. Naylor transforms on (0, 1) and (1, ∞) for spherical coordinates. In [1] Naylor also introduces two transforms designed for use with spherical coordinates. On (0, 1), consider a function g satisfying conditions (i) and (ii) of § 2.1. We define the transform N_2 as follows:

$$N_2\{g(r); r \to s\} = G_2(s) \triangleq \int_0^1 (r^s - r^{-s-1})g(r)\,dr \tag{22}$$

on the strip $-\alpha - 1 < \operatorname{Re} s < \alpha$. Letting

$$g_2(r) \triangleq \begin{cases} rg(r) , & 0 < r < 1, \\ -g(1/r), & r > 1, \end{cases} \tag{23}$$

one has

$$N_2\{g\} = \mathscr{M}\{g_2\}. \tag{24}$$

The inversion formula is obtained directly, namely,

$$g(r) = \int_L r^{-s-1}G_2(s)\,ds, \qquad 0 < r < 1, \tag{25}$$

where L denotes a line, $\operatorname{Re} s = \lambda$ with $-\alpha - 1 < \lambda < \alpha$. The important operational property is[5]

$$N_2\left\{\frac{d}{dr}[r^2g'(r)]\right\} = s(s + 1)G_2(s) - (2s + 1)g(1). \tag{26}$$

Similarly if $g(r)$ is defined on $(1, \infty)$, the counterpart of $\mathscr{N}_4$ is given by

$$N_4\{g(r)\} = N_2\{h(r)\}, \tag{27}$$

where $h(r) = -(1/r)g(1/r), 0 < r < 1$.

5. Applications. The Naylor transforms $\mathscr{N}_1$, $\mathscr{N}_2$ and N_2, described in the preceding sections, are particularly applicable to Laplace's equation in polar and spherical coordinates where the region is a finite sector or finite spherical cone. In the use of the transforms, a variety of boundary conditions can be handled particularly on the circular or spherical surface, $r = 1$.

The following example is presented primarily to exhibit the use of some of the transform properties of §§ 2. A second paper is planned in which solutions to some boundary value problems for finite spherical cones will be given.

We consider the steady-state heat problem in a finite sector (or wedge), where $T(r, \theta)$ satisfies

$$r^2T_{rr} + rT_r + T_{\theta\theta} = 0, \qquad 0 < r < 1, \quad |\theta| < \alpha < \pi, \tag{28}$$

[5] There is a transform analogous to $\mathscr{N}_1$, employing a + instead of − in (22). Unfortunately,

$$N_1\left\{\frac{d}{dr}[r^2g'(r)]\right\} = s(s + 1)G_1(s) + 2g'(1) + g(1),$$

involving both $g'(1)$ and $g(1)$.

with boundary conditions

(29) $$T(r, \pm\alpha) = T_0 = \text{const.}; \qquad T_r(1, \theta) + hT(1, \theta) = 0, \quad h > 0.$$

Because of the radiation boundary conditions at $r = 1$, we do not apply a Naylor transform directly to $T(r, \theta)$. Instead, we observe that the function

(30) $$v(r, \theta) = rT_r + hT$$

will satisfy Laplace's equation, if $T(r, \theta)$ does, and hence we consider the related problem

(31) $$r^2 v_{rr} + rv_r + v_{\theta\theta} = 0, \qquad 0 < r < 1, \quad |\theta| < \alpha < \pi,$$

with

(32) $$v(r, \pm\alpha) = hT_0, \qquad v(1, \theta) = 0.$$

Letting $V(s, \theta) = \mathcal{N}_2\{v(r, \theta); r \to s\}$, one obtains from the application of the $\mathcal{N}_2$ transform,

(33) $$\frac{d^2 V}{d\theta^2} + s^2 V = 0, \quad \text{with } V(s, \pm\alpha) = 2hT_0/s.^6$$

Thus

(34) $$V(s, \theta) = \frac{2hT_0 \cos \theta s}{s \cos \alpha s}.$$

Letting $k = \pi/2\alpha$, one finds that

(35) $$\mathcal{M}\{r^k/(1 + r^{2k})\} = \mathcal{N}_1\{r^k/(1 + r^{2k})\} = \alpha \sec \alpha s.$$

The application of convolution formula (14), with $g(r) = 1$, yields[6]

(36) $$\mathcal{N}_2^{-1}\left\{\frac{2\alpha \sec \alpha s}{s}\right\} = \frac{1}{k}\left[\frac{\pi}{2} - 2 \arctan r^k\right] = h(r).$$

Since $h(1/r) = -h(r)$, we employ Theorem 2 to obtain (see [2])

(37) $$v(r, \theta) = hT_0\left[1 - \frac{2}{\pi} \arctan \frac{2r^k \cos k\theta}{1 - r^{2k}}\right], \qquad 0 < r < 1.$$

Integration of (30), with $v(r, \theta)$ given by (37), yields

(38) $$T(r, \theta) = T_0\left[1 - \frac{2h}{\pi r^h}\int_0^r x^{h-1} \arctan \frac{2x^k \cos k\theta}{1 - x^{2k}} dx\right].$$

An equivalent form is

(39) $$T(r, \theta) = T_0\left[1 - \frac{4h}{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{r^{(2n+1)k} \cos (2n + 1)k\theta}{(2n + 1)[(2n + 1)k + h]}\right].$$

REFERENCES

[1] DEREK NAYLOR, *On a Mellin type integral transform*, J. Math. Mech., 12 (1963), pp. 265–274.

[2] W. J. HARRINGTON, *A property of Mellin transforms*, SIAM Rev., 9 (1967), pp. 542–547.

[3] MAURICE HEINS, *Selected Topics in the Classical Theory of Function of a Complex Variable*, Holt, Rinehart and Winston, New York, 1962.

[6] This is meaningful in the limiting sense indicated in footnote 1.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

THE RAYLEIGH-FABER-KRAHN THEOREM FOR THE CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS*

CATHERINE BANDLE†

Abstract. This paper is concerned with functionals which were introduced by Nehari and also discussed by Coffman in connection with the study of nonlinear boundary value problems. Their behavior under the Schwarz symmetrization is studied, and an isoperimetric inequality analogous to that of Rayleigh–Faber–Krahn for the fundamental frequency of a vibrating membrane is derived.

Introduction. Let Ω be a bounded region in R^n for which the Green's function for the Laplace operator exists. We shall write P for an arbitrary point in R^n and R_+ for the positive real axis. Let $F(s, P)$ be a positive function on $\bar{R}_+ \times \Omega$ with the following properties:

(A) $F(\cdot, x)$ is continuous on $\bar{R}_+$ for almost all $x \in \Omega$. $F(s, \cdot)$ is measurable for all $s \in \bar{R}_+$.

(B) There exists a positive number ε such that for almost all $P \in \Omega$ and for all $s_1 < s_2$,

$$s_1^{-\varepsilon}F(s_1, P) \leqq s_2^{-\varepsilon}F(s_2, P).$$

We define the function $G(t, P)$ by

$$G(t, P) = \int_0^t F(s, P)\,ds,$$

and consider the functional

$$H(v) = \mathscr{D}(v) - \int_\Omega G(v^2, P)\,dx$$

(dx = volume element in R^n, $\mathscr{D}(v) = \int_\Omega \operatorname{grad}^2 v\,dx$, $(x^1, x^2, \cdots, x^n)$ are Cartesian coordinates), within the class Γ of piecewise continuously differentiable functions which vanish on the boundary $\partial\Omega$ and are not identically zero in Ω. This note will be concerned with the functional

$$\Lambda(\Omega) = \min_v H(v),$$

where v ranges over all functions in Γ satisfying the side condition

$$\mathscr{D}(v) = \int_\Omega v^2F(v^2, P)\,dx. \tag{1}$$

Following Nehari we call $\Lambda(\Omega)$ the *characteristic value.*

Nehari [9] showed that if $\Omega = (a, b)$, there exists a function $u \in \Gamma$ subject to (1) which minimizes $H(v)$. Furthermore this function is a solution of the differential

* Received by the editors October 7, 1971, and in revised form January 7, 1972.

† Department of Mathematics, Stanford University, Stanford, California 94305. This work was supported by the National Science Foundation under Grant GU-2056.

equation $u'' + uF(u^2, P) = 0$ in (a, b), $u(a) = 0$ and $u(b) = 0$. It might be observed that the minimum of $H(v)$ without additional restriction on v does not exist in general. Coffman [2] generalized Nehari's result for the case where $\Omega \subset R^n$, $n \geqq 2$. It can be stated as follows:

Let $F(s, P)$ be locally Hölder continuous on $\bar{R}_+ \times \Omega$, and suppose that there are positive constants σ, c and $\gamma < 2/(n-2)$ such that $F(s, P) \leqq cs^\gamma + \sigma$ for all $s \in R_+$. (In R^2 there is no restriction on γ.) If we assume further that (A) and (B) hold, then $\Lambda(\Omega)$ exists, and the minimizing function u is of class C^2 in Ω, and solves the Dirichlet problem $\Delta u + uF(u^2, P) = 0$ in Ω, $u = 0$ on $\partial\Omega$ ($\Delta = \sum_{i=1}^n \partial^2/(\partial x^i)^2$ is the Laplacian). If $\Omega \subset R^2$, then more general results can be found in [6], [1]. In a physical system $\Lambda(\Omega)$ corresponds to the energy.

Example. Consider a membrane which covers at rest a region of the (ξ, η)-plane. We assume that the interior of the membrane is subject to an external force $(0, 0, -zF(z^2, x))$ $(x = (\xi, \eta))$; the membrane is fixed on the boundary. Let the deformation normal to the equilibrium plane be denoted by $u(x, y)$ and suppose that this deformation is small. If the modulus of elasticity is 1, then the equilibrium position is described by the equation $\Delta u + uF(u^2, x) = 0$ in Ω, $u = 0$ on $\partial\Omega$. The total energy is $\frac{1}{2}\Lambda(\Omega)$.

In this paper we shall derive a bound for $\Lambda(\Omega)$ which depends only on F and on the volume of Ω.

1. Nehari [8] proved that for every function $v \in \Gamma$ satisfying (1) the inequality

$$H(v) \geqq \frac{\varepsilon}{1+\varepsilon} \int_\Omega v^2 F(v^2, P)\, dx$$

holds. $\Lambda(\Omega)$ is therefore bounded from below. It was pointed out in [8] that for every function $v \in \Gamma$ there exists a constant $\alpha \neq 0$ such that αv satisfies the side condition (1). This is an immediate consequence of (B) and the fact that $\lim_{s \to 0} F(s, P) = 0$ and $\lim_{s \to \infty} F(s, P) = \infty$.

In order to estimate $\Lambda(\Omega)$ we shall use the following property of $H(v)$.

LEMMA 1. *If v satisfies* (1), *then the inequality*

$$H(\alpha v) \leqq H(v)$$

holds for every real number α.

Proof. Since $F(s, P)$ is nondecreasing, $G(s, P)$ is concave and hence $G(s_0, P) - G(s_1, P) \geqq (s_0 - s_1)F(s_1, P)$. Thus, observing (1) we have

$$\begin{aligned} H(\alpha v) - H(v) &= (\alpha^2 - 1)\mathscr{D}(v) - \int_\Omega \{G(\alpha^2 v^2, P) - G(v^2, P)\}\, dx \\ &\leqq (\alpha^2 - 1)\mathscr{D}(v) - \int_\Omega (\alpha^2 v^2 - v^2) F(v^2, P)\, dx = 0. \end{aligned}$$

From this simple lemma it follows that

$$\Lambda(\Omega) = \min_{v = 0 \text{ on } \partial\Omega} \max_\alpha H(\alpha v).$$

This minimum property and a reflection argument show that the minimizing function is positive. It is also easily seen that $\Lambda(\Omega)$ is a monotonic decreasing functional of the domain.

For the following considerations we shall need the *Schwarz symmetrization* [12]. By this symmetrization a domain $B \subseteq R^n$ is transformed into a n-sphere B^* with the center at the origin and the same volume as B. A positive measurable function f on B with $f = 0$ on ∂B is transformed into a function f^* on B^* in the following way: Let B_t denote the region $B_t = \{P \in B; f(P) > t\}$. f^* is the radially symmetrical function with $f^* > t$ on B_t^* and $f^* = t$ on ∂B_t^*. The next result is based on the inequality of Rayleigh–Faber–Krahn [12] for vibrating membranes.

THEOREM 1. *Let $F(s, P) = F(s)$ satisfy* (A) *and* (B) *and be independent of P. Then among all regions $\Omega \subseteq R^n$ with a given volume the n-sphere yields the minimal value of* $\Lambda(\Omega)$.

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions in Γ, subject to the side condition (1), and with the property

$$\Lambda(\Omega) = \lim_{n\to\infty} H(u_n).$$

We denote by u_n^* the function obtained from u_n after the Schwarz symmetrization. For each u_n^* we determine a number α_n such that

$$\mathscr{D}_{\Omega^*}(\alpha_n u_n^*) = \int_{\Omega^*} \alpha_n^2 u_n^{*2} F(\alpha_n^2 u_n^{*2})\, dx.$$

It follows from the definition of u_n^* that $\int_\Omega G(\alpha_n^2 u_n^2)\, dx = \int_{\Omega^*} G(\alpha_n^2 u_n^{*2})\, dx$. Since the symmetrization diminishes the Dirichlet integral, we have $\mathscr{D}_\Omega(\alpha_n u_n) \geqq \mathscr{D}_{\Omega^*}(\alpha_n u_n^*)$ (see [12]), and thus by Lemma 1,

$$\begin{aligned}\Lambda(\Omega) &\geqq \lim_{n\to\infty}\left[\mathscr{D}_\Omega(\alpha_n u_n) - \int_\Omega G(\alpha_n^2 u_n^2)\, dx\right] \\ &\geqq \liminf_{n\to\infty}\left[\mathscr{D}_{\Omega^*}(\alpha_n u_n^*) - \int_{\Omega^*} G(\alpha_n u_n^{*2})\, dx\right].\end{aligned}$$

This inequality together with the minimum property of $\Lambda(\Omega^*)$ proves that $\Lambda(\Omega) \geqq \Lambda(\Omega^*)$.

Remark 1. The same arguments show that $\Lambda(\Omega)$ is diminished by the Steiner symmetrization [12].

Remark 2. Suppose that Ω is a sphere and that the minimizing function u of the variational problem exists. Then u is radially symmetric and nonincreasing in $r(r^2 = \sum_{i=1}^n (x^i)^2)$. From this fact it is not difficult to obtain a lower bound for the maximal value of the function u which solves the Dirichlet problem $\Delta u + uF(u^2) = 0$ in $\Omega = \{x; |x| \leqq R\}$, $u = 0$ on $\partial\Omega$, and yields the minimum of $H(v)$. As an example we consider the case where $F(s) = s^m$ and $\Omega = \{(x, y);\ x^2 + y^2 \leqq 1\}$. u has the representation

$$u(z) = \frac{-1}{2\pi}\iint_\Omega \ln|z - z'| u^{2m+1}(z')\, dA_{z'}, \tag{2}$$

$$z = x + iy, \quad z' = x' + iy', \quad dA_{z'} = dx'\, dy'.$$

Since $\max_{z\in\Omega} u(z) = u(0)$, and since u is decreasing, we have

$$u(0) \geqq 2^{1/m}.$$

Remark 3. Some growth conditions on $F(s)$ are necessary in order to obtain a minimizing function of class $C^2(\Omega)$. Indeed, consider the functional

$$\Lambda(\Omega) = \min_{v=0 \text{ on } \partial\Omega} \max_{\alpha} \left\{ \mathscr{D}(\alpha v) - \frac{1}{m+1} \int_\Omega (\alpha v)^{2m+2}\, dx \right\}, \tag{3}$$

and suppose that the minimizing function u is of class $C^2(\Omega)$. It is therefore a solution of the corresponding Euler equation $\Delta u + u^{2m+1} = 0$ in Ω, $u = 0$ on $\partial\Omega$. If $u(r)$ is the solution for $\Omega_1 = \{x; |x| \leqq 1\}$, then $t^{-1/m}u(r/t)$ is the solution for the sphere $\Omega_t = \{x; |x| \leqq t\}$. An easy computation yields

$$\Lambda(\Omega_t) = t^{-(2m+2)/m+n}\Lambda(\Omega_1), \tag{4}$$

where n is the dimension of the space. Since $\Lambda(\Omega_t)$ is a monotonic functional of t, we must have $m \leqq 2/(n-2)$. This condition was obtained by Pohozaev [11] in a different way. If we compute the value for α, then (3) becomes

$$\Lambda(\Omega) = \min_{v=0 \text{ on } \partial\Omega} \frac{m}{m+1} \left[\frac{\mathscr{D}(v)^{m+1}}{\int_\Omega v^{2m+2}\, dx} \right]^{1/m}. \tag{5}$$

We now consider the case $n = 3$, $m > 2$. If we take

$$v = \begin{cases} \cos \pi k r/2 & \text{in } [0, 1/k], \\ 0 & \text{in } [1/k, 1], \end{cases}$$

$k > 1$, then v is admissible for the variational characterization (5) of $\Lambda(\Omega_1)$. The computation shows that the right side of (5) tends to zero if $k \to \infty$. Hence $\Lambda(\Omega_1) = 0$, and by the same argument and the monotonicity of $\Lambda(\Omega)$, we can prove that $\Lambda(\Omega) = 0$ for an arbitrary domain Ω. If $n = 3$ and $m = 2$, there exists a constant $\beta > 0$ such that

$$\Lambda(\Omega) = \beta \quad \text{for all } \Omega.$$

Because of an inequality by Ladyzhenskaja [5]:

$$\{\mathscr{D}(v)\}^3 \geqq 48 \int_\Omega v^6\, dx^1\, dx^2\, dx^3,$$

it follows that $\Lambda(\Omega) > 0$ for all Ω. If we can show that β is the same for all circles, then the assertion will be proved. Let $t_1 \leqq t_2$, and $u_n(r)$ be a sequence of radially symmetrical functions subject to (1) such that

$$\Lambda(\Omega_{t_2}) = \lim_{n\to\infty} H(u_n).$$

The functions $v_n = t_0^{-1/2}u_n(r/t_0)$ with $t_0 = t_2/t_1$ are admissible for the variational characterization of $\Lambda(\Omega_{t_1})$. Hence

$$\Lambda(\Omega_{t_1}) \leqq \lim_{v_n\to\infty} H(v_n) = \Lambda(\Omega_{t_2}).$$

On the other hand, we have from the monotonicity that

$$\Lambda(\Omega_{t_2}) \leqq \Lambda(\Omega_{t_1}).$$

2. Let $\Omega \subseteq R^2$, and consider functions $F(s, P)$ of the form

$$F(s, P) = \rho(P)F_0(s) + \sigma(P),$$

where $\rho(P)$ and $\sigma(P)$ are positive on Ω and $F_0(s)$ satisfies (A) and (B) of § 1. If the least eigenvalue of the membrane problem $\Delta u + \lambda \sigma u = 0$ in Ω, $u = 0$ on $\partial\Omega$ exceeds 1, then it is possible to find for each function $w \in \Gamma$ a constant $\alpha \neq 0$ such that (1) holds for $v = \alpha w$ (see [2], [3], [8], [9]). Under all these assumptions $\Lambda(\Omega)$ exists. If $F(s, P)$ is locally Hölder continuous on $\bar{R}_+ \times \Omega$, then there is a minimizing function u which solves the boundary value problem $\Delta u + u\{\rho(P)F_0(u^2(P)) + \sigma(P)\} = 0$ in Ω, $u = 0$ on $\partial\Omega$ (see [2]). It may be observed that $\lambda_1 > 1$ is also necessary for the existence of the function u. Indeed, since u does not change sign, it can be interpreted as the first eigenfunction of the problem $\Delta v + \mu m(P)v = 0$ in Ω, $v = 0$ on $\partial\Omega$, where $m(P) = \rho(P)F_0(u^2(P)) + \sigma(P)$. We have $m(P) > \sigma(P)$, and by the monotonicity of the eigenvalues $1 = \mu_1 < \lambda_1$ (see [4]).

We shall use the following notations:

$$M_\rho(B) = \iint_B \rho\, dx\, dy, \qquad M_\sigma(B) = \iint_B \sigma\, dx\, dy,$$

where $B \subseteq \Omega$ is an arbitrary domain, and x, y are the Cartesian coordinates. Let $r = \sqrt{x^2 + y^2}$, and K be an arbitrary real number. Then we define

$$g_K(r) = \begin{cases} \dfrac{4}{|K|(1 + r^2)^2} & \text{if } K > 0, \\ \dfrac{4}{|K|(1 - r^2)^2} & \text{if } K < 0, \\ 1 & \text{if } K = 0, \end{cases}$$

and $M_{gK}(B) = \iint_B g_K\, dx\, dy$. Let $F_0(s)$ be fixed, and consider $\Lambda(\Omega) = \Lambda(\Omega, \sigma, \rho)$ as a function of Ω, σ and ρ. Ω_σ^* denotes the circle with the property

$$\iint_{\Omega_\sigma^*} g_K\, dx\, dy = \iint_\Omega \sigma\, dx\, dy,$$

and Ω_ρ^* is defined in an analogous way. The next result is a generalization of Theorem 1 of § 1. It is related to some extensions of the Rayleigh–Faber–Krahn inequality for inhomogeneous membranes [1], [10].

In order to simplify the proof we shall assume that there exists a function $u \in C^2$ belonging to Γ and subject to (1) which yields the minimum of $H(v)$. Otherwise we have to consider a minimizing sequence as we did in the proof of Theorem 1.

THEOREM 2. *Suppose that Ω is simply connected, $\lambda_1 > 1$ and that there exists a number K such that the following inequalities hold in Ω:*

$$\frac{-\Delta \ln \rho}{2\rho} \leqq K, \quad \frac{-\Delta \ln \sigma}{2\sigma} \leqq K, \quad 4\pi - KM_\rho > 0 \quad \text{and} \quad 4\pi - KM_\sigma > 0.$$

(a) *If $\Omega_\rho^* \subseteq \Omega_\sigma^*$, and if the first eigenvalue of the problem $\Delta u + \mu g_K u = 0$ in Ω_σ^*, $u = 0$ on $\partial\Omega_\sigma^*$, exceeds 1, then we have, for fixed $F_0(s)$,*

$$\Lambda(\Omega, \sigma, \rho) \geqq \Lambda(\Omega_\sigma^*, g_K, (M_\rho/M_\sigma)g_K),$$

$M_\rho = M_\rho(\Omega)$, $M_\sigma = M_\sigma(\Omega)$.

(b) *If* $\Omega_\sigma^* \subseteq \Omega_\rho^*$, *and if the first eigenvalue of the problem* $\Delta u + \mu(M_\sigma/M_\rho)g_K u = 0$ *in* Ω_ρ^*, $u = 0$ *on* $\partial\Omega_\rho^*$, *exceeds* 1, *then*

$$\Lambda(\Omega, \sigma, \rho) \geqq \Lambda(\Omega_\rho^*, (M_\sigma/M_\rho)g_K\, dx\, dy.$$

Proof of part (a). If f is an arbitrary positive function, let $B^*_{(f)}$ denote the circle with center at the origin and the property that

$$\iint_B f\, dx\, dy = \iint_{B^*_{(f)}} g_K\, dx\, dy.$$

Let $\Omega(t) = \{P \in \Omega; u(P) \geqq t\}$, and let $u^*_{(f)}$ be the radially symmetrical function on $\Omega^*_{(f)}$ such that $u^*_{(f)} \geqq t$ in $\Omega^*_{(f)}(t)$ and $u^*_{(f)} = t$ on $\partial\Omega^*_{(f)}(t)$. We shall write $c = M_\sigma/M_\rho$, $\bar\rho = c\rho$ and $h(P) = \max\{u^*_{(\bar\rho)}(P), u^*_{(\sigma)}(P)\}$. Since $\mu_1 > 1$, there exists a number α such that

$$\mathscr{D}_{\Omega^*_{(\sigma)}}(\alpha h) = \iint_{\Omega^*_{(\sigma)}} \alpha^2 h^2 \{F_0(\alpha^2 h^2)c^{-1}\, g_K + g_K\}\, dx\, dy.$$

The proof is based on the following lemma [1].

LEMMA 2. *Let* v *be an arbitrary positive function in* Ω *which vanishes on the boundary* $\partial\Omega$. *Let* $G(t)$ *be the domain* $\{P \in \Omega; v(P) \geqq t\}$. *If a positive function* f *satisfies in* Ω *the inequalities* $(-\Delta \ln f)/2f \leqq K$ *and* $4\pi - K\iint_\Omega f\, dx\, dy > 0$, *then for every* $(t_1, t_2)\,(t_1 \leqq t_2)$,

$$\iint_{G(t_1)\setminus G(t_2)} \operatorname{grad}^2 v\, dx\, dy \geqq \iint_{G^*_{(f)}(t_1)\setminus G^*_{(f)}(t_2)} \operatorname{grad}^2 v^*_{(f)}\, dx\, dy. \tag{6}$$

Because of the assumptions regarding σ, it follows therefore that

$$\iint_{\Omega(t_1)\setminus\Omega(t_2)} \operatorname{grad}^2 u\, dx\, dy \geqq \iint_{\Omega^*_{(\sigma)}(t_1)\setminus\Omega^*_{(\sigma)}(t_2)} \operatorname{grad}^2 u^*_{(\sigma)}\, dx\, dy \tag{7}$$

for all $t_1 \leqq t_2$.

Because of $c \geqq 1$, we have $(-\Delta \ln \bar\rho)/2\bar\rho \leqq K/c \leqq K$. Since $4\pi - K\iint_\Omega \bar\rho\, dx\, dy = 4\pi - KM_\sigma \geqq 0$, we can apply Lemma 2 to $u^*_{(\bar\rho)}$, and we obtain

$$\iint_{\Omega(t_1)\setminus\Omega(t_2)} \operatorname{grad}^2 u\, dx\, dy \geqq \iint_{\Omega^*_{(\bar\rho)}(t_1)\setminus\Omega^*_{(\bar\rho)}(t_2)} \operatorname{grad}^2 u^*_{(\bar\rho)}\, dx\, dy. \tag{8}$$

From (7) and (8) we conclude that

$$\mathscr{D}_\Omega(\alpha u) \geqq \mathscr{D}_{\Omega^*_{(\sigma)}}(\alpha h). \tag{9}$$

The following relations are immediate consequences of the definition of $u^*_{(\bar\rho)}$ and $u^*_{(\sigma)}$:

$$\iint_\Omega \left\{\int_0^{\alpha^2 u^2} F_0(s)\, ds\right\}\rho\, dx\, dy = \iint_{\Omega^*_{(\sigma)}} \left\{\int_0^{\alpha^2 u^{*2}_{(\bar\rho)}} F_0(s)\, ds\right\} c^{-1} g_K\, dx\, dy \tag{10}$$

and

$$\iint_\Omega \alpha^2 u^2 \sigma\, dx\, dy = \iint_{\Omega^*_{(\sigma)}} \alpha^2 u^{*2}_{(\sigma)} g_K\, dx\, dy. \tag{11}$$

From (10) and (11) and the monotonicity of $F_0(s)$ we have

$$\iint\limits_{\Omega} G(\alpha^2 u^2)\,dx\,dy \leqq \iint\limits_{\Omega^*_{(\sigma)}} \left[\left\{c^{-1}\int_0^{\alpha^2 h^2} F_0\,ds\right\} + \alpha^2 h^2\right] g_K\,dx\,dy,$$

and by (9) and the same arguments as in the proof of Theorem 1,

$$\Lambda(\Omega,\sigma,\rho) \geqq \Lambda(\Omega^*_{(\sigma)}, g_K, (M_\rho/M_\sigma)g_K).$$

Part (b) can be proved in a similar way:

Example. Consider functions σ and ρ such that $\Delta \ln \sigma \geqq 0$, $\Delta \ln \rho \geqq 0$, $M_\sigma/M_\rho \geqq 1$ and $M_\sigma \leqq \pi j_0^2$ ($j_0 = 2.4048 \cdots$, first zero of the Bessel function of order zero). We have $K = 0$ and $g_K = 1$. From the inequality of Nehari for inhomogeneous membranes [10] it follows that $\lambda_1 \geqq \pi j_0^2/M_\sigma \geqq 1$, and from the Rayleigh–Faber–Krahn inequality $\mu_1 \geqq \pi j_0^2/M_\sigma \geqq 1$. Hence, Theorem 2 yields

$$\Lambda(\Omega,\sigma,\rho) \geqq \Lambda(\Omega^*_{(\sigma)}, 1, M_\rho/M_\sigma),$$

where

$$\Omega^*_{(\sigma)} = \{(x,y) \in R^2;\ \sqrt{x^2+y^2} \leqq \sqrt{M_\sigma/\pi}\}.$$

Acknowledgments. The author wishes to thank Professor Z. Nehari for having suggested the problem and for his help during the preparation of this paper. She is also indebted to Professor C. V. Coffman for many helpful discussions.

REFERENCES

[1] C. BANDLE, *Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie*, Comm. Math. Helv., 46 (1971), pp. 182–213.

[2] C. V. COFFMAN, *An existence theorem for a class of nonlinear integral equations with applications to a nonlinear elliptic boundary value problem*, J. Math. Mech., 18 (1968), pp. 411–420.

[3] ———, *A minimum–maximum principle for a class of non-linear integral equations*, J. Analyse Math., 22 (1969), pp. 391–419.

[4] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. 1, Interscience, New York, 1965.

[5] O. A. LADYZHENSKAJA, *The Mathematical Theory of Viscous Incompressible Flow*, rev. ed., Gordon and Breach, New York, 1963.

[6] N. LEVINSON, *Positive eigenfunctions for $\Delta u + \lambda f(u) = 0$*, Arch. Rational Mech. Anal., 11 (1962), pp. 258–272.

[7] Z. NEHARI, *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc., 95 (1960), pp. 101–123.

[8] ———, *Characteristic values associated with a class of nonlinear second order differential equations*, Acta Math., 105 (1961), pp. 141–175.

[9] ———, *On a class of nonlinear integral equations*, Math. Z., 72 (1959), pp. 175–183.

[10] ———, *On the principal frequency of a membrane*, Pacific J. Math., 8 (1958), pp. 285–293.

[11] S. T. POHOZAEV, *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk SSSR, 165 (1965), pp. 36–39 = Soviet Math. Dokl., 5 (1965), pp. 1408–1411.

[12] G. PÓLYA AND G. SZEGÖ, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, 1951.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

A GENERATING OPERATOR FOR SOLUTIONS OF CERTAIN PARTIAL DIFFERENCE AND DIFFERENTIAL EQUATIONS*

JOAN ROHRER HUNDHAUSEN†

Abstract. This paper concerns an algebraic method for generating additional solutions of an n-dimensional homogeneous linear partial difference equation from a known solution. A parallel theory is developed for the continuous case via the Taylor series expansion; the pertinent partial differential equation is linear and homogeneous of order ℓ, with constant coefficients. In both the discrete and the continuous cases, a generating operator is introduced and is shown to commute with the given difference or differential operator, respectively. Applications are presented for both cases.

1. Introduction. In the context of this paper, *generation* refers to an operation performed upon a known solution of a difference or differential equation to produce another solution. Generating processes have been devised for various special forms of difference operators; these include methods of differentiation, integration [2] and convolution of solutions [3], [4]. Continuous analogues of many of these processes may be applied to corresponding differential operators. In particular, algebraic generating processes for harmonic and polyharmonic difference operators have been studied by Duffin and Shelly [5]. The content of this paper extends a result developed in [5] for polyharmonic operators to a more general class of linear partial difference operators with constant coefficients; it is also shown that a parallel theory holds for linear partial differential operators with constant coefficients which are homogeneous of order ℓ. A generating operator is explicitly displayed for each case. The transition between the treatments of the discrete and the continuous cases is provided by the vehicle of the Taylor series.

Although perhaps the more orthodox approach is to derive discrete analogues from the better-known continuous theorems, here is a case which exemplifies a statement appearing in the Editor's Foreword to the text by Miller [7]: "It is possible to derive theorems about differential equations from theorems on difference operators, and the methods might be more transparent in the latter case." Thus the discussion of the discrete case precedes that of the continuous case.

Some applications of the generating process are presented in § 4. The generating operator is developed for the case of the n-dimensional Laplacian operator; both continuous and discrete versions are considered. A particularly interesting application in the discrete case lies in the context of the theory of discrete analytic functions. Here a modification of the generating operator coincides with an operator introduced by Duffin [2] which is useful in generating a sequence of discrete analytic polynomials.

2. The generating operator in the context of difference equations. In preparation for development of the theory in the discrete case, we impose a cubical grid of width h upon n-dimensional Euclidean space. Nodes of the grid structure are

* Received by the editors September 16, 1971.

† Los Alamos Scientific Laboratory of the University of California, Los Alamos, New Mexico 87544. This work was supported by the U.S. Atomic Energy Commission under Contract W-7405-ENG-36.

denoted by points $x = (x_1, x_2, \cdots, x_n)$, where $x_i = \ell_i h$, ℓ_i integer, $i = 1, 2, \cdots, n$. Complex-valued functions $\varphi(x)$ defined at these nodes are designated as discrete or lattice functions. In order to prevent formulas from becoming too cumbersome, we adopt where advantageous the notation introduced by Schwartz [8]. For two vectors

$$x = (x_1, x_2, \cdots, x_n), \qquad m = (m_1, m_2, \cdots, m_n)$$

where the m_i are integers, x^m denotes the monomial $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, $m!$ denotes the product $|m_1|!|m_2|! \cdots |m_n|!$, and $|m|$ denotes the sum $|m_1| + |m_2| + \cdots + |m_n|$. With the designation e_i as the characteristic vector having 1 in the ith position and 0 elsewhere, the fundamental translation operators $X_i^{m_i}$ may be defined concisely as

$$X_i^{m_i}\varphi(x) = \varphi(x + hm_i e_i), \qquad m_i \text{ integer}, \quad i = 1, 2, \cdots, n.$$

The translation operators are clearly linear and commutative, and $X_i^0 \varphi = I\varphi = \varphi$. Also

$$X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} \varphi(x) = X^m \varphi(x) = \varphi\left(x + h \sum_{i=1}^{n} m_i e_i\right).$$

For differentiation, the operational symbols $D_i = \partial/\partial x_i$, $i = 1, 2, \cdots, n$, can be combined into a gradient vector $D = (D_1, \cdots, D_n)$, and for $k = (k_1, k_2, \cdots, k_n)$, where the k_i are nonnegative integers, the general derivatives of $\varphi(x)$ may be abbreviated to $D^k \varphi(x)$. The Taylor series expansion for $X^m \varphi(x)$ becomes

$$X^m \varphi(x) = \sum_{\ell=0}^{\infty} h^\ell \sum_{|k|=\ell} \frac{1}{k!} m^k D^k \varphi(x).$$

Let M represent a linear difference operator of the form

$$M = \sum_m a_m X^m, \tag{1}$$

where m ranges over a finite set of vectors having integer entries and the coefficients $a_m = a_{m_1 m_2 \cdots m_n}$ are complex constants. We are concerned with the family of solutions of the homogeneous difference equation $M\varphi(x) = 0$. Anticipating a form of Taylor series expansion for M, we introduce the associated or *derived* operators

$$M_k \equiv \sum_m a_m m^k X^m.$$

Noting that $M_k(1) = \sum_m a_m m^k$, we may exhibit the relationship between M and its derived operators via the Taylor series expansion as follows:

$$M = \sum_m a_m X^m = \sum_{\ell=0}^{\infty} h^\ell \sum_{|k|=\ell} \frac{1}{k!} M_k(1) D^k. \tag{2}$$

The derived operators themselves have the expansions

$$M_s = \sum_{\ell=0}^{\infty} h^\ell \sum_{|k|=\ell} \frac{1}{k!} M_{s+k}(1) D^k,$$

where $s = (s_1, s_2, \cdots, s_n)$ with s_i a nonnegative integer. Also for $p = (p_1, p_2, \cdots, p_n)$, p_i nonnegative integer, the relation $X^m[x_i \varphi(x)] = (x_i + hm_i) X^m \varphi(x)$ is the basis for derivation of the general formula

$$(3)\qquad M_k[x^p\varphi(x)] = \left[\sum_{r_1=0}^{p_1}\sum_{r_2=0}^{p_2}\cdots\sum_{r_n=0}^{p_n}\binom{p_1}{r_1}\binom{p_2}{r_2}\cdots\binom{p_n}{r_n}x^{p-r}M_{k+r}\right]\varphi(x),$$

where the vector $r = (r_1, r_2, \cdots, r_n)$.

We now define the generating operators in the discrete context.

DEFINITION. $\tilde{M}_{ij} \equiv x_jM_{e_i} - x_iM_{e_j}$, $i \neq j$, $i, j = 1, 2, \cdots, n$.

For future reference we cite the Taylor series representation

$$(4)\qquad \tilde{M}_{ij} = \sum_{\ell=0}^{\infty} h^\ell \sum_{|k|=\ell}\frac{1}{k!}[x_jM_{k+e_i}(1) - x_iM_{k+e_j}(1)]D^k.$$

The following theorem shows that the operators $\tilde{M}_{ij}$ are useful in generating additional solutions of the difference equation $M\varphi(x) = 0$ from a known solution $f(x)$. The proof involves a simple application of (3) and is based upon the condition that the relation $Mf(x) = 0$ holds in a suitably extensive region of discrete Euclidean space, viz., a region containing at least each point $(x + h\sum_{i=1}^n m_ie_i)$, where $m = (m_1, m_2, \cdots, m_n)$ appears in the summation formula (1) for M. Several corollaries follow almost immediately from the theorem; their proofs depend upon extensions of this condition and for the sake of brevity are omitted. Finally, we make the basic assumption that the relation $Mf(x) = 0$ holds in a sufficiently extensive region of discrete Euclidean space and do not repeat it in the statement of the theorem or the corollaries.

THEOREM. *If* $Mf(x) = 0$, *then* $M[\tilde{M}_{ij}f(x)] = 0$, *for* $i \neq j$, $i, j = 1, 2, \cdots, n$.

Proof.

$$\begin{aligned} M[\tilde{M}_{ij}f(x)] &= x_jMM_{e_i}f(x) + M_{e_j}M_{e_i}f(x) - x_iMM_{e_j}f(x) - M_{e_i}M_{e_j}f(x) \\ &= (x_jM_{e_i} - x_iM_{e_j})Mf(x) \\ &= \tilde{M}_{ij}Mf(x) = 0. \end{aligned}$$

The notation M^q indicates that the operator M is to be applied q times in succession; for example, $M^2 = \sum_m\sum_k a_ma_kX^{m+k}$, where m and k range over the same finite set of vectors. Formula (3) is helpful in developing successive powers of the operators $\tilde{M}_{ij}$.

COROLLARY 1. *If* $M^qf(x) = 0$, *then* $M^q[\tilde{M}_{ij}f(x)] = 0$ *for* $i \neq j$, $i, j = 1, 2, \cdots, n$; $q = 1, 2, \cdots$.

COROLLARY 2. *If* $Mf(x) = 0$, *then* $M[\tilde{M}_{ij}^qf(x)] = 0$, $i \neq j$, $i, j = 1, 2, \cdots, n$; $q = 1, 2, \cdots$.

Here a sequence of additional solutions may be generated by repeated application of the generating operator; the proof follows easily by induction. Also different generating operators may be applied successively to a known solution, thus generating additional families of solutions.

COROLLARY 3. *If* $Mf(x) = 0$, *then* $M[\tilde{M}_{ij}[\tilde{M}_{rs}f(x)]] = 0$, $i \neq j$, $r \neq s$, $i, j, r, s = 1, 2, \cdots, n$.

3. The generating operator in the context of differential equations. The representation (2) illustrates the fact that a difference operator M is always an approximation to a linear homogeneous partial differential operator $\mathscr{M}$ of order ℓ having the form

$$\mathcal{M} = \sum_{|k|=\ell} \alpha_k D^k, \tag{5}$$

where $\alpha_k = \alpha_{k_1 k_2 \cdots k_n}$ are complex coefficients; i.e.,

$$\lim_{h \to 0} \frac{M - M(1)I}{h^q} = \mathcal{M}, \qquad q \geqq 1. \tag{6}$$

The exact value of the integer q and the exact form of $\mathcal{M}$ are uniquely determined by the values $M_k(1)$, again emphasizing the essential role played by the derived operators of M. On the other hand, the great variety of difference expressions (and translations thereof) which may be used to approximate derivatives renders possible the approximation of a given form $\mathcal{M}$ by many different forms M. For further details see Collatz [1].

Because of this approximation relationship between the discrete and continuous cases, the theory of the generating operator as developed in § 2 has a parallel in the continuous case; the natural vehicle of transition is the Taylor series. The schematic diagram in Fig. 1 serves as a brief outline of the process by which the generating operators $\tilde{\mathcal{M}}_{ij}$ corresponding to a given operator $\mathcal{M}$ are developed. We assume $\mathcal{M}$ as given in the form (5).

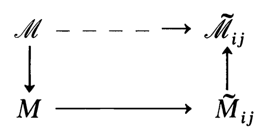

FIG. 1

Reference to the Taylor series representation (2) for M together with the desired approximation relationship $M - M(1)I = h^\ell \sum_{|k|=\ell} \alpha_k D^k + O(h^{\ell+1})$ demonstrates the necessity of the conditions

$$M_k(1) = 0 \qquad \text{for } 1 \leqq |k| < \ell, \tag{7}$$

$$M_k(1) = \alpha_k k! \quad \text{for } |k| = \ell. \tag{8}$$

These in turn may be used together with the Taylor series representation (4) of the corresponding generating operators $\tilde{M}_{ij}$ to determine, in the spirit of the approximation relationship (6), corresponding generating operators $\tilde{\mathcal{M}}_{ij}$. Thus for a vector $p = (p_1, p_2, \cdots, p_n)$ having nonnegative entries, condition (7) enables us to write

$$\tilde{M}_{ij} = h^{\ell-1} \sum_{|p|=\ell-1} \frac{1}{p!}[x_j M_{p+e_i}(1) - x_i M_{p+e_j}(1)]D^p + O(h^\ell),$$

$$i \neq j, \quad i, j = 1, \cdots, n.$$

Noting that $|p + e_i| = |p| + |e_i| = \ell$, we use relation (8) to obtain

$$\alpha_{p+e_i}(p + e_i)! = M_{p+e_i}(1) \quad \text{for } |p| = \ell - 1.$$

Finally after simplifying the factorial expressions we define the generating operators $\tilde{\mathcal{M}}_{ij}$.

DEFINITION

$$\tilde{\mathcal{M}}_{ij} = \sum_{|p|=\ell-1} [x_j \alpha_{p+e_i}(p_i + 1) - x_i \alpha_{p+e_j}(p_j + 1)]D^p,$$

$$i \neq j, \quad i, j = 1, 2, \cdots, n.$$

We thus have a family of homogeneous differential operators with the property $\tilde{\mathscr{M}}_{ij} = -\tilde{\mathscr{M}}_{ji}$; they are useful in generating additional solutions of the differential equation $\mathscr{M}\varphi(x) = 0$ from a known solution $f(x)$. Indeed, some members of the family may be null operators; it is not difficult to discern from the definition that $\tilde{\mathscr{M}}_{ij}$ will be a null operator whenever derivatives with respect to x_i and x_j are entirely absent from each term of the given $\mathscr{M}$.

The following generating theorem may be proved independently of its discrete counterpart in § 2. The basis for the proof is the commutation relationship between $\mathscr{M}$ and $\tilde{\mathscr{M}}_{ij}$ which we state as a lemma.

LEMMA. *Let $f(x) \in C^{2\ell-1}[R]$, where R is some region of Euclidean n-space. Then*

$$\mathscr{M}[\tilde{\mathscr{M}}_{ij}f(x)] = \tilde{\mathscr{M}}_{ij}[\mathscr{M}f(x)].$$

Proof.

$$\begin{aligned}
\mathscr{M}[\tilde{\mathscr{M}}_{ij}f] &= \sum_{|k|=\ell} \alpha_k D^k \left\{ \sum_{|p|=\ell-1} [x_j \alpha_{p+e_i}(p_i + 1) - x_i \alpha_{p+e_j}(p_j + 1)] D^p f \right\} \\
&= \sum_{|k|=\ell} \alpha_k \sum_{|p|=\ell-1} [\alpha_{p+e_i}(p_i + 1) D^k(x_j D^p f) - \alpha_{p+e_j}(p_j + 1) D^k(x_i D^p f)] \\
&= \sum_{|k|=\ell} \alpha_k \sum_{|p|=\ell-1} [\alpha_{p+e_i}(p_i + 1)(x_j D^{k+p} f + k_j D^{k+p-e_j} f) \\
&\qquad - \alpha_{p+e_j}(p_j + 1)(x_i D^{k+p} f + k_i D^{k+p-e_i} f)] \\
&= \sum_{|p|=\ell-1} \left\{ (\alpha_{p+e_i}(p_i + 1) x_j D^p - \alpha_{p+e_j}(p_j + 1) x_i D^p) \sum_{|k|=\ell} \alpha_k D^k f \right. \\
&\qquad \left. + \sum_{|k|=\ell} \alpha_k (\alpha_{p+e_i}(p_i + 1) k_j D^{k+p-e_j} f - \alpha_{p+e_j}(p_j + 1) k_i D^{k+p-e_i} f) \right\} \\
&= \tilde{\mathscr{M}}_{ij}(\mathscr{M}f) + \sum_{|p|=\ell-1} \sum_{|k|=\ell} [\alpha_k \alpha_{p+e_i}(p_i + 1) k_j D^{k+p-e_j} f \\
&\qquad - \alpha_k \alpha_{p+e_j}(p_j + 1) k_i D^{k+p-e_i} f].
\end{aligned}$$

A careful rearrangement of indices enables us to write the lengthy expression in the form

$$\sum_{|p|=\ell-1} \sum_{|k|=\ell-1} \alpha_{k+e_j} \alpha_{p+e_i}(p_i + 1)(k_j + 1) D^{k+e_j+p-e_j} f$$
$$- \sum_{|p|=\ell-1} \sum_{|k|=\ell-1} \alpha_{k+e_i} \alpha_{p+e_j}(p_j + 1)(k_i + 1) D^{k+e_i+p-e_i} f$$

which can be seen to vanish identically.

The lemma readily establishes the proof of the following theorem.

THEOREM. *Let $f(x) \in C^{2\ell-1}[R]$ such that $\mathscr{M}f(x) = 0$ in R. Then*

$$\mathscr{M}[\tilde{\mathscr{M}}_{ij}f(x)] = 0 \quad \text{in } R, \qquad i \neq j, \quad i, j = 1, \cdots, n.$$

Letting $\mathscr{M}^q$ and $\tilde{\mathscr{M}}^q_{ij}$ denote repeated applications of the respective operators, we may state three corollaries, counterparts of those stated for the discrete case. The proofs, similar to those indicated for the discrete case, are omitted.

COROLLARY 1′. *If $f(x) \in C^{(q+1)\ell-1}[R]$ and $\mathscr{M}^q f(x) = 0$ in R, then*

$$\mathscr{M}^q[\tilde{\mathscr{M}}_{ij}f(x)] = 0 \quad \text{in } R \quad \text{for } i \neq j, \quad i, j = 1, 2, \cdots, n, \quad q = 1, 2, \cdots.$$

COROLLARY 2'. *If* $f(x) \in C^{\ell + q(\ell - 1)}[R]$ *and* $\mathscr{M}f(x) = 0$ *in* R, *then*

$$\mathscr{M}[\tilde{\mathscr{M}}_{ij}^{q} f(x)] = 0 \quad \text{in } R \quad \text{for } i \neq j, \quad i, j = i, \cdots, n, \quad q = 1, 2, \cdots.$$

COROLLARY 3'. *If* $f(x) \in C^{3\ell - 2}[R]$ *and* $\mathscr{M}f(x) = 0$ *in* R, *then*

$$\mathscr{M}[\tilde{\mathscr{M}}_{ij} \tilde{\mathscr{M}}_{rs} f(x)] = 0 \quad \text{in } R \quad \text{for } i \neq j, \quad r \neq s, \quad i, j, r, s = 1, \cdots, n.$$

4. Examples and applications. As a first example we consider the n-dimensional Laplacian operator $\Delta \equiv \sum_{i=1}^{n} \partial^2/\partial x_i^2$ and a discrete harmonic operator having relatively simple form, viz.,

$$M = \sum_{m} a_m X^m = \sum_{i=1}^{n} (X_i + X_i^{-1}) - 2nI.$$

Although this example has been treated by Duffin and Shelly [5], it is nonetheless instructive to place it within the present context of Taylor series expansion and derived operators and from these to demonstrate the form of the corresponding generating operators. The simple form of the operator enables us to write for the Taylor expansion

$$Mf(x) = M(1)f(x) + h \sum_{i=1}^{n} M_{e_i}(1) \frac{\partial f}{\partial x_i} + \frac{h^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{e_i + e_j}(1) \frac{\partial^2 f}{\partial x_i \partial x_j} + \cdots.$$

Using the convention $0^0 = 1$ one can easily verify that

$$M(1) = 0,$$

$$M_{e_i} = X_i - X_i^{-1}, \qquad M_{e_i}(1) = 0, \qquad i = 1, 2, \cdots, n;$$

$$M_{e_i + e_j} = \begin{cases} 0 & \text{for } i \neq j, \\ X_i + X_i^{-1} & \text{for } i = j, \end{cases} \qquad M_{e_i + e_j}(1) = 2\delta_{ij}, \qquad i, j = 1, 2, \cdots, n$$

(using the Kronecker delta symbol), so that in fact,

$$Mf(x) = h^2 \Delta f(x) + O(h^4).$$

The generating operators become

$$\tilde{M}_{ij} = x_j(X_i - X_i^{-1}) - x_i(X_j - X_j^{-1}),$$

and it is interesting to note that these simulate via the Taylor expansion (4) the corresponding family of operators

$$\tilde{\Delta}_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \qquad i \neq j, \quad i, j = 1, 2, \cdots, n.$$

The latter may be recognized as analogues of the components of the vector $\mathbf{r} \times$ grad in three dimensions, which are known to generate harmonic functions when applied to a harmonic function.

An interesting application of the discrete version of the generating operator lies in the context of discrete analytic function theory. We recall that a complex function f is termed analytic in the continuous theory when $\partial f/\partial \bar{z} = 0$, where $\partial f/\partial \bar{z} = \frac{1}{2}(\partial f/\partial x + i\, \partial f/\partial y) = 0$ is the complex form of the Cauchy–Riemann equations. By analogy, a discrete analytic function f satisfies the equation $Lf = 0$

in some region of the discrete xy-plane, where L is a linear difference operator with constant coefficients. A crucial property of L is that it is an approximation to $\partial/\partial\bar{z}$; for detailed treatment of other properties of discrete analytic operators, see Duffin [2] and Hundhausen [6].

Thus the theory developed in this paper is useful in generating a sequence of discrete analytic functions via Corollary 2; we cite an important example of the process for the case of a particular operator L. For the discrete analytic operator $L = I + iX - XY - iY$, Duffin [2] introduces the operator

$$Z = \tfrac{1}{4}[z(I + X + XY + Y) - i\bar{z}(I - X + XY - Y)]$$

and shows that if $Lf = 0$, then $L(Zf) = 0$ also. Algebraic simplification and use of the relation $Lf = 0$ show that Z is a variation of $\tilde{L}_{12}$; indeed, $Zf = (\frac{1}{2} - \frac{1}{2}i)\tilde{L}_{12}f$. To achieve greater symmetry relative to the point of application, Duffin forms a new operator $\underset{\sim}{Z}$ from the average of Z applied at the four points z, $z - 1$, $z - i$, and $z - 1 - i$, and finally establishes the interesting relation

$$\underset{\sim}{Z}z^{(n)} = z^{(n+1)}. \tag{9}$$

Here $z^{(n)}$ is the nth member of the sequence of discrete analytic polynomials, which were originally defined by a process of recursive indefinite discrete integration with $z^{(0)} \equiv 1$; relation (9) provides an alternate (and simpler) method of generating this particular sequence of functions.

Finally we note that for discrete analytic operators L, the corresponding generating operators $\tilde{L}_{12}$ are aptly symbolized by Z since the conditions imposed upon $L - L(1)I$ in order that it approximate $\partial/\partial\bar{z}$, viz., $L_{e_2}(1) = iL_{e_1}(1) \neq 0$, are exactly those which determine $\tilde{L}_{12}$ as an approximation to multiplication by z. More precisely, the Taylor expansions (2) and (4) become

$$[L - L(1)I]f = L_{e_1}(1)\,\partial f/\partial\bar{z} + O(h^2);$$

$$\tilde{L}_{12}f = iL_{e_1}(1)zf + O(h).$$

Thus in this particular context the generating process may be considered a simulation of multiplication in the continuous case.

REFERENCES

[1] L. Collatz, *Numerical Treatment of Differential Equations*, 3rd ed., Springer-Verlag, Berlin, 1960.

[2] R. J. Duffin, *Basic properties of discrete analytic functions*, Duke Math. J., 23 (1956), pp. 335–363.

[3] R. J. Duffin and C. S. Duris, *A convolution product for discrete function theory*, Ibid., 31 (1964), pp. 199–220.

[4] R. J. Duffin and Joan Rohrer, *A convolution product for the solutions of partial difference equations*, Ibid., 35 (1968), pp. 683–698.

[5] R. J. Duffin and E. P. Shelly, *Difference equations of polyharmonic type*, Ibid., 25 (1958), pp. 209–238.

[6] J. R. Hundhausen, *A generalization of discrete analytic and harmonic functions*, J. Math. Anal. Appl., 25 (1969), pp. 628–652.

[7] K. S. Miller, *Linear Difference Equations*, W. A. Benjamin, New York, 1968.

[8] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1957–59.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

A VOLTERRA EQUATION WITH PARAMETER*

KENNETH B. HANNSGEN†

Abstract. We discuss the Volterra integral equation $x'(t) + \lambda\int_0^t a(t-\tau)x(\tau)\,d\tau = k$, $\lambda \geqslant \lambda_0 > 0$. We find conditions under which solutions are bounded on $\{0 \leqslant t < \infty\}$, uniformly in λ. We deduce results on the asymptotic behavior of certain Volterra equations in Hilbert space arising, for example, in viscoelasticity.

1. Introduction. In this paper we discuss boundedness of solutions of the real Volterra equation

$$x'(t) + \lambda\int_0^t a(t-\tau)x(\tau)\,d\tau = k, \quad x(0) = x_0, \qquad 0 \leqq t < \infty \tag{1.1}$$

(primes denote differentiation with respect to t).

Let $x(t) = x(t, \lambda)$ denote the solution of (1.1), where $0 < \lambda_0 \leqq \lambda < \infty$ and x_0 and k are prescribed constants. Theorems 2, 3 and 4 below give conditions ensuring that

$$|x(t, \lambda)| \leqq B(|x_0| + |k|), \qquad 0 \leqq t < \infty, \quad \lambda_0 \leqq \lambda < \infty, \tag{1.2}$$

where B depends only on $a(t)$ and λ_0. The conditions on $a(t)$ will include the following.

(H) *$a(t)$ is continuous, nonnegative, nonincreasing, and convex on* $(0, \infty)$, $0 < a(0+) \leqq \infty$, $\int_0^1 a(t)\,dt < \infty$, *and* $a(t) \to 0$ *as* $t \to \infty$.

In [5] we showed that for fixed positive λ, $x(t, \lambda)$ is a bounded function of t, provided (H) holds; if in addition $a(t)$ is differentiable on $(0, \infty)$, then

$$x(t, \lambda) \to 0 \quad \text{as } t \to \infty. \tag{1.3}$$

Our interest in the uniform estimate (1.2) comes from questions of asymptotic behavior of solutions of certain integral equations in Hilbert space. Let L denote a symmetric linear operator defined on a dense subspace $\mathbf{K}$ of a real, separable Hilbert space $\mathbf{H}$. Assume that $\langle L\chi, \chi\rangle \geqq \lambda_0\langle\chi, \chi\rangle$, $\chi \in \mathbf{K}$, and that the inverse L^{-1} of L is a compact operator on $\mathbf{H}$. Consider the equation

$$y(t) + L\int_0^t h(t-\tau)y(\tau)\,d\tau = \mu + t\nu, \tag{1.4}$$

where μ and ν are prescribed elements of $\mathbf{H}$ and $h(t) = \int_0^t a(\tau)\,d\tau$. Following A. Friedman [3], we expand $y(t)$ in terms of eigenvectors of L; then the expansion coefficient $y_k(t)$ satisfies (1.1), where $\lambda = \lambda_k$ is the corresponding eigenvalue. Theorem 1 below says that this method yields information about $y(t)$ when (1.2) holds.

We discuss (1.4) and related work of C. M. Dafermos [1], [2] and A. Friedman and M. Shinbrot [4] in § 2. We state our main results, the sufficient conditions for (1.2), in § 3; proofs follow in §§ 4 and 5.

* Received by the editors October 7, 1971, and in revised form January 21, 1972.

† Department of Mathematics, University of California, Los Angeles, California 90024. This work was supported by the National Science Foundation under Grant GP-27973.

Throughout this paper we let $u(t) = u(t, \lambda)$ and $w(t) = w(t, \lambda)$ denote respectively the solutions of

(1.5) $$u'(t) + \lambda \int_0^t a(t - \tau)u(\tau)\, d\tau = 0, \qquad u(0) = 1$$

and

(1.6) $$w'(t) + \lambda \int_0^t a(t - \tau)w(\tau)\, d\tau = 1, \qquad w(0) = 0.$$

One easily checks that

(1.7) $$x(t) = x_0 u(t) + kw(t) \quad \text{and} \quad w'(t) = u(t).$$

Then (1.2) holds whenever $|u(t, \lambda)| + |w(t, \lambda)| \leqq B\ (0 \leqq t < \infty, \lambda_0 \leqq \lambda < \infty)$.

2. Equations in Hilbert space. Let $\mathbf{H}$ be as in § 1, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\{\chi_k\}_{k=0}^{\infty}$ be a complete orthonormal set of eigenvectors of the operator L, with corresponding eigenvalues $\{\lambda_k\}$.

THEOREM 1. *Let $a(t)$ satisfy condition* (H), *and let $h(t) = \int_0^t a(\tau)\, d\tau$. Suppose $|u(t, \lambda)| \leqq M$ and $|w(t, \lambda)| \leqq M$ $(0 \leqq t < \infty, \lambda_0 \leqq \lambda < \infty)$. Let*

(2.1) $$y_k(t) = \mu_k u(t, \lambda_k) + \nu_k w(t, \lambda_k),$$

where $\mu_k = \langle \mu, \chi_k \rangle$ and $\nu_k = \langle \nu, \chi_k \rangle$. The series

(2.2) $$y(t, \mu, \nu) = \sum_{k=0}^{\infty} y_k(t)\chi_k$$

converges in $\mathbf{H}$, *uniformly in t, to the unique continuous solution of* (1.4). *Moreover,*

(2.3) $$\|y(t, \mu, \nu)\|^2 \leqq 2M^2(\|\mu\|^2 + \|\nu\|^2).$$

The following is an immediate consequence of uniform convergence in (2.2).

COROLLARY 1. *Let the hypotheses of Theorem* 1 *hold, and assume that $a(t)$ is differentiable on* $(0, \infty)$, *so that* (1.3) *holds. Then $\|y(t, \mu, \nu)\| \to 0$ as $t \to \infty$.*

Proof of Theorem 1. Uniform convergence in (2.2) and the estimate (2.3) follow from the inequality $y_k^2(t) \leqq 2M^2(\mu_k^2 + \nu_k^2)$. Then $y(t) = y(t, \mu, \nu)$ is continuous. Setting $x = y_k$, $\lambda = \lambda_k$, $k = \nu_k$, $x_0 = \mu_k$ in (1.1), integrating, and dividing by λ_k, we obtain the identity

(2.4) $$\lambda^{-1} y_k(t) + \int_0^t h(t - \tau) y_k(\tau)\, d\tau = \lambda_k^{-1}(\mu_k + t\nu_k).$$

Multiply χ_k by both sides of (2.4) and sum over k; this yields

(2.5) $$L^{-1} y(t) + \int_0^t h(t - \tau) y(\tau)\, d\tau = L^{-1}(\mu + t\nu).$$

Thus L can be applied to both sides of (2.5); this gives (1.4). Conversely, if $v = \sum v_k(t)\chi_k$ satisfies (1.4), v_k satisfies (2.4), so $v_k = y_k$. This proves Theorem 1.

We remark that our proof follows the proof of Theorem 4.1 of [3]; as in that paper, only the uniform boundedness of the family $\{u(t, \lambda), w(t, \lambda)\}$ was used here, and the sum (2.2) could be replaced by an integral for the case where L is self-adjoint but L^{-1} is not compact.

We also note that if $\|L^r\mu\|^2 + \|L^r v\|^2 = \sum_{k=0}^{\infty} \lambda_k^{2r}(\mu_k^2 + v_k^2) < \infty, r > 0$, the same argument shows that $L^r y(t, \mu, v) = \sum_{k=0}^{\infty} \lambda_k^r y_k(t)\chi_k$ converges uniformly.

Under the conditions of Theorem 1, the equations $R(t)\mu = y(t, \mu, 0)$ and $S(t)v = y(t, 0, v)$ define $R(t)$ and $S(t)$ as uniformly bounded, strongly continuous operator-valued functions; moreover

$$R(t) + L\int_0^t h(t-\tau)R(\tau)\,d\tau = I$$

(I = identity). Friedman and Shinbrot [4] analyze this operator equation in Banach space in the case where $h(0) > 0$ and $h' \in L^1(0, \infty)$; in Theorem 1 above, $h(0) = 0$.

Consider the more general nonhomogeneous equation

$$z(t) + L\int_0^t h(t-\tau)z(\tau)\,d\tau = \mu + tv + F(t), \tag{2.6}$$

where $F(t) = \int_0^t f(\tau)\,d\tau + \int_0^t\int_0^\tau g(\sigma)\,d\sigma\,d\tau$. The operators $R(t)$ and $S(t)$ are resolvents for this equation; some calculation shows that the solution of (2.6) is

$$z(t) = R(t)\mu + S(t)v + \int_0^t R(t-\tau)f(\tau) + S(t-\tau)g(\tau)\,d\tau. \tag{2.7}$$

As an application of our results, consider the equation

$$z''(t) + (a(0) + c)Lz(t) + \int_0^t a'(t-\tau)Lz(\tau)\,d\tau = g(t), \tag{2.8}$$

where (H) holds, $a(0) < \infty$, and $c \geqq 0$. Dafermos [1], [2] obtains results on asymptotic behavior for a wide class of equations arising in viscoelasticity and including (2.8) with $c > 0$ as a simple particular case. If instead $c = 0$, integration shows that a solution of (2.8) is a solution of (2.6) ($f = 0$, $v = z'(0)$, $\mu = z(0)$); then by formula (2.7), $\|z(t)\|$ is uniformly bounded on $\{0 \leqq t < \infty\}$ when the hypotheses of Theorem 1 hold and $\int_0^\infty \|g(t)\|\,dt < \infty$.

3. The scalar equation. We state our results in terms of u and w. In the following, B denotes a finite, positive constant, independent of t and λ; its value may change from line to line.

THEOREM 2. *If condition* (H) *holds, then* $|u(t, \lambda)| \leqq \sqrt{2}$ $(0 \leqq t < \infty, 0 < \lambda < \infty)$.

THEOREM 3. *If condition* (H) *holds and* $a(t) \in L^1(0, \infty)$, *then*

$$|w(t,\lambda)| \leqq B, \qquad 0 \leqq t < \infty, \quad \lambda_0 \leqq \lambda < \infty. \tag{3.1}$$

THEOREM 4. *Suppose condition* (H) *holds,* $a(t)$ *is twice differentiable on* $(0, \infty)$, *and* $a(t) \notin L^1(0, \infty)$. *We have three alternative cases*:

(i) *If* $a(0+) < \infty$ *and* a'' *is bounded away from zero on every finite interval* $(0, L]$, *then* (3.1) *holds.*

(ii) *If* $a''(t)$ *is nonincreasing on* $(0, \infty)$, *and if*

$$a(t) = O(t^{-\beta}) \quad \text{and} \quad \frac{-1}{a'(t)} = O(t^{\beta}), \qquad t \to 0, \tag{3.2}$$

for some β, $0 < \beta < 1$, *then* (3.1) *holds.*

(iii) *If* $a''(t)$ *is nonincreasing on* $(0, \infty)$, *and if*

$$\frac{-1}{a'(t)} = O(t), \qquad t \to 0, \tag{3.3}$$

then (3.1) *holds.*

In view of (1.7), our results give sufficient conditions for (1.2).

Theorem 2 gives a uniform bound on u for all positive λ. No such bound is possible for w. In fact, integration of (1.6) shows that

$$w(t, \lambda) = t - \lambda \int_0^t h(t - \tau) w(\tau, \lambda)\, d\tau. \tag{3.4}$$

Then if $|w(t, \lambda)| \leqq B$, (3.4) says that $w(t, \lambda) \to t$ as $\lambda \to 0$ for each fixed t, a contradiction.

Our proofs of Theorems 2 and 3 come essentially from the work of J. J. Levin [7], which deals with the case of fixed λ. For Theorem 4 we show that some estimates in [5] can be made uniform in λ.

4. Proofs of Theorems 2 and 3. For Theorem 2, recall [8, p. 230] that when $a(t)$ satisfies condition (H), we may write $a(t) = \int_\infty^t \alpha(\tau)\, d\tau$, where $\alpha(\tau)$ is a nonpositive, nondecreasing function and $\alpha(t) = \alpha(t+), 0 < t < \infty$. With $u(t) = u(t, \lambda)$, define

$$V(t) = \frac{1}{2} u^2(t) + \frac{1}{2} \lambda a(t) \left(\int_0^t u(\tau)\, d\tau \right)^2 - \frac{1}{2} \lambda \int_0^t \left[\int_{t-\tau}^t u(s)\, ds \right]^2 \alpha(\tau)\, d\tau.$$

Direct computation using (1.1) (see [6]) shows that $V(t) - V(0) = \int_0^t V_1(\tau)\, d\tau$ with $V_1(\tau) \leqq 0$ a.e. Therefore

$$u^2(t) \leqq 2V(t) \leqq 2V(0) = 2u^2(0) = 2,$$

as asserted.

For Theorem 3, choose $n > 0$ such that $\lambda_0[\int_0^n - 2\int_n^\infty] a(t)\, dt = \gamma > 0$. Choose an integer $N > n$ such that $2N\gamma > 3$. Fix $\lambda \geqq \lambda_0$, and let $w(t) = w(t, \lambda)$. Suppose there exists $T > 0$ such that $|w(T)| = 4N$ and $|w(t)| \leqq |w(T)|$ $(0 \leqq t \leqq T)$. Since $|w'| = |u| \leqq 2$ by Theorem 2, $|w(t)| \geqq 2N$ $(T - N \leqq t \leqq T)$. (Note that $T - N > 0$, since $w(0) = 0$.) But

$$1 - w'(T) = \left[\int_0^{T-N} + \int_{T-N}^T \right] \lambda a(T - \tau) w(\tau)\, d\tau,$$

$$\left| \int_0^{T-N} \lambda a(T - \tau) w(\tau)\, d\tau \right| \leqq 4N\lambda \int_N^\infty a(t)\, dt$$

and

$$\left| \int_{T-N}^T \lambda a(T - \tau) w(\tau)\, d\tau \right| \geqq 2N\lambda \int_0^N a(t)\, dt.$$

Hence,

$$3 \geqq |1 - w'(T)| \geqq 2N\lambda \left[\int_0^N - 2 \int_N^\infty \right] a(t)\, dt$$

$$\geqq 2N\gamma > 3.$$

We conclude that no such T exists and $|w(t)| = |w(t, \lambda)| \leqq 4N$. This proves Theorem 3.

5. Proof of Theorem 4. The proof depends on the integral representation

$$\pi w(t, \lambda) = \int_0^\infty \operatorname{Re}\left\{\frac{e^{i\tau t}}{i\tau[\lambda\varphi(\tau) - i\lambda\psi(\tau) + i\tau]}\right\} d\tau, \qquad t > 0, \tag{5.1}$$

which we derived in [5]. Here

$$\varphi(\tau) = \lim_{R\to\infty} \int_0^R a(t) \cos \tau t \, dt$$

and

$$\psi(\tau) = \lim_{R\to\infty} \int_0^R a(t) \sin \tau t \, dt.$$

Briefly, we showed in [5] that the Laplace transform $W(s, \lambda) = \int_0^\infty e^{st} w(t, \lambda)\, dt$ is analytic in $\{\operatorname{Re} s > 0\}$ and continuous in $\{\operatorname{Re} s \geqq 0\}$ except possibly at $s = 0$; moreover $W(s, \lambda) = [s(s + \lambda A(s))]^{-1}$, where $A(s)$ is the Laplace transform of $a(t)$. The complex inversion formula

$$2\pi w(t, \lambda) = e^{\sigma t} \int_{-\infty}^\infty e^{i\tau t} W(\sigma + i\tau, \lambda)\, d\tau, \qquad \sigma > 0, \quad t > 0,$$

holds; a contour shift, together with some estimates near $s = 0$ and a change of variable, yield the representation

$$\pi w(t, \lambda) = \int_0^\infty \operatorname{Re}\{e^{i\tau t} W(i\tau, \lambda)\}\, d\tau.$$

Since $\varphi(\tau)$ and $-\psi(\tau)$ are the real and imaginary parts of $A(i\tau)$, we obtain (5.1).

Let $\varphi_1(\tau) = \int_0^{\pi/2\tau} a(t) \cos \tau t\, dt$, and let $\varphi_2(\tau) = \varphi(\tau) - \varphi_1(\tau)$. Using the monotonicity and convexity of $a(t)$, we proved several facts about $\varphi_1, \varphi_2, \varphi$ and ψ in [5]; we collect these facts in the following lemma.

LEMMA 1. *With $a(t)$ as in Theorem 4, the following relations hold* $(0 < \tau < \infty)$:

$$0 \leqq \psi(\tau) \leqq \int_0^{\pi/\tau} a(t)\, dt, \tag{5.2}$$

$$0 < \frac{1}{2}\int_0^{\pi/3\tau} a(t)\, dt \leqq \varphi_1(\tau), \tag{5.3}$$

$$0 \leqq -\varphi_2(\tau) \leqq \psi(\tau) \leqq 4\varphi_1(\tau), \tag{5.4}$$

$$\varphi_1(\tau) \leqq \varphi(\tau) + \psi(\tau), \tag{5.5}$$

$$\varphi(\tau) > 0. \tag{5.6}$$

For details, see Lemmas 3 and 5 and inequalities (3.23) and (3.24) of [5].

Since $a(t) \notin L^1(0, \infty)$, $\varphi_1(\tau) \to \infty$ as $\tau \to 0$, by (5.3). Choose $\rho > 0$ such that

$$\varphi_1(\tau) \geqq 2\tau/\lambda_0, \qquad 0 < \tau \leqq \rho. \tag{5.7}$$

Since

$$\frac{d}{d\tau}\varphi_1(\tau) = -\int_0^{\pi/2\tau} ta(t)\sin\tau t\,dt \leqq 0,$$

$1/\varphi_1(\tau) \downarrow 0$ as $\tau \downarrow 0$; in particular $1/\varphi_1(\tau)$ is of bounded variation on $[0, \rho]$, and by the familiar theorem concerning the kernel $(\sin \tau t)/\tau$ [9, p. 64] it follows that

$$\lim_{t\to\infty}\int_0^\rho \frac{\sin\tau t}{\tau\varphi_1(\tau)}\,d\tau = 0. \tag{5.8}$$

We write (5.1) as

$$\pi w(t) = \int_0^\rho + \int_\rho^{T(\lambda)} + \int_{T(\lambda)}^\infty = I_1 + I_2 + I_3, \tag{5.9}$$

where $T(\lambda)$ will be determined separately for each case of Theorem 4.

LEMMA 2. *In cases* (i), (ii), *and* (iii), $|I_1| \leqq B\,(0 \leqq t < \infty, \lambda_0 \leqq \lambda < \infty)$.

Proof. Rewriting the integrand in (5.1), we see that

$$\begin{aligned} I_1 = \int_0^\rho \cos\tau t\left[\frac{\lambda\psi(\tau)}{\tau D^2(\tau,\lambda)}\right]d\tau - \int_0^\rho \frac{\cos\tau t}{D^2(\tau,\lambda)}\,d\tau \\ + \int_0^\rho \frac{\sin\tau t}{\tau}\,\frac{\lambda\varphi(\tau)}{D^2(\tau,\lambda)}\,d\tau, \end{aligned} \tag{5.10}$$

where $D(\tau, \lambda) = |\lambda\varphi(\tau) - i\lambda\psi(\tau) + i\tau|$. (The existence of these three integrals was proved in [5]; the first inequality in (5.11) below and the fact that $\varphi_1(\tau) \to \infty$ as $\tau \to 0$ provide the required estimates.)

Using elementary inequalities for complex numbers, we find that

$$2^{-1/2}\lambda[\varphi(\tau) + \psi(\tau) - \tau/\lambda] \leqq D(\tau,\lambda) \leqq \lambda[\varphi_1(\tau) + \varphi_2(\tau) + \psi(\tau) + \tau/\lambda_0].$$

Using (5.5), (5.7) and (5.4), we see that

$$\lambda\varphi_1(\tau)/2\sqrt{2} \leqq D(\tau,\lambda) \leqq 10\lambda\varphi_1(\tau), \qquad 0 < \tau \leqq \rho. \tag{5.11}$$

Then we can estimate the middle term in (5.10) as follows:

$$\left|\int_0^\rho \cos\tau t D^{-2}(\tau,\lambda)\,d\tau\right| \leqq 8\lambda_0^{-2}\int_0^\rho \varphi_1^{-2}(\tau)\,d\tau = B. \tag{5.12}$$

Moreover,

$$0 \leqq \frac{\lambda\psi(\tau)}{\tau D^2(\tau,\lambda)} \leqq \frac{8\psi(\tau)}{\lambda\tau\varphi_1^2(\tau)} \leqq \frac{800\lambda\psi(\tau)}{\tau D^2(\tau,\lambda)}, \qquad 0 < \tau \leqq \rho. \tag{5.13}$$

The existence of the first integral in (5.10) implies that the last expression in (5.13) is in $L^1(0, \rho)$; hence so are the other two expressions. Therefore,

$$\left|\int_0^\rho \cos\tau t\left[\frac{\lambda\psi(\tau)}{\tau D^2(\tau,\lambda)}\right]d\tau\right| \leqq 8\lambda_0^{-1}\int_0^\rho \frac{\psi(\tau)\,d\tau}{\tau\varphi_1^2(\tau)} = B. \tag{5.14}$$

We write the last term in (5.10) as

$$J(t,\lambda) = \frac{1}{\lambda}\int_0^\rho \frac{\sin\tau t}{\tau\varphi_1(\tau)}\,d\tau + \int_0^\rho E(\tau,\lambda)\sin\tau t\,d\tau,$$

where

$$E(\tau, \lambda) = \{\lambda^2\varphi_1\varphi - \lambda^2\varphi^2 - [\lambda\psi - \tau]^2\}/\tau\lambda\varphi_1 D^2(\tau, \lambda).$$

Straightforward estimates using (5.3) and (5.4) yield the inequality

$$|E(\tau, \lambda)| \leqq B[\lambda\tau^{-1}\psi(\tau) + 1]/\lambda D^2(\tau, \lambda).$$

From (5.11) and the fact that the expressions in (5.13) are in $L^1(0, \rho)$ as functions of τ, we conclude that

$$\left| \int_0^\rho E(\tau, \lambda) \sin \tau t \, d\tau \right| \leqq B_1 \int_0^\rho \varphi_1^{-2}(\tau)[\tau^{-1}\psi(\tau) + \lambda_0^{-1}] \, d\tau = B.$$

On the other hand, (5.8) shows that

$$\left| \frac{1}{\lambda} \int_0^\rho \frac{\sin \tau t}{\tau\varphi_1(\tau)} d\tau \right| \leqq B;$$

therefore $|J(t, \lambda)| \leqq B\,(0 < t \leqq \infty,\ \lambda_0 \leqq \lambda < \infty)$. In view of (5.10), (5.12), and (5.14), this proves Lemma 2.

Next we prove (3.1) in case (i) of Theorem 4. We set

$$T(\lambda) = \max\{\rho, (2\pi a(0)\lambda)^{1/2}\} \tag{5.15}$$

in (5.9).

We estimate I_3 first. By (5.2) and the monotonicity of $a(t)$,

$$0 \leqq \lambda\psi(\tau) \leqq \lambda a(0)\pi/\tau \leqq T^2(\lambda)/2\tau \leqq \tau/2, \qquad T(\lambda) \leqq \tau < \infty.$$

Then (with $D(\tau, \lambda)$ as in the proof of Lemma 2) we have

$$D(\tau, \lambda) \geqq \tau - \lambda\psi(\tau) \geqq \tau/2, \qquad T(\lambda) \leqq \tau < \infty,$$

so $|I_3| \leqq 2\int_\rho^\infty \tau^{-2} \, d\tau = B$.

For I_2, integration by parts shows that

$$\tau\varphi(\tau) = \lim_{R\to\infty} \left\{ -\int_0^R \sin \tau t \, a'(t) \, dt \right\}, \tag{5.16}$$

since $a(\infty) = 0$. Equation (5.16) is also valid when $a(0+) = \infty$, because condition (H) implies that $ta(t) \leqq \int_0^t a(\tau) \, d\tau \to 0$ as $t \to 0$. Let

$$N(\tau) = \text{greatest integer} \leqq 2\tau/\rho. \tag{5.17}$$

Note that $N(\tau) \geqq \tau/\rho$ when $\tau \geqq \rho$. Since $a(t)$ is convex, $-a'(t)$ is nonincreasing and

$$\begin{aligned} \tau\varphi(\tau) &\geqq \int_0^{2N(\tau)\pi/\tau} \sin \tau t[-a'(t)] \, dt \\ &= \sum_{k=0}^{N(\tau)-1} \int_0^{\pi/\tau} \sin \tau t\{-a'[t + 2k\pi/\tau] + a'[t + (2k+1)\pi/\tau]\} \, dt. \end{aligned} \tag{5.18}$$

Now $2N(\tau)\pi/\tau \leqq 4\pi/\rho$. Since a'' is bounded away from zero on finite intervals, there exists $\eta > 0$ such that $-a'(t) + a'(t + \varepsilon) \geqq \eta\varepsilon$ if $0 < t < t + \varepsilon \leqq 4\pi/\rho$. Then

$$\tau\varphi(\tau) \geqq \tau^{-1}\eta\pi \sum_{k=0}^{N(\tau)-1} \int_0^{\pi/\tau} \sin \tau t \, dt = 2\pi\eta N(\tau)/\tau^2 \geqq 2\pi\eta/\tau\rho, \qquad \tau \geqq \rho.$$

Using this and (5.15), we make the estimate

$$|I_2| \leqq \frac{1}{\lambda}\int_\rho^{T(\lambda)} \frac{d\tau}{\tau\varphi(\tau)} \leqq \frac{\rho}{4\pi\eta\lambda}[T^2(\lambda) - \rho^2]$$
$$\leqq \rho a(0)/2\eta = B.$$

These estimates for I_3 and I_2, together with Lemma 2, establish (3.1) in case (i).

LEMMA 3. *In cases* (ii) *and* (iii) *of Theorem* 4, *there is a positive number Q such that*

$$\tau\varphi(\tau) \geqq Q\tau^{\beta-1}, \qquad \tau \geqq \rho, \tag{5.19}$$

where β comes from (3.2) *in case* (ii) *and $\beta = 1$ in case* (iii).

Proof. We may assume that

$$-a'(t) \geqq \delta t^{-\beta}, \qquad 0 < t \leqq 4\pi/\rho, \tag{5.20}$$

where $\delta > 0$. Define $N(\tau)$ as in (5.17). We again have relation (5.18). Using the mean value theorem and the fact that $a''(t)$ is nonincreasing in cases (ii) and (iii), we make the estimate

$$\begin{aligned}\tau\varphi(\tau) &\geqq \sum_{k=1}^{N(\tau)} \int_0^{\pi/\tau} \sin \tau t\; a''(2k\pi/\tau)\pi/\tau\, dt \\ &= 2\pi\tau^{-2} \sum_{k=1}^{N(\tau)} a''(2k\pi/\tau).\end{aligned} \tag{5.21}$$

But $2[N(\tau) + 1]\pi/\tau > 4\pi/\rho$, so

$$\begin{aligned}2\pi\tau^{-1} \sum_{k=1}^{N(\tau)} a''(2k\pi/\tau) &\geqq \int_{2\pi/\tau}^{4\pi/\rho} a''(t)\, dt \\ &= a'(4\pi/\rho) - a'(2\pi/\tau),\end{aligned} \tag{5.22}$$

since the first expression is an upper sum for the integral.

Since $\varphi(\tau) > 0$ for all τ (see (5.6)), we deduce (5.19) from (5.20), (5.21) and (5.22). This proves Lemma 3.

Now consider case (ii) of Theorem 4. Choose $\gamma > 0$ such that $a(t) \leqq \gamma t^{-\beta}$ $(0 < t \leqq \pi/\rho)$ and define

$$T(\lambda) = \max\{\rho, (2\Delta\lambda)^{1/(2-\beta)}\},$$

where $\Delta = \gamma\pi^{(1-\beta)}/(1 - \beta)$.

By Lemma 3 we have

$$|I_2| \leqq \frac{1}{\lambda}\int_\rho^{T(\lambda)} \frac{d\tau}{\tau\varphi(\tau)} \leqq \frac{Q}{\lambda}\int_\rho^{T(\lambda)} \tau^{1-\beta}\, d\tau$$
$$\leqq \frac{Q}{\lambda(2-\beta)}[(2\Delta\lambda)^{1/(2-\beta)}]^{2-\beta} = B.$$

For $\tau \geqq T(\lambda)$,

$$0 \leqq \lambda\psi(\tau) \leqq \lambda\gamma\int_0^{\pi/\tau} t^{-\beta}\, dt \leqq [\tfrac{1}{2}\Delta^{-1}\tau^{2-\beta}]\Delta\tau^{\beta-1} = \tfrac{1}{2}\tau,$$

where the first two inequalities come from (5.2). Therefore, with $D(\tau, \lambda)$ as in the proof of Lemma 2, we have $D(\tau, \lambda) \geqq \tau - \lambda\psi(\tau) \geqq \tau/2\,(\tau \geqq T(\lambda))$ and $|I_3| \leqq 2\int_\rho^\infty \tau^{-2}\,d\tau = B$. Lemma 2 again provides a bound for I_1, so (3.1) holds in case (ii).

Finally, we estimate I_2 and I_3 in case (iii). By (5.2), $\psi(\tau) \to 0$ as $\tau \to \infty$. Choose a number ω so large that $|\psi(\tau)| \leqq 1$ if $\tau \geqq \omega$, and let $T(\lambda) = \max\{\rho, \omega, 2\lambda\}$. Then $D(\tau, \lambda) \geqq \tau - \lambda\psi(\tau) \geqq \tau/2\,(\tau \geqq T(\lambda))$ and $|I_3| \leqq 2\int_\rho^\infty \tau^{-2}\,d\tau = B$. For I_2 we use (5.19) with $\beta = 1$. We have

$$|I_2| \leqq \frac{1}{\lambda}\int_\rho^{T(\lambda)} \frac{d\tau}{\tau\varphi(\tau)} \leqq 2Q = B.$$

This completes the proof of Theorem 4.

REFERENCES

[1] C. M. Dafermos, *An abstract Volterra equation with application to linear viscoelasticity*, J. Differential Equations, 7 (1970), pp. 554–569.

[2] ———, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal., 37 (1970), pp. 297–308.

[3] A. Friedman, *Monotonicity of solutions ov Volterra integral equations in Banach space*, Trans. Amer. Math. Soc., 138 (1969), pp. 129–148.

[4] A. Friedman and M. Shinbrot, *Volterra integral equations in Banach space*, Ibid., 126 (1967), pp. 131–179.

[5] K. B. Hannsgen, *Indirect Abelian theorems and a linear Volterra equation*, Ibid., 142 (1969), pp. 539–555.

[6] ———, *On a nonlinear Volterra equation*, Michigan Math. J., 16 (1969), pp. 365–376.

[7] J. J. Levin, *The asymptotic behavior of the solution of a Volterra equation*, Proc. Amer. Math. Soc., 14 (1963), pp. 534–541.

[8] I. P. Natanson, *Theory of Functions of a Real Variable*, vol. II, Frederick Ungar, New York, 1960.

[9] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, N.J., 1946.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

ON THE EQUATION OF EULER–POISSON–DARBOUX*

D. W. BRESTERS†

Abstract. Weak solutions of the initial value problem for the EPD equation are constructed using distributional methods. After taking the Fourier transform with respect to the space variables we obtain an equation related to the Bessel differential equation which can easily be solved. The inverse transforms are then found using some results obtained earlier by the author.

It is shown that for values of the parameter λ which are greater than $n - 1$ (n being the space dimension) the solution is the same as the one obtained by Weinstein [16]. However, the method of this paper can be used for all values of the parameter. Also the exceptional values $\lambda = -1, -3, -5, \cdots$ fit in quite naturally. Conditions for the regularity of the solutions are given for all values of λ.

1. Introduction. Consider the hyperbolic differential equation

$$\left(\Delta - \frac{\partial^2}{\partial t^2} - \frac{\lambda}{t}\frac{\partial}{\partial t}\right)u(t, x; \lambda) = 0, \qquad t > 0, \tag{1.1}$$

with initial conditions

$$\begin{aligned} &\lim_{t \downarrow 0} u(t, x; \lambda) = \phi(x), \\ &\lim_{t \downarrow 0} \frac{\partial}{\partial t} u(t, x; \lambda) = 0, \end{aligned} \tag{1.2}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, $x = (x_1, x_2, \cdots, x_n)$ and λ is a real or complex parameter. For $\lambda \neq 0$ the problem considered here is a *singular* Cauchy problem.

The case considered most frequently is obviously the one where $\lambda = 0$. Equation (1.1) then turns into the n-dimensional wave equation. For $\lambda = 0$, (1.1) appears in several branches of applied mathematics such as the transonic flow of compressible fluids. For $\lambda = \frac{1}{3}$, (1.1) corresponds to Tricomi's equation. If we replace Δ by $-\Delta$, we obtain an elliptic equation which appears in generalized axially symmetric potential theory and has applications in hydrodynamics and the theory of elasticity.

Equation (1.1) is generally referred to as the equation of Euler–Poisson–Darboux (abbreviated as the EPD equation). It is almost impossible to mention all publications on the EPD equation. Hence we restrict ourselves to the following rather arbitrary survey. References [7], [9] and [15] are of historic interest. The most essential steps forward have been done by Weinstein [16], [17]. His studies were followed by several others from the "Maryland-School" of which we mention Diaz and Weinberger [8], Martin [14] and Blum [1], [2]. Several recent publications still depend on the work of Weinstein (for example, Young [18]). The papers mentioned above all give solutions in the classical sense. For a treatment in the distributional sense we refer the reader to Lions [13] and Carroll [5]. Lions uses "operateurs de transmutation" (after an idea of Delsarte) and Carroll applies

* Received by the editors March 23, 1971.

† Institute of Applied Mathematics, University of Amsterdam, Roetersstraat 15, Amsterdam, The Netherlands.

Fourier transformation with respect to the space variables only. Neither constructs solutions but both derive theorems on existence, uniqueness and convexity properties of the solutions.

It is the aim of this paper to show that the method applied by Carroll can also be used to construct the solution of the Cauchy problem (1.1)–(1.2). In the author's opinion the greatest advantage of this method is that it gives solutions for all values of λ. Weinstein considers first positive integer values of λ which are greater than $n - 1$; the solutions obtained for these values of λ are then used to obtain solutions in the other cases by means of a generalized method of descent and recurrence formulas.

The situation for $\lambda < 0$, where the solution is no longer uniquely determined, can also be clarified by the method of this paper and the exceptional values $\lambda = -1, -3, -5, \cdots$ fit in quite naturally. We shall use Fourier transformation with respect to the space variables only and it will appear that the required inverse transforms can easily be found from the tables of Fourier transforms as given by Gel'fand and Shilov [10] and by the author [4].

In the subsequent section we denote by "distribution" a generalized function defined on the space S of testing functions which decrease, together with all their derivatives, faster than any negative power of $|x|$ as $|x| \to \infty$. In this case the Fourier transform is a one–one mapping of the dual space S' into itself.

Finally it may be remarked that the method of solution introduced in this paper can be modified in order to solve the Cauchy problem for the equation

$$\left(\Delta - \frac{\partial^2}{\partial t^2} - \frac{\lambda}{t}\frac{\partial}{\partial t}\right)u(t,x;\lambda) = c^2 u(t,x;\lambda). \tag{1.3}$$

This problem has been studied by Young [18] by means of methods related to those of Weinstein.

It is the author's intention to study problem (1.3) in detail in a subsequent paper.

2. Solution of the Cauchy problem.

2.1. Preliminaries. We consider again the singular Cauchy problem:

$$\begin{aligned} &\left(\Delta_x - \frac{\partial^2}{\partial t^2} - \frac{\lambda}{t}\frac{\partial}{\partial t}\right)u(t,x;\lambda) = 0, \qquad t > 0, \\ &u(0,x;\lambda) = \phi(x), \\ &u_t(0,x;\lambda) = 0. \end{aligned} \tag{2.1}$$

Applying Fourier transformation with respect to the space variables only we obtain

$$\left(k^2 + \frac{\partial^2}{\partial t^2} + \frac{\lambda}{t}\frac{\partial}{\partial t}\right)\tilde{u}(t,k) = 0, \tag{2.2}$$

$$\tilde{u}(0,k;\lambda) = \tilde{\phi}(k), \tag{2.3}$$

$$\tilde{u}_t(0,k;\lambda) = 0, \tag{2.4}$$

where $k = (k_1, k_2, \cdots, k_n)$ corresponds to $x = (x_1, \cdots, x_n)$, $k^2 = k_1^2 + k_2^2 + \cdots + k_n^2$, $\tilde{u}(t, k; \lambda) = F_x[u(t, x; \lambda)]$ and $\tilde{\phi}(k) = F_x[\phi(x)]$.

First we construct a solution $\tilde{G}_\lambda(t, k)$ of (2.2) which satisfies

$$\lim_{t \to 0} \tilde{G}_\lambda(t, k) = 1, \tag{2.5}$$

$$\lim_{t \to 0} \frac{\partial}{\partial t} \tilde{G}_\lambda(t, k) = 0. \tag{2.6}$$

When $\tilde{G}_\lambda(t, k)$ is found, the solution of (2.2), (2.3), (2.4) is

$$\tilde{u}(t, k; \lambda) = \tilde{G}_\lambda(t, k) \cdot \tilde{\phi}(k) \tag{2.7}$$

and the solution of (2.1) is then given by

$$u(t, x; \lambda) = F_k^{-1}[\tilde{G}_\lambda(t, k)] * \phi(x) = G_\lambda(t, x) * \phi(x), \tag{2.8}$$

where F_k^{-1} denotes the inverse Fourier transform with respect to $k = (k_1, k_2, \cdots, k_n)$ and the symbol $*$ denotes convolution with respect to $x = (x_1, \cdots, x_n)$ only.

We call $G_\lambda(t, x)$ the *fundamental solution* of the Cauchy problem (2.1). It satisfies the EPD equation with initial condition $\phi(x) = \delta(x)$. In order to prove that (2.8) actually represents the solution of problem (2.1) we shall show in § 2.3 that:

(i) $G_\lambda(t, x)$ is a well-defined distribution in S'_x (i.e., the dual of the space S_x of testing functions depending on x) which is twice continuously differentiable with respect to the parameter t for all $t \geqq 0$.

(ii) The convolution product in (2.8) exists for a large class of functions and distributions $\phi(x)$.

2.2 Construction of $G(t, k)$. By the transformation

$$\tilde{G}_\lambda(t, k) = t^{(1-\lambda)/2} h(t, k), \tag{2.9}$$

equation (2.2) becomes

$$\frac{d^2 h}{dt^2} + t^{-1} \frac{dh}{dt} + \left(k^2 - \frac{((1-\lambda)/2)^2}{t^2} \right) h = 0, \tag{2.10}$$

that is, the Bessel differential equation.

The following two cases are considered separately:

Case I. $(1 - \lambda)/2$ noninteger.

Case II. $(1 - \lambda)/2$ integer.

For future use we summarize below some general formulas for Bessel functions of the first kind $J_\nu(z)$ and the second kind $Y_\nu(z)$:

$$\lim_{z \to 0} z^{-\nu} J_\nu(z) = \frac{1}{2^\nu \Gamma(\nu + 1)}, \qquad \nu \neq \text{neg. integer}, \tag{2.11}$$

$$J_{-n}(z) = (-1)^n J_n(z), \qquad n \text{ integer}, \tag{2.12}$$

$$\frac{d}{dz}\{z^{-\nu}J_\nu(z)\} = -z^{-\nu}J_{\nu+1}(z), \tag{2.13}$$

$$\frac{d}{dz}\{z^{\nu}J_\nu(z)\} = z^{\nu}J_{\nu-1}(z), \tag{2.14}$$

$$\lim_{z\to 0} z^\nu J_\nu(z) = \begin{cases} 0 & \text{for} \quad \nu > 0, \\ \infty & \text{for} \quad \nu < 0, \\ 1 & \text{for} \quad \nu = 0, \end{cases} \tag{2.15}$$

$$\lim_{z\to 0} z^\nu Y_\nu(z) = -\frac{2^\nu\Gamma(\nu)}{\pi}, \qquad \nu > 0. \tag{2.16}$$

Formulas (2.12), (2.13), (2.14) hold for $Y_\nu(z)$ as well.

We start with Case I: $(1-\lambda)/2$ noninteger (λ even). The general solution of (2.10) is then given by:

$$h(t,k) = AJ_{(1-\lambda)/2}(|k|\cdot t) + BJ_{(\lambda-1)/2}(|k|\cdot t), \tag{2.17}$$

where A and B are arbitrary complex numbers which may depend on $|k| = \sqrt{k_1^2 + \cdots + k_n^2}$. Hence,

$$\tilde{G}_\lambda(t,k) = At^{(1-\lambda)/2}J_{(1-\lambda)/2}(|k|\cdot t) + Bt^{(1-\lambda)/2}J_{(\lambda-1)/2}(|k|\cdot t). \tag{2.18}$$

Now A and B should be chosen such that $G(t,k)$ satisfies conditions (2.5) and (2.6). Using formulas (2.11)–(2.15) we easily obtain that

$$B = |k|^{(1-\lambda)/2}2^{(\lambda-1)/2}\Gamma\left(\frac{\lambda+1}{2}\right),$$

while $A = 0$ for $\lambda \geqq 0$ but remains undetermined for $\lambda < 0$. Hence we have for $\lambda < 0$ ($\lambda \neq 2l+1$, $l = 0, \pm 1, \pm 2, \cdots$):

$$\tilde{G}_\lambda(t,k) = 2^{(\lambda-1)/2}\Gamma\left(\frac{\lambda+1}{2}\right)(|k|\cdot t)^{(1-\lambda)/2}J_{(\lambda-1)/2}(|k|\cdot t) + A\cdot t^{(1-\lambda)/2}J_{(1-\lambda)/2}(|k|\cdot t), \tag{2.19}$$

while for $\lambda \geqq 0$ (and $\neq 2l+1$) the second term vanishes ($A = 0$).

Next we consider *Case* II: $(1-\lambda)/2$ integer (that is, λ odd). We obtain

$$\tilde{G}_\lambda(t,k) = C\cdot t^{(1-\lambda)/2}J_{(\lambda-1)/2}(|k|\cdot t) + Dt^{(1-\lambda)/2}Y_{(\lambda-1)/2}(|k|\cdot t), \tag{2.20}$$

where C and D are arbitrary numbers which may depend on $|k|$. Using again formulas (2.11)–(2.16) we obtain that conditions (2.5) and (2.6) are satisfied if we take

$$C = 2^{(\lambda-1)/2}\Gamma\left(\frac{\lambda+1}{2}\right)|k|^{(1-\lambda)/2} \text{ for } \lambda = 1, 3, 5, \cdots,$$

$$D = 0$$

and

$$C = \text{arbitrary},$$
$$D = -\pi 2^{(\lambda-1)/2}/\Gamma((1-\lambda)/2)|k|^{(1-\lambda)/2} \quad \text{for } \lambda = -1, -3, -5, \cdots.$$

Summarizing our results we have

$$\tilde{G}_\lambda(t,k) = 2^{(\lambda-1)/2}\Gamma\left(\frac{\lambda+1}{2}\right)(|k|\cdot t)^{(1-\lambda)/2}J_{(\lambda-1)/2}(|k|\cdot t) \quad \text{for all } \lambda \geqq 0, \tag{2.21}$$

$$\begin{aligned}\tilde{G}_\lambda(t,k) &= 2^{(\lambda-1)/2}\Gamma\left(\frac{\lambda+1}{2}\right)(|k|\cdot t)^{(1-\lambda)/2}J_{(\lambda-1)/2}(|k|\cdot t)\\ &\quad + At^{(1-\lambda)/2}J_{(1-\lambda)/2}(|k|\cdot t) \quad \text{for } \lambda < 0, \quad \lambda \neq -1, -3, -5, \cdots\end{aligned} \tag{2.22}$$

$$\begin{aligned}\tilde{G}_\lambda(t,k) &= \frac{-\pi 2^{(\lambda-1)/2}}{\Gamma((1-\lambda)/2)}(|k|\cdot t)^{(1-\lambda)/2}Y_{(1-\lambda)/2}(|k|\cdot t)\\ &\quad + B\cdot t^{(1-\lambda)/2}J_{(1-\lambda)/2}(|k|\cdot t) \qquad \text{for} \quad \lambda = -1, -3, -5, \cdots,\end{aligned} \tag{2.23}$$

where A and B are arbitrary complex numbers. It follows immediately that a unique solution for the Cauchy problem (2.1) is certainly not possible for negative values of λ.

The difference between two solutions of the problem (2.5), (2.6), (2.7) is always of the type

$$C\cdot t^{(1-\lambda)/2}J_{(1-\lambda)/2}(|k|\cdot t), \tag{2.24}$$

where C is an arbitrary complex number which may depend on k. Also we remark that for $\lambda = -1, -3, -5, \cdots$ the solution will be of a different character than the solutions for other values of λ, due to the occurrence of a Bessel function of the second kind $Y_\nu(z)$ which is singular at $z = 0$. We shall return to this case after we have studied the inverse Fourier transformation of the obtained solutions $\tilde{G}_\lambda(t,k)$.

3. The fundamental solution $G_\lambda(t,x)$ for $\lambda \neq -1, -3, \cdots$. In § 2.2 we obtained that for $\lambda \neq -1, -3, -5, \cdots$,

$$G_\lambda(t,k) = 2^{(\lambda-1)/2}\Gamma\left(\frac{\lambda+1}{2}\right)(|k|\cdot t)^{(1-\lambda)/2}J_{(\lambda-1)/2}(|k|\cdot t) \tag{3.1}$$

is a solution of (2.2) which satisfies conditions (2.5) and (2.6). Moreover, (3.1) gives a unique solution in the case where $\lambda \geqq 0$.

For the solution of our original Cauchy problem (2.1) we need the inverse Fourier transform of (3.1) with respect to $k = (k_1, k_2, \cdots, k_n)$. We use the following formula from the table of Fourier transforms as given by Gel'fand and

Shilov [10]:

$$
\begin{aligned}
F\left[\frac{(m^2+P)_+^\mu}{\Gamma(\mu+1)}\right] &= \frac{2^{\mu+n/2+1}\pi^{n/2-1}m^{n/2+\mu}}{\sqrt{|\Delta|}} \\
&\cdot\Bigg[-\sin(\mu+q/2)\pi\cdot\frac{K_{\mu+n/2}(mQ_+^{1/2})}{Q_+^{(\mu+n/2)/2}}+\frac{\pi}{2\sin(\mu+n/2)\pi} \\
&\cdot\Bigg\{\sin(\mu+q/2)\pi\cdot\frac{J_{\mu+n/2}(mQ_-^{1/2})}{Q_-^{(\mu+n/2)/2}}+\sin\frac{n-q}{2}\pi \\
&\cdot\frac{J_{\mu-n/2}(mQ_-^{1/2})}{Q_-^{(\mu+n/2)/2}}\Bigg\}\Bigg],
\end{aligned} \tag{3.2}
$$

where P is a real quadratic form $\sum_{r,s=1}^n g^{rs}x_rx_s = (x, \Gamma x)$, $\Delta = \det\Gamma$, Γ is the matrix (g^{rs}) and q is the number of terms with negative sign in the canonical form of P. $J_\nu(z)$ and $K_\nu(z)$ denote the usual Bessel functions. Q is the dual of the quadratic form P, that is, $Q = (x, \Gamma^{-1}x)$, while $Q_+ = Q$ for $Q > 0$ and vanishes for $Q < 0$. Terms with Q_+ vanish for $Q < 0$. $Q_- = |Q|$ for $Q < 0$ and vanishes for $Q > 0$.

Now if we take

$$
\begin{gathered}
P = -x_1^2 - x_2^2 - \cdots - x_n^2 \quad (= -x^2), \\
Q = -k_1^2 - k_2^2 - \cdots - k_n^2 \quad (= k^2), \\
p = 0, \quad q = n, \quad \mu = \frac{\lambda - n - 1}{2}, \quad m = t, \quad |\Delta| = 1,
\end{gathered}
$$

and denote by $|k|$ the square root $\sqrt{k^2}$, we obtain:

$$
F_x\left[\frac{(t^2-x^2)_+^{(\lambda-n-1)/2}}{\Gamma((\lambda-n+1)/2)}\right] = \frac{2^{(\lambda+1)/2}\pi^{n/2-1}t^{(\lambda-1)/2}}{|k|^{(\lambda-1)/2}}J_{(\lambda-1)/2}(|k|\cdot t), \tag{3.3}
$$

and consequently,

$$
G_\lambda(t,x) = t^{1-\lambda}\frac{\Gamma((\lambda+1)/2)}{\pi^{n/2}}\frac{(t^2-x^2)_+^{(\lambda-n-1)/2}}{\Gamma((\lambda-n+1)/2)} \tag{3.4}
$$

and the solution of problem (2.1) is given by

$$
u(t,x;\lambda) = t^{1-\lambda}\frac{\Gamma((\lambda+1)/2)}{\pi^{n/2}}\frac{(t^2-x^2)_+^{(\lambda-n-1)/2}}{\Gamma((\lambda-n+1)/2)} * \phi(x). \tag{3.5}
$$

It should be remembered that (3.5) gives the solution for all $\lambda \neq -1, -3, -5, \cdots$ but that only for $\lambda \geqq 0$ the solution is uniquely determined.

In the case where $(\lambda - n - 1)/2$ is a negative integer we use the fact that

$$
\left.\frac{(t^2-x^2)_+^\nu}{\Gamma(\nu+1)}\right|_{\nu=-l} = \delta^{(l-1)}(t^2-x^2). \tag{3.6}
$$

We remark again that all distributions appearing in our solutions are defined on the space S_x of testing functions depending on $x = (x_1, \cdots, x_n)$ only. The variable t only appears as a parameter on which the solution depends continuously for $t \geqq 0$. The latter property is based on the fact that the Fourier transform is a continuous linear one–one mapping and on the properties of Bessel functions. In the same way we can conclude that the distribution $G_\lambda(t, x)$ is twice continuously differentiable with respect to the parameter t for all $t > 0$.

Next we consider the existence of the convolution in (3.5). Since $(t^2 - x^2)_+^\mu$ has as its support the interior of the sphere $|x| < t$, we may conclude that the convolution exists for arbitrary $\phi(x) \in S'$ (see, for example, de Jager [12, pp. 24–25]). Hence, a distributional solution of our problem exists for arbitrary $\phi(x) \in S'$.

For a solution in the classical sense we shall have to impose some conditions of differentiability on the function $\phi(x)$. In that case $u(t, x; \lambda)$ should be a twice continuously differentiable function of x for all $t > 0$ (remember that it is already such a function with respect to t).

We consider the expression

$$\frac{\partial^m}{\partial x_i^m}(t^2 - x^2)_+^\nu = \frac{\partial^m}{\partial x_i^m}\theta(t^2 - x^2)(t^2 - x^2)^\nu, \tag{3.7}$$

where $m = [\nu + 1]$, that is, the largest integer which is smaller than or equal to $\nu + 1$. For k positive the symbol $\partial^{-k}/\partial x^{-k}$ denotes the primitive of order k with respect to x (see, for example, [10, pp. 118–124]). It is easily seen that (3.7) contains $\delta(t^2 - x^2)$ as its most singular part. It follows from the definition of $\delta(t^2 - x^2)$ that $\delta(t^2 - x^2) * \phi(x)$ is defined and regular for $\phi(x) \in C^0$ (that is, $\phi(x)$ continuous). Hence for the existence in the classical sense of

$$\frac{\partial^m}{\partial x_i^m}(t^2 - x^2)_+^\nu * \phi(x)$$

we need only assume that $\phi(x) \in C^0$. Consequently,

$$(t^2 - x^2)_+^{(\lambda - n - 1)/2} * \phi(x)$$

is a C^2-function if we require that $\phi(x)$ be

$$2 - \left[\frac{\lambda - n - 1}{2} + 1\right] = 2 - \left[\frac{\lambda - n + 1}{2}\right]$$

times continuously differentiable. This result is in agreement with that of Weinstein [16]. For $\lambda > n - 1$ the solution as given by (3.5) is easily rewritten in the form in which it has originally been presented by Weinstein (cf. [16]).

If we put for the surface of the unit sphere in R_s

$$\frac{2\pi^{s/2}}{\Gamma(s/2)} = \omega_s,$$

we have from (2.29),

$$u(t, x; \lambda) = \frac{\omega_{\lambda - n + 1}}{\omega_{\lambda + 1}} t^{1 - \lambda}(t^2 - x^2)_+^{(\lambda - n - 1)/2} * \phi(x),$$

or equivalently,

$$u(t,x;\lambda) = \frac{\omega_{\lambda-n+1}}{\omega_{\lambda+1}} \int_{\xi^2<1} (1-\xi^2)^{(\lambda-n-1)/2} \phi(x+\xi t)\, d\xi, \tag{3.8}$$

where $\xi^2 = \sum_{j=1}^n \xi_j^2$, $x + \xi t = (x_1 + \xi_1 t, x_2 + \xi_2 t, \cdots, x_n + \xi_n t)$ and $d\xi = d\xi_1 \cdot d\xi_2 \cdots d\xi_n$.

We describe briefly the way in which Weinstein derives formula (3.8). He starts with the solution for $\lambda = n - 1$ which can easily be found. Next, (3.8) is proved for integer values of λ which are $\geqq n - 1$ and then the validity of (3.8) for noninteger values of λ, which are greater than $n - 1$, is shown by means of, as Weinstein calls it, the generalized method of descent. Then recurrence formulas relating the $u(t, x; \lambda)$ for different values of λ are used to obtain the solution for values of λ, smaller than $n - 1$ and not equal to $-1, -3, -5, \cdots$.

We remark that the distributional attack used in this paper produces the solution for all $\lambda \neq -1, -3, -5, \cdots$ at once. We have already seen that for $\lambda < 0$ the solution is not uniquely determined. The difference between two such solutions is always of the form:

$$F_k^{-1}[C \cdot t^{(1-\lambda)/2} J_{(1-\lambda)/2}(|k| \cdot t)], \tag{3.9}$$

where C may depend on k. Consequently, the difference between two arbitrary solutions for $\lambda < 0$ is always of the form:

$$t^{1-\lambda} G_{2-\lambda}(t,x) * \psi(x) = t^{1-\lambda} u(t,x;2-\lambda), \tag{3.10}$$

where $G_{2-\lambda}(t, x)$ is the fundamental solution of the Cauchy problem (2.1) with $2 - \lambda$ as the value for the parameter, and $\psi(x)$ is an arbitrary function or distribution belonging to S'. Hence, $u(t, x; 2 - \lambda)$ is the solution of some Cauchy problem (2.1) with arbitrary initial condition $\psi(x)$.

It is easily checked that $u(t, x; 2 - \lambda)$ for all $\lambda < 0$ has the properties

$$\begin{aligned} &\lim_{t\to 0} t^{-\lambda/2} u(t,x;2-\lambda) = 0, \\ &\lim_{t\to 0} t^{-\lambda/2-1} u_t(t,x;2-\lambda) = 0. \end{aligned} \tag{3.11}$$

Conditions (3.11) are the ones given by Blum [2] for the difference of two solutions.

It should be noted that in the presentation of these conditions by Weinstein [16] or Hadamard [11, Chap. VIII], $u(t, x; 2 - \lambda)$ does not denote the solution of a Cauchy problem of type (2.1) but an arbitrary solution of the EPD equation. In our presentation $u(t, x; 2 - \lambda)$ is considered as the solution of a Cauchy problem and then conditions (3.11) are automatically satisfied. It is also easily seen that conditions (3.11) when applied to the general solution $\tilde{G}_\lambda(t, k)$ as given by (2.18) or (2.20) lead to a difference between two arbitrary solutions for $\lambda < 0$ which is of the form (3.10). Hence, Blum's conditions (3.11) are equivalent to the condition that $u(t, x; 2 - \lambda)$ is the solution of some Cauchy problem (2.1), that is, $u(t, x; 2 - \lambda)$ satisfies

$$\begin{aligned} &\lim_{t\to 0} u(t,x;2-\lambda) \quad \text{exists in the distributional sense in } S', \\ &\lim_{t\to 0} u_t(t,x;2-\lambda) = 0. \end{aligned} \tag{3.12}$$

Finally, we remark that (3.10) with conditions (3.11) or (3.12) also gives the form of the difference between two arbitrary solutions in the case where $\lambda = -1, -3, \cdots$.

We investigate this case more extensively in the next section.

4. A fundamental solution for the exceptional values of the parameter λ. In this case a nonunique solution of the Cauchy problem (2.1) will always contain a term

$$\frac{-\pi 2^{(\lambda-1)/2}}{\Gamma((1-\lambda)/2)} F_k^{-1}[(|k|\cdot t)^{(1-\lambda)/2} Y_{(1-\lambda)/2}(|k|\cdot t)]. \tag{4.1}$$

We should find the inverse transform occurring in formula (4.1). The calculations can be simplified, however, if we make use of the arbitrariness of the solution for $\lambda < 0$.

It is clear from § 3 that

$$\frac{-\pi 2^{(\lambda-1)/2}}{\Gamma((1-\lambda)/2)} F_k^{-1}[(|k|\cdot t)^{(1-\lambda)/2}\{Y_{(1-\lambda)/2}(|k|\cdot t) \pm iJ_{(1-\lambda)/2}(|k|\cdot t)\}] \tag{4.2}$$

will also be a fundamental solution of our problem. Hence we can also take

$$G_\lambda(t,x) = \frac{\pi i 2^{(\lambda-1)/2}}{\Gamma((1-\lambda)/2)} F_k^{-1}[(|k|\cdot t)^{(1-\lambda)/2} H^{(1)}_{(1-\lambda)/2}(|k|\cdot t)]$$

or

$$G_\lambda(t,x) = \frac{-\pi i 2^{(\lambda-1)/2}}{\Gamma((1-\lambda)/2)} F_k^{-1}[(|k|\cdot t)^{(1-\lambda)/2} H^{(2)}_{(1-\lambda)/2}(|k|\cdot t)],$$

where $H^{(1)}_\nu = J_\nu + iY_\nu$ and $H^{(2)}_\nu = J_\nu - iY_\nu$ are the Hankel functions of the first and second kind.

In order to obtain the inverse Fourier transforms we use the table of Fourier transforms as given by Gel'fand and Shilov [10, (17), (18), p. 351]. If we take

$$P = -x_1^2 - x_2^2 - \cdots - x_n^2 = -x^2,$$
$$Q = -k_1^2 - k_2^2 - \cdots - k_n^2 = -k^2,$$
$$p = 0, \quad q = n, \quad c = t,$$

we obtain after some simple calculations

$$G_\lambda(t,x) = \frac{\Gamma((n-\lambda+1)/2)}{\Gamma((1-\lambda)/2)\pi^{n/2}} t^{1-\lambda} e^{\pm n\pi i/2}(t^2 - x^2 \pm i0)^{(\lambda-n-1)/2}. \tag{4.3}$$

Using the results obtained by the author in [4] (or an earlier paper [3]), it is easily seen that for n even formula (4.3) can be written as a combination of

$$(t^2 - x^2)^{(\lambda-n-1)/2} \quad \text{and} \quad \delta^{(n-\lambda-1)/2}(t^2 - x^2)$$

while for n odd we obtain a combination of

$$(t^2 - x^2)_-^{(\lambda-n-1)/2} \quad \text{and} \quad (t^2 - x^2)_+^{(\lambda-n-1)/2}.$$

In both cases the first term is connected with

$$F_k^{-1}[(|k| \cdot t)^{(1-\lambda)/2} Y_{(1-\lambda)/2}(|k| \cdot t)].$$

Hence, this first term is essential in so far that it will always be present in the solution. Consequently, the fundamental solution is no longer concentrated within the sphere $|x| < t$ and the existence of the convolution

$$G_\lambda(t, x) * \phi(x) \tag{4.4}$$

is no longer ensured. However it follows from the behavior of

$$(|k| \cdot t)^{(1-\lambda)/2} Y_{(1-\lambda)/2}(|k| \cdot t)$$

that every distribution $\tilde{\phi}(k) \in S'$ is a multiplier for this distribution. Consequently the convolution (4.4) exists for arbitrary $\phi(x) \in S'$. For a solution in the classical sense, $\phi(x)$ will have to satisfy certain conditions of differentiability. It is easily seen from (4.3) that these conditions are the same as those derived in § 3 as far as the variables $(x_1, x_2, \cdots, x_n)$ are concerned. But even then the solution can display a singular character with respect to the variable t at $t = 0$.

It follows immediately from (4.1) and (2.14) that

$$\left(\frac{\partial}{\partial t}\right)^{1-\lambda} F_x[u(t, x; \lambda)], \qquad \lambda = -1, -3, \cdots,$$

behaves for $t \to 0$ like

$$|k|^{1-\lambda} Y_0(|k| \cdot t) \cdot \tilde{\phi}(k).$$

Consequently $(\partial/\partial t)^{1-\lambda} u(t, x; \lambda)$ has a logarithmic singularity at $t = 0$. This singularity vanishes if $|k|^{1-\lambda}\tilde{\phi}(k) = 0$, that is, if

$$\Delta^{(1-\lambda)/2}\phi(x) = 0.$$

It follows that a solution for $\lambda = -1, -3, -5, \cdots$ has logarithmic singularities in certain of its derivatives unless we assume that the initial condition $\phi(x)$ is polyharmonic of order $(1 - \lambda)/2$. This is a well-known result which was obtained earlier by Weinstein [16], Diaz and Weinberger [8] and Blum [1].

REFERENCES

[1] E. K. Blum, *The Euler–Poisson–Darboux equation in the exceptional cases*, Proc. Amer. Math. Soc., 5 (1954), pp. 511–520.

[2] ———, *The solution of the EPD equation for negative values of the parameter*, Duke Math. J., 21 (1954), pp. 257–269.

[3] D. W. Bresters, *On distributions connected with quadratic forms*, SIAM J. Appl. Math., 16 (1968), pp. 563–581.

[4] ———, *Initial value problems for iterated wave operators*, Thesis, Enschede, 1969.

[5] R. W. Carroll, *Some singular Cauchy problems*, Ann. Mat. Pura Appl., Ser IV, 56 (1961), pp. 1–31.

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. II, Interscience, New York, 1966.

[7] G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. II, Gauthier–Villars, Paris, 1915.

[8] J. B. Diaz and H. F. Weinberger, *A solution of the singular initial value problem for the Euler–Poisson–Darboux equation*, Proc. Amer. Math. Soc., 4 (1953), pp. 703–715.

[9] L. EULER, *Institutiones Calculi Integralis*, vol. III, Petropoli, 1770.

[10] J. M. GEL'FAND AND G. J. SHILOV, *Verallgemeinerte Funktionen*, Bd. I., Deutscher Verlag der Wissenschaften, Berlin, 1960.

[11] J. HADAMARD, *La théorie des équations aux derivées partielles*, Ed. Scientifiques, Pekin, 1964.

[12] E. M. DE JAGER, *Applications of Distributions in Mathematical Physics*, 2ᵈ ed., Math. Centre Tracts no. 10, Amsterdam, 1969.

[13] J. L. LIONS, *Opérateurs de transmutation singuliers et équations d'Euler Poisson Darboux généralisées*, Rend. Sem. Mat. Fis. Milano, 28 (1959), pp. 124–137.

[14] M. H. MARTIN, *Riemann's method and the problem of Cauchy*, Bull. Amer. Math. Soc., 57 (1951), pp. 238–249.

[15] S. D. POISSON, *Mémoire sur l'integration des équations linéaires aux derivées partielles*, J. de l'Ecole Polytechnique, 12 (1823), no. 19.

[16] A. WEINSTEIN, *On the wave equation and the equation of Euler–Poisson*, Proc. Symposia Appl. Math., vol. 5, McGraw-Hill, New York, 1954, pp. 137–147.

[17] ———, *Sur le problème de Cauchy pour l'équation de Poisson et l'équation des ondes*, C.R. Acad. Sci. Paris, 234 (1952), pp. 2584–2585.

[18] F. C. YOUNG, *On a generalized EPD equation*, J. Math. Mech., 18 (1969), pp. 1167–1175.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

IMPROPERLY POSED INITIAL VALUE PROBLEMS FOR SELF-ADJOINT HYPERBOLIC AND ELLIPTIC EQUATIONS*

DAVID COLTON†

Abstract. Integral representations are obtained for the solution to Cauchy's problem for hyperbolic equations along a convex time-like surface, the exterior characteristic initial value problem for hyperbolic equations, and Cauchy's problem for elliptic equations along an analytic surface. Each of these problems is improperly posed in the real domain and hence our representations are constructed by integrating over appropriate regions in the space of one and several complex variables.

1. Introduction. Until about twenty years ago the problem of constructing approximate solutions to improperly posed initial value problems in partial differential equations was ignored by most mathematicians on the basis that such problems did not correspond to meaningful physical phenomena and hence such efforts were at best misguided and at worst fruitless. However, during the past two decades it came to be realized that such problems do in fact arise in mathematical physics. One such appearance is in the form of inverse free boundary problems in fluid mechanics (cf. [14]), and another is in boundary value problems where part of the boundary is inaccessible to measurement and hence the boundary data is incomplete (cf. [24], [26]). The physical origin of these problems has led to two different mathematical approaches.

In the case of inverse free boundary value problems the interest lies in constructing a "catalogue" of explicit solutions, and hence analytic data is prescribed on some analytic surface and it is desired to construct an approximate solution to a well-defined initial value problem. On the other hand, in the situation where the boundary data is incomplete, the initial data is not known exactly and approximations are constructed by assuming an a priori bound on the solution and then applying a Rayleigh–Ritz procedure [29].

Alternatively one can assume that the initial data itself satisfies an a priori bound, approximate it by a polynomial in some appropriate region (cf. [23]), and then treat the resulting initial value problem in the manner developed for inverse free boundary problems.

In all approaches the basic problem remains the same: the initial value problem is improperly posed in the sense that the solution does not depend continuously on the (real) initial data and hence one cannot approximate the solution by simply constructing the solution corresponding to approximate initial data.

In this paper we consider three classic examples of improperly posed initial value problems in partial differential equations: Cauchy's problem for hyperbolic equations along a time-like manifold [3], [21], [22], [11, pp. 754–760]; the exterior characteristic initial value problem for hyperbolic equations [12], [18], [25]; and Cauchy's problem for elliptic equations [4], [5], [6], [14], [17]. Each of these

* Received by the editors November 23, 1971, and in revised form March 7, 1972.

† Department of Mathematics, Indiana University, Bloomington, Indiana 47401, and Department of Mathematics, University of Glasgow, Glasgow, Scotland. This research was supported in part by the National Science Foundation under Grant GP-27232 and in part by the Science Research Council while the author was a visiting research fellow at the University of Glasgow.

problems is improperly posed in the sense that the solution (if it exists) does not depend continuously on the initial data and possesses coherence properties [12], [14], [21], [11, pp. 754–760]. (It should be noted, however, that in the case of analytic coefficients uniqueness is no problem since it is assured by Holmgren's theorem [12], [21], [13, pp. 185–188].)

We shall first treat in detail the exterior characteristic initial value problem and Cauchy's problem along a time-like manifold for the self-adjoint hyperbolic equation

$$u_{x_1x_1} = u_{x_2x_2} + u_{x_3x_3} + q(x_1, x_2, x_3)u - f(x_1, x_2, x_3), \tag{1.1}$$

where $q(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ are analytic functions of their independent variables. We shall then briefly show how to modify these results to treat Cauchy's problem for the elliptic equation

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + q(x_1, x_2, x_3)u = f(x_1, x_2, x_3). \tag{1.2}$$

For the special case of equation (1.1) when $q \equiv 0$ (i.e., the wave equation) the problems we are considering have been studied by Pucci [25] and Cannon [3] who showed existence, uniqueness, and continuous dependence on the data (in the complex domain) under the assumption that the initial data was analytic in one of its variables and differentiable to a sufficiently high order in the remaining variable (our results show that in the case of Cauchy's problem the smoothness conditions imposed by Cannon on the initial data can be weakened somewhat). It should also be noted that in the case of Cauchy's problem similar results had previously been given for general hyperbolic equations in two space variables by Titt [27] through the use of contraction mapping and majorization arguments.

However, our aim (and that of Cannon and Pucci) is more ambitious in that we want to obtain the solution as a linear functional of the data when the data is analytic in one of its variables and is prescribed either along a smooth time-like surface or on intersecting characteristic planes. Such an approach is advantageous in that it leads in a natural manner to results on existence, continuous dependence on the initial data, and approximation procedures. In the special case when the manifold on which the initial data is prescribed is noncharacteristic and analytic, and when the initial data is analytic in all of its independent variables, our work can be compared in some respects to that of Hill [19] and Garabedian [13, pp. 211–224].

Our results for hyperbolic equations and their analogue for elliptic equations in three independent variables are of additional interest in that they provide integral operators analogous to those of Riemann and Vekua in two independent variables [13], [30]. In the elliptic case these operators have several advantages (and some disadvantages) over the author's previous construction of integral operators in [7] (which can be viewed as an extension of Bergman's operators in two independent variables [1]) and a brief comparison of these two approaches will be discussed in § 3. It should be noted that in the elliptic case it is assumed that the initial data and the initial surface are analytic, and hence in this case the initial value problem under consideration could be solved locally via the Cauchy–Kowalewski theorem (cf. [20, pp. 116–119]). However, in addition to no longer being able to represent the solution by quadrature, this approach is far too tedious for practical application, and even if a series solution is constructed it may not converge in the full region where the solution is needed in a particular example (cf. [29]).

2. The hyperbolic equation (1.1). We shall now construct integral representations of the solutions to the Cauchy problem along a time-like manifold and the exterior characteristic initial value problem for (1.1). For convenience's sake we make the assumption that $q(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ are entire functions of the (complex) variables x_1, x_2 and x_3. It will be clear from our analysis that this assumption can be relaxed to assuming only $q(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ to be analytic in some polydisc in $\mathbb{C}^3$, the space of three complex variables. We also need the following definition [27].

DEFINITION 2.1. A function $g(x_1, x_2)$ of two real variables x_1 and x_2 is said to be *partially analytic with respect to* x_1 for $x_1 = a$ in the interval $\alpha \leqq x_2 \leqq \beta$ provided it can be represented by a series of the form

$$g(x_1, x_2) = b_0(x_2) + b_1(x_2)(x_1 - a) + b_2(x_2)(x_1 - a)^2 + \cdots \tag{2.1}$$

whose coefficients are continuous functions of x_2 in the interval $\alpha \leqq x_2 \leqq \beta$ and provided that the series (2.1) converges absolutely and uniformly for $\alpha \leqq x_2 \leqq \beta$, $|x_1 - a| \leqq \gamma$. The region $\alpha \leqq x_2 \leqq \beta$, $|x_1 - a| \leqq \gamma$ is known as the *region of partial analyticity*. The extension to more variables is evident.

We now introduce the coordinates

$$x = x_3 - x_1, \qquad y = x_1 + x_3, \qquad z = x_2 \tag{2.2}$$

and rewrite (1.1) in the form

$$L[u] \equiv u_{zz} + 4u_{xy} + Q(x, y, z)u = F(x, y, z), \tag{2.3}$$

where $F(x, y, z) = f(x_1, x_2, x_3)$ and $Q(x, y, z) = q(x_1, x_2, x_3)$. Let u and v be "well-behaved" functions to be prescribed shortly. Integrate the identity

$$vL[u] - uL[v] = (2u_y v - 2uv_y)_x + (2u_x v - 2uv_x)_y + (vu_z - uv_z)_z \tag{2.4}$$

over the torus $D \times \Omega$, where Ω is the circle $|z - \xi| = \delta > 0$ in the complex plane and D is the region in the Euclidean plane $\mathbb{R}^2$ bounded by a contour C consisting of a vertical segment C_1 joining a point B on the smooth, monotonically decreasing curve $y = y(x)$ to a point P above this curve, plus a horizontal segment C_2 joining P to a point A on $y = y(x)$, plus the arc C_3 defined by $y = y(x)$ joining A and B (see Fig. 1).

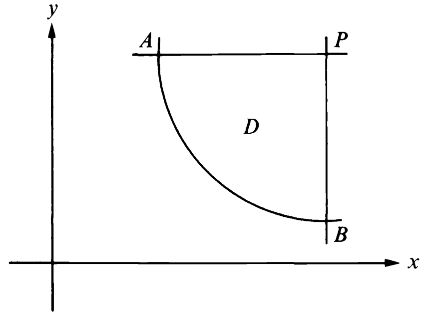

FIG. 1

Note that the integrals are to be interpreted in the sense of the calculus of exterior differential forms (cf. [2], [13, pp. 167, 213]), which attaches a meaning to them even when the differential dz is complex. Note also that the cylinder $y = y(x)$ in Euclidean three-space $\mathbb{R}^3$ is time-like with respect to the hyperbolic equation (2.3). For our purpose it is important that the curve $y = y(x)$ be monotonically decreasing and hence that the region D be as in Fig. 1 rather than as in Fig. 2. This is because of the fact that the curve AB in Fig. 2 is not time-like but space-like. Furthermore, we shall later on allow the curve C_3 to degenerate to a segment of the vertical characteristic plane through A and a segment of the horizontal characteristic plane through B. In the case of Fig. 1 this will correspond to an exterior characteristic initial value problem, whereas for Fig. 2 this becomes a (well-posed) interior characteristic value problem.

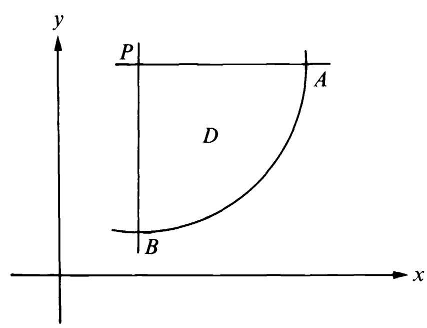

FIG. 2

The result of integrating (2.4) over the torus $D \times \Omega$, and then preforming an integration by parts on the right-hand side of the resulting identity, is, in the notation of the calculus of exterior differential forms,

$$\begin{aligned}(2.5)\qquad &\iiint_{D\times\Omega} (vL[u] - uL[v])\, dx\, dy\, dz \\ &\quad + \int_\Omega [2v(A,z)u(A,z) + 2v(B,z)u(B,z) - 4v(P,z)u(P,z)]\, dz \\ &\quad + 4 \iint_{C_1\times\Omega} uv_y\, dy\, dz - 4 \iint_{C_2\times\Omega} uv_x\, dx\, dz \\ &\quad + 2 \iint_{C_3\times\Omega} [(uv_y - vu_y)\, dy\, dz - (uv_x - vu_x)\, dx\, dz] = 0,\end{aligned}$$

where we have made use of the fact that $dx\, dy = 0$ on $\partial D \times \Omega$. Note that an expression of the form $v(A, z)$ is a function of three independent variables, i.e., $v(A, z) = v(x, y, z)$, where (x, y) are the Cartesian coordinates of the point A in $\mathbb{R}^2$.

We now choose u and v such that equation (2.5) reduces to an expression for the solution u of $L[u] = f$ satisfying prescribed Cauchy data on a smooth convex surface, where C_3 is the intersection of this surface with the plane $z = \zeta$, i.e., C_3 is a function of ζ. It is further assumed that the normal to the initial surface is never parallel to the z-axis and that C_3 is an analytic function of ζ. First let u be a twice continuously differentiable solution of $Lu = f$, where u and its partial derivatives of order less than or equal to two are partially analytic with respect to z in some neighborhood of the curve $y = y(x)$ and such that u satisfies prescribed Cauchy data on this curve. For v we construct a fundamental solution of $L[v] = 0$ which satisfies the boundary conditions

$$v_y = 0 \quad \text{on } C_1 \times \Omega, \tag{2.6}$$

$$v_x = 0 \quad \text{on } C_2 \times \Omega, \tag{2.7}$$

and such that at the point $(P, z) = (\xi, \eta, z)$,

$$v(P, z) = \frac{1}{8\pi i(z - \zeta)} + \text{analytic function of } (z - \zeta). \tag{2.8}$$

Note that conditions (2.6) and (2.7) are analogues to the boundary conditions satisfied by the Riemann function in two independent variables, and imply that in (2.5) the integrals over $C_1 \times \Omega$ and $C_2 \times \Omega$ vanish. We shall now show that the function v exists and possesses the necessary regularity properties for it to be substituted into (2.5).

Recall [13, pp. 152–168] that a fundamental solution $S = S(x, y, z; \xi, \eta, \zeta)$ of $L[u] = 0$ is of the form

$$S = U/R + W, \tag{2.9}$$

where $R = \sqrt{(z - \zeta)^2 + (x - \xi)(y - \eta)}$, $U = \sum_{l=0}^{\infty} U_l R^{2l}$, and W is a regular solution of $L[u] = 0$. The terms U_l, $l = 0, 1, 2, \cdots$, can be computed recursively. When the coefficients of the differential equation are entire, so is U, both as a function of (x, y, z) and the parameter point (ξ, η, ζ) (cf. [13, pp. 161, 167]). The term U_0 is given by the formula

$$U_0 = P_{00} \exp\left(-\int_0^s (C - 3/2)\frac{ds}{s}\right), \tag{2.10}$$

where (in the case of (2.3)) s is a parameter measured along the geodesics of the metric whose arc length element ds is given by the quadratic form

$$ds^2 = dx^2 + 4\,dx\,dy, \tag{2.11}$$

C is defined by

$$C = \frac{1}{4}\left[\frac{\partial^2 R^2}{\partial z^2} + 4\frac{\partial^2 R^2}{\partial x \partial y}\right], \tag{2.12}$$

and P_{00} is a constant. Equations (2.10) and (2.12) imply $U_0 = P_{00}$, a constant. We choose $P_{00} = 1/(8\pi i)$. Hence we have

$$S = \frac{1}{8\pi i R} + \sum_{l=1}^{\infty} U_l R^{2l-1} + W. \tag{2.13}$$

Now let us look at the singularities of $1/R$ in the complex z-plane for x and y in the region D of Fig. 1. In this case $(x - \xi)(y - \eta) \geqq 0$. If we cut the complex z-plane along a line parallel to the imaginary axis between $\zeta \pm i\sqrt{(x - \xi)(y - \eta)}$, $1/R$ is an analytic function of z outside this cut. (Note that if the region of integration were the region D in Fig. 2, we would have $(x - \xi)(y - \eta) \leqq 0$ and the complex z-plane would have had to be cut along the real axis.) In particular, $1/R$ is analytic for $|z - \zeta|^2 > |(x - \xi)(y - \eta)|$, i.e., for

$$\frac{|(x - \xi)(y - \eta)|}{|z - \zeta|^2} < 1. \tag{2.14}$$

Hence if W is, for example, an entire solution of $L[u] = 0$, S is regular for all points (x, y, z) and (ξ, η, ζ) satisfying the inequality (2.14). Thus if the point (ξ, η, ζ) is sufficiently near to the curve $y = y(x)$, S can be substituted for v in (2.5). The range of validity of (2.5) with S substituted for v can now be extended by analytic continuation, provided S satisfies (2.6) and (2.7) and the domain of regularity (as a function of z) of the Cauchy data is known.

We now turn our attention to choosing W such that (2.6) and (2.7) are satisfied by S. From (2.13) and the definition of R it is seen that one way this can be accomplished is to construct a solution W of $L[u] = 0$ satisfying the boundary conditions

$$W = -\sum_{l=1}^{\infty} U_l(z - \zeta)^{2l-1} \tag{2.15}$$

on the characteristic plane $x = \xi$, and

$$W = -\sum_{l=1}^{\infty} U_l(z - \zeta)^{2l-1} \tag{2.16}$$

on the characteristic plane $y = \eta$. (Note that $1/(8\pi iR)$ satisfies the boundary conditions (2.6)–(2.8). Furthermore, due to the form of equations (2.6)–(2.8), there exist boundary conditions different from (2.15) and (2.16) that could be chosen to define the function W.) This defines a characteristic initial value problem for $L[u] = 0$ with analytic (in fact entire) initial data. Hence from Hormander's generalized Cauchy–Kowalewski theorem [20, pp. 116–119] we can construct an entire solution W of $L[u] = 0$ which satisfies the initial data (2.15) and (2.16). Equation (2.13) now gives a suitable function $v = S$ to be substituted into (2.5). Note that from (2.13) we have that S satisfies condition (2.8). In the special case when $q = \text{const.} = \lambda^2$ a possible choice for the function $S = S_\lambda$ is

$$S_\lambda = \frac{\cos \lambda R}{8\pi i R}. \tag{2.17}$$

Now in (2.5) let $v = S$ and let u be a twice continuously differentiable solution of $L[u] = f$ whose partial derivatives of order less than or equal to two are partially analytic with respect to z. From (2.8) we have

$$4\int_\Omega v(P, z)u(P, z)\, dz = u(\xi, \eta, \zeta). \tag{2.18}$$

Hence (2.5) becomes

$$
\begin{aligned}
u(\xi,\eta,\zeta) = &+ 2\int_{\Omega} [S(A,z;\xi,\eta,\zeta)u(A,z) + S(B,z;\xi,\eta,\zeta)u(B,z)]\,dz \\
&- 2\iint_{C_3\times\Omega} [u(x,y,z)S_x(x,y,z;\xi,\eta,\zeta) \\
&\qquad - S(x,y,z;\xi,\eta,\zeta)u_x(x,y,z)]\,dx\,dz \\
&+ 2\iint_{C_3\times\Omega} [u(x,y,z)S_y(x,y,z;\xi,\eta,\zeta) \\
&\qquad - S(x,y,z;\xi,\eta,\zeta)u_y(x,y,z)]\,dy\,dz \\
&+ \iiint_{D\times\Omega} S(x,y,z;\xi,\eta,\zeta)F(x,y,z)\,dx\,dy\,dz.
\end{aligned} \tag{2.19}
$$

Equation (2.19) is the desired integral representation of u in terms of its Cauchy data along a smooth time-like convex surface, where C_3 denotes the intersection of this surface with the plane $z = \zeta$. Equation (2.19) also shows that at the point (ξ, η, ζ), $u(\xi, \eta, \zeta)$ depends continuously on its Cauchy data in $C_3 \times G$, where G is an arbitrarily small neighborhood containing the branch line $\zeta \pm i\sqrt{(x - \xi)(y - \eta)}$ for all points $(x, y) \in C_3$.

The solution of the exterior characteristic initial value problem for $L[u] = f$ can now be obtained in a manner analogous to the method used to solve the characteristic initial value problem for hyperbolic equations in two variables [13, p. 131] by setting $v = S(x, y, z; \xi, \eta, \zeta)$ in (2.4) and integrating this identity over the rectangle $ATBP$ in Fig. 3. In other words, we allow the curve C_3 to degenerate onto the characteristics $C_4 = AT$ and $C_5 = TB$ (where C_4 and C_5 are independent of ζ).

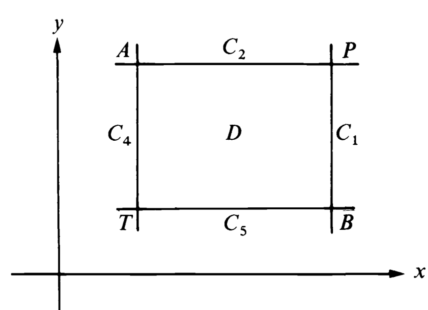

FIG. 3

Performing this deformation, and integrating by parts along the characteristics to eliminate the partial derivatives of u there, leads to

$$
\begin{aligned}
u(\xi,\eta,\zeta) = &+ 4\int_{\Omega} [S(A,z;\xi,\eta,\zeta)u(A,z) + S(B,z;\xi,\eta,\zeta)u(B,z) \\
&- S(T,z;\xi,\eta,\zeta)u(T,z)]\,dz \qquad \text{(cont.)}
\end{aligned}
$$

$$(2.20)\qquad \begin{aligned} &+ 4 \iint\limits_{C_4 \times \Omega} S_y(x, y, z; \xi, \eta, \zeta) u(x, y, z)\, dy\, dz \\ &- 4 \iint\limits_{C_5 \times \Omega} S_x(x, y, z; \xi, \eta, \zeta) u(x, y, z)\, dx\, dz \\ &+ \iiint\limits_{D \times \Omega} S(x, y, z; \xi, \eta, \zeta) F(x, y, z)\, dx\, dy\, dz. \end{aligned}$$

Equation (2.20) gives the integral representation of the solution u of $L[u] = f$ as a linear functional of its initial data on two intersecting characteristic planes which is valid in the wedge bisected by the plane $y = x$ and bounded by the two characteristic planes, i.e., equation (2.20) gives the solution of the exterior characteristic initial value problem.

3. The elliptic equation (1.2). Similar integral representations to those developed in § 2 for hyperbolic equations can also be found for the elliptic equation (1.2), provided we make the further assumptions that the initial data is analytic in each of its independent variables and that, in the case of Cauchy's problem, the surface on which the data is prescribed is also analytic. To see this we make use of the fact that twice continuously differentiable solutions of (1.2) are analytic functions of their independent variables (cf. [13, p. 164]) and introduce the change of variables

$$(3.1)\qquad x = x_1, \qquad z = x_2 + ix_3, \qquad z^* = x_2 - ix_3$$

defining a nonsingular map of $\mathbb{C}^3$ into itself. The elliptic equation (1.2) can then be written as

$$(3.2)\qquad u_{xx} + 4u_{zz^*} + Q(x, z, z^*)u = F(x, z, z^*),$$

which is formally of the same hyperbolic form as equation (2.3). Repeating the analysis of § 2 now leads to the integral representations (2.19) and (2.20) (with z replaced by x, x replaced by z, and y replaced by z^*) for the solution of the Cauchy and complex Goursat problems, respectively. (In the case of Cauchy's problem, $z = z(z^*)$ is the expression in conjugate coordinates of the intersection of the plane $x = \zeta$ with the initial surface.) In this case our analysis is reminiscent of Vekua's [15], [30] and Henrici's [15], [17] development of the analytic theory of elliptic equations in two independent variables. It is also similar to the integral operators constructed by Colton in [7] (see also [8], [9], [10], [16] and [28]).

The operators constructed in this paper have several advantages over the approach used in [7]:

(i) The form of the integral representations arises in a natural manner.

(ii) The integral representation of the solution to Cauchy's problem can be readily obtained. In particular, this considerably improves upon the results in [10] where the Cauchy data was required to be prescribed on the plane $x_1 = 0$ instead of on an analytic surface as in the present work, and where furthermore the coefficient $q(x_1, x_2, x_3)$ was required to be independent of x_1.

(iii) The nonhomogeneous equation can be treated.

On the other hand, several disadvantages must be mentioned. One of these is that difficulties arise in treating non-self-adjoint equations since the leading (singular) term of the fundamental solution S in general no longer satisfies the Goursat data as it does in the self-adjoint case. Extensions to higher dimensions also run into difficulties due to logarithmic terms appearing in the construction of S in an even number of independent variables and also due to the fact that the geodesic distance R between two points no longer has a pole-like singularity along the characteristics. The author is at present looking into these problems, and the results will hopefully be reported in a future paper.

We finally note in passing that different representations than those obtained in this paper can be derived for the solutions to improperly posed Cauchy problems for elliptic and hyperbolic equations by means of an appropriate change of variables in the complex domain and use of a fundamental solution (cf. [3], [13, pp. 614–621]). In this case the fundamental solution is not required to satisfy prescribed boundary data along the characteristics. On the other hand, new problems are created since the representation now includes terms involving the derivative of an improper integral and/or the finite parts of divergent integrals.

Note added in proof. The fact that the solution W of $L[u] = 0$ satisfying the Goursat data (2.15), (2.16) is entire follows from the results of Jan Persson in his paper *Linear Goursat problems for entire functions when the coefficients are variable*, Ann. Scoula Norm. Sup. Pisa, 23 (1969), pp. 87–98.

REFERENCES

[1] S. BERGMAN, *Integral Operators in the Theory of Linear Partial Differential Equations*, Springer-Verlag, Berlin, 1961.

[2] R. C. BUCK, *Advanced Calculus*, McGraw-Hill, New York, 1956.

[3] J. R. CANNON, *A Cauchy problem for the wave equation for a space-like region*, Matematiche (Catania), 16 (1961), pp. 43–50.

[4] D. COLTON, *Cauchy's problem for almost linear elliptic equations in two independent variables*, J. Approx. Theory, 3 (1970), pp. 66–71.

[5] ———, *Cauchy's problem for almost linear elliptic equations in two independent variables. II*, Ibid., 4 (1971), pp. 288–294.

[6] ———, *Cauchy's problem for a class of fourth order elliptic equations in two independent variables*, Applicable Anal., 1 (1971), pp. 13–22.

[7] ———, *Integral operators for elliptic equations in three independent variables. I*, Ibid., to appear.

[8] ———, *Integral operators for elliptic equations in three independent variables. II*, Ibid., to appear.

[9] ———, *Bergman operators for elliptic equations in three independent variables*, Bull. Amer. Math. Soc., 77 (1971), pp. 752–756.

[10] D. COLTON AND R. P. GILBERT, *An integral operator approach to Cauchy's problem for* $\Delta_{p+2}u(x) + F(x)u(x) = 0$, this Journal, 2 (1971), pp. 113–132.

[11] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. II, John Wiley, New York, 1962.

[12] F. G. FRIEDLANDER, *On an improperly posed characteristic initial value problem*, J. Math. Mech., 16 (1967), pp. 907–915.

[13] P. R. GARABEDIAN, *Partial Differential Equations*, John Wiley, New York, 1964.

[14] ———, *Applications of the theory of partial differential equations to problems of fluid mechanics*, Modern Mathematics for the Engineer: Second Series, Edwin Beckenbach, ed., McGraw-Hill, New York, 1961, pp. 347–372.

[15] R. P. GILBERT, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

[16] R. P. GILBERT AND C. Y. LO, *On the approximation of solutions of elliptic partial differential equations in two and three dimensions*, this Journal, 2 (1971), pp. 17–30.

[17] P. HENRICI, *A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients*, Z. Angew. Math. Phys., 8 (1957), pp. 169–203.

[18] R. HERSH AND Y. W. CHEN, *The three-dimensional wave equation in a characteristic quarter-space*, J. Math. Mech., 17 (1967), pp. 449–459.

[19] C. D. HILL, *Linear functionals and the Cauchy–Kowalewski theorem*, Ibid., 19 (1969), pp. 271–277.

[20] L. HORMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1964.

[21] F. JOHN, *On linear partial differential equations with analytic coefficients*, Comm. Pure Appl. Math., 2 (1949), pp. 209–253.

[22] ———, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Ibid., 13 (1960), pp. 551–585.

[23] K. MILLER, *Least squares methods for ill-posed problems with a prescribed bound*, this Journal, 1 (1970), pp. 52–74.

[24] L. E. PAYNE, *On some non well-posed problems for partial differential equations*, Numerical Solutions of Nonlinear Differential Equations, Donald Greenspan, ed., John Wiley, New York, 1966, pp. 239–263.

[25] C. PUCCI, *Su un problema esterno par la equazione delle onde*, Ann. Mat. Pura Appl., 56 (1961), pp. 69–78.

[26] ———, *Some topics in parabolic and elliptic equations*, University of Maryland Lecture Series 36, College Park, 1958.

[27] E. W. TITT, *An initial value problem for all hyperbolic partial differential equations of second order with three independent variables*, Ann. of Math., 40 (1939), pp. 862–891.

[28] B. L. TJONG, *Operators generating solutions of* $\Delta_3\psi(x, y, z) + F(x, y, z)\psi(x, y, z) = 0$ *and their properties*, Analytic Methods in Mathematical Physics, R. P. Gilbert and R. G. Newton, eds., Gordon and Breach, New York, 1970.

[29] M. VAN DYKE, *The supersonic blunt-body problem—review and extension*, J. Aero/Space Sciences, 25 (1958), pp. 485–496.

[30] I. N. VEKUA, *New Methods for Solving Elliptic Equations*, John Wiley, New York, 1967.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

CYCLICLY RELATED DIFFERENTIAL SYSTEMS*

G. J. ETGEN AND N. C. WONG†

Abstract. This paper is concerned with first order, linear, vector-matrix differential systems in which the coefficient matrix is an $n \times n$ g-circulant matrix whose entries are real-valued, Lebesgue integrable functions. The objective of the paper is to establish the general solution of such systems and to exhibit their exponential nature. Extensive use is made of the work of C. M. Ablow and J. L. Brenner on circulant matrices. The results obtained in this paper extend the work of W. M. Whyburn on cyclicly related functional equations and the work of E. J. Pellicciaro on cyclicly related differential equations.

1. Introduction. This paper is concerned with establishing the general solution of vector-matrix differential equations of the form

$$y' = B(x) \cdot y, \tag{1}$$

where $B(x)$ is an $n \times n$ g-circulant matrix whose entries $b_0(x), b_1(x), \cdots, b_{n-1}(x)$ are real-valued and Lebesgue integrable on the interval $[a, b]$. By an $n \times n$ g-circulant matrix with entries $b_0, b_1, \cdots, b_{n-1}$, we mean an $n \times n$ matrix whose first row is the vector $(b_0, b_1, \cdots, b_{n-1})$ and which has the property that each succeeding row is obtained from its immediate predecessor by a cyclic shift of g-columns to the right.

This work is motivated by the results of W. M. Whyburn [5], P. Barnhard and E. J. Pellicciaro [2], and W. E. Baxter and E. J. Pellicciaro [3]. In particular, in [5], Whyburn considered the cyclicly related system of equations

$$y_i' = \sum_{k=1}^{n} a_k(x) y_{i+m+hk}, \qquad i = 1, 2, \cdots, n, \tag{2}$$

where m and h are given integers, the subscripts $i + m + hk$ are reduced modulo n, and the functions $a_k(x)$ are Lebesgue integrable on $[a, b]$. Whyburn obtained the general solution of (2) in terms of exponential functions. In [2] Barnhard and Pellicciaro consider the more general system

$$y_i' = \sum_{k=1}^{m} a_k(x) y_{i+\alpha(k)}, \qquad i = 1, 2, \cdots, n, \tag{3}$$

where α is a function whose domain is the set $\{1, 2, \cdots, m\}$ and whose range is a subset of the set $\{1, 2, \cdots, n\}$. They show that the system (3) can be expressed in the vector-matrix form

$$y' = A(x)y, \tag{4}$$

where $A(x)$ is an $n \times n$ 1-circulant whose entries $c_0(x), c_1(x), \cdots, c_{n-1}(x)$ are certain sums of the functions $a_1(x), \cdots, a_m(x)$. Baxter and Pellicciaro [3] continued the study of cyclicly related systems by considering the vector-matrix equation (1) in the two cases: $B(x)$ a 1-circulant and $B(x)$ an $(n - 1)$-circulant.

* Received by the editors March 16, 1971, and in final revised form March 2, 1972.

† Department of Mathematics, University of Houston, Houston, Texas 77004.

The purpose of this paper is to obtain the general solution of certain systems of the form (1) where g is any integer. Of course, in view of the definition of a circulant matrix, we can, without loss of generality, assume $0 \leqq g \leqq n - 1$. The matrix methods employed in this paper make extensive use of the results of C. M. Ablow and J. L. Brenner [1]. In the context of [2] and [5], our results may be interpreted as establishing the general solution of certain cyclicly related differential systems of the form

$$y_i' = \sum_{k=1}^{m} a_k(x) y_{g \cdot i + \alpha(k)}, \qquad i = 1, 2, \cdots, n, \tag{5}$$

where, as above, the functions $a_k(x)$ are Lebesgue integrable on $[a, b]$, α is a function with domain and range $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$, respectively, and the subscripts $g \cdot i + \alpha(k)$ are reduced modulo n. In particular, it is easily verified that the system of equations (5) can be equivalently written in the vector-matrix form (1) where $B(x)$ is an $n \times n$ g-circulant with entries $b_j(x), j = 0, 1, \cdots, n - 1$, given by

$$b_j(x) = \sum_k a_k(x), \qquad g + \alpha(k) \equiv j + 1 \pmod n.$$

In order to obtain the general solution of (1) for arbitrary g, $0 \leqq g \leqq n - 1$, we distinguish two cases: $(g, n) = 1$, i.e., g and n relatively prime, and $(g, n) > 1$. The next section considers the case $(g, n) = 1$ and § 3 handles the case $(g, n) > 1$. We conclude the paper with some immediate generalizations of § 2 and § 3.

2. The case $(g, n) = 1$. Using the methods of [1], we give the following definition.

DEFINITION 1. Let $(g, n) = 1$ and let $\sim$ be the relation defined on the residue classes $0, 1, \cdots, n - 1 \pmod n$ as follows: $h_1 \sim h_2$ if and only if there exists a nonnegative integer q such that $h_1 \equiv h_2 \cdot g^q \pmod n$.

It is readily verified that $\sim$ is an equivalence relation separating the residue classes $0, 1, \cdots, n - 1 \pmod n$ into the equivalence classes $[h_0], [h_1], \cdots, [h_k]$. We shall assume that $[h_0]$ is the equivalence class containing 0 and it is easily seen that $[h_0]$ contains exactly one element, namely, 0. As shown in [1] the elements in $[h_j]$, $1 \leqq j \leqq k$, are: $h_j, h_j \cdot g, \cdots, h_j \cdot g^{q(j)} \pmod n$, where $q(j)$ is the least nonnegative integer such that $h_j \cdot g^{q(j)+1} \equiv h_j \pmod n$. Thus, associated with each equivalence class $[h_j], j = 0, 1, \cdots, k$, there is a nonnegative integer $q(j)$ such that $q(0) = 0$, $\sum_{j=0}^{k} [q(j) + 1] = n$. We note, in addition, that in the case $g = 1$, $q(j) = 0, j = 0, 1, \cdots, k, k = n - 1$, and the equivalence classes generated by $\sim$ are merely the n residue classes modulo n. For notational convenience we shall denote the elements in the equivalence class $[h_j]$ by $h_j = h_{j0}$, $h_j \cdot g = h_{j1}, \cdots$, $h_j g^{q(j)} = h_{jq(j)}, j = 0, 1, \cdots, k$.

The following theorem has been established by Ablow and Brenner [1, Thm. 4.1].

THEOREM 2.1. *Let B be any $n \times n$ g-circulant, $(g, n) = 1$. Let $[h_0], [h_1], \cdots, [h_k]$ be the equivalence classes generated by $\sim$. Let $\gamma(\rho)$ be the n-component column vector whose components are $1, \rho^p, \rho^{2p}, \cdots, \rho^{(n-1)p}$, where $\rho = \exp(2\pi i/n)$. Let M be the $n \times n$ matrix whose columns are the vectors $\gamma(h_0), \gamma(h_{10}), \gamma(h_{11}), \cdots, \gamma(h_{1q(1)})$,*

$\gamma(h_{21}), \cdots, \gamma(h_{kq(k)})$. *Then M is nonsingular and $M^{-1}BM$ has the block diagonal form $M^{-1}BM = \operatorname{diag}[R_0, R_1, \cdots, R_k]$, where for each integer $j, j = 0, 1, \cdots, k$, R_j is the $[q(j) + 1] \times [q(j) + 1]$ matrix having the form*

$$R_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & w(h_{jq(j)}) \\ w(h_{j0}) & 0 & \cdots & 0 & 0 \\ 0 & w(h_{j1}) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & w(h_{jq(j)-1}) & 0 \end{bmatrix}$$

provided $q(j) > 0$ and $R_j = [w(h_{j0})]$ if $q(j) = 0$, where

$$w(h_{jp}) = b_0 + b_1\rho^{h_{jp}} + b_2\rho^{2h_{jp}} + \cdots + b_{n-1}\rho^{(n-1)h_{jp}}. \tag{6}$$

Our next definition gives a property which will be required of the g-circulant matrices $B(x)$ of (1) on $[a, b]$.

DEFINITION 2. Let B be an $n \times n$ g-circulant matrix, $(g, n) = 1$, with entries $b_0, b_1, \cdots, b_{n-1}$. Let $[h_0], [h_1], \cdots, [h_k]$ be the equivalence classes generated by $\sim$. The matrix B has *property σ* provided the entries of B whose subscripts belong to the equivalence class $[h_j]$ are equal, $j = 0, 1, \cdots, k$, that is, $b_{h_{j0}} = b_{h_{j2}} = b_{h_{jq(j)}}$, $j = 0, 1, \cdots, k$.

We now apply Theorem 2.1 to the matrix differential equation (1).

THEOREM 2.2 *Let $B(x)$ be an $n \times n$ g-circulant matrix, $(g, n) = 1$, with entries $b_0(x), b_1(x), \cdots, b_{n-1}(x)$. Let $[h_0], [h_1], \cdots, [h_k]$ be the equivalence classes generated by the equivalence relation $\sim$. If $B(x)$ has property σ on $[a, b]$, then for each integer j, $0 \leqq j \leqq k$,*

$$w(h_{jp}) = w(h_{j0}) = \theta_j(x) = b_0(x) + \sum_{s=1}^{k} b_{h_s}(x)[\rho^{h_j h_{s0}} + \cdots + \rho^{h_j h_{sq(s)}}], \tag{7}$$

$p = 1, 2, \cdots, q(j)$, and each block $R_j(x)$ in the block diagonal form $M^{-1}B(x)M$ has the form

$$R_j(x) = \theta_j(x) \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{8}$$

Proof. Referring to [1, Thm. 4.1], we have the system of equations

$$\begin{aligned} B(x)\gamma(h_{j0}) &= w(h_{j0})\gamma(h_{j1}), \\ B(x)\gamma(h_{j1}) &= w(h_{j1})\gamma(h_{j2}), \\ &\vdots \\ B(x)\gamma(h_{jq(j)}) &= w(h_{jq(j)})\gamma(h_{j0}), \end{aligned} \tag{9}$$

$j = 0, 1, \cdots, k$. Now assuming that $B(x)$ has property σ, we have, from (6), for each j, $0 \leqq j \leqq k$,

$$w(h_{jp}) = b_0(x) + \sum_{s=1}^{k} b_{h_s}(x)[\rho^{h_{jp}h_{s0}} + \rho^{h_{jp}h_{s1}} + \cdots + \rho^{h_{jp}h_{sq(s)}}]$$

for $p = 0, 1, \cdots, q(j)$. We claim that $w(h_{jp}) = w(h_{j0}) = \theta_j(x)$, $p = 1, 2, \cdots, q(j)$. Choose any positive integer p, $1 \leqq p \leqq q(j)$, and any s, $1 \leqq s \leqq k$. Then for some integer t, $1 \leqq t \leqq q(s)$, $p + t \equiv 0 \pmod{(q(s) + 1)}$. Thus we see that the exponents

$$h_{jp}h_{s0} = h_j g^p h_s, \quad h_{jp}h_{s1} = h_j g^p h_s g, \quad \cdots, \quad h_{jp}h_{sq(s)} = h_j g^p h_s g^{q(s)}$$

may also be interpreted as

$$h_{j0}h_{sp}, \quad h_{j0}h_{s(p+1)}, \quad \cdots, \quad h_{j0}h_{s(p+t)} \equiv h_{j0}h_{s0}, \quad h_{j0}h_{s1}, \quad \cdots, \quad h_{j0}h_{s(p-1)}$$

and $w(h_{jp}) = w(h_{j0})$. It is clear from (7) and (6) that if $B(x)$ has property σ, then $\theta_j(x) = w(h_{j0})$ for each j. We conclude, therefore, that the system of equations (9) has the form

$$\begin{aligned} B(x)\gamma(h_{j0}) &= \theta_j(x)\gamma(h_{j1}), \\ B(x)\gamma(h_{j1}) &= \theta_j(x)\gamma(h_{j2}), \\ &\vdots \\ B(x)\gamma(h_{jq(j)}) &= \theta_j(x)\gamma(h_{j0}) \end{aligned} \tag{10}$$

and the theorem follows.

Our next theorem shows that each of the blocks $R_j(x)$ obtained in Theorem 2.2 can be diagonalized.

THEOREM 2.3. *If R is a $[q + 1] \times [q + 1]$ matrix having the form* (7), *then there exists a nonsingular $[q + 1] \times [q + 1]$ constant matrix N such that $RN = ND$, where D is a diagonal matrix having diagonal entries $\theta(x)r_q, \theta(x)r_q^2, \cdots, \theta(x)r_q^q$, $\theta(x)$, and $r_q = \exp(2\pi i/(q + 1))$, a primitive $(q + 1)$st root of unity.*

Proof. Let p be a positive integer such that $1 \leqq p \leqq q + 1$ and consider the $(q + 1)$-component column vector N_p whose components are $r_q^{q+1-p}, r_q^{2(q+1-p)}$, $\cdots, 1$, respectively. It is easily verified that N_p is a characteristic vector of R corresponding to the characteristic root $\theta(x)r_q^p$. Consequently, the $[q + 1] \times [q + 1]$ matrix N whose columns are $N_1, N_2, \cdots, N_{q+1}$ satisfies the equation $RN = ND$, where D is the diagonal matrix having entries $\theta(x)r_q, \theta(x)r_q^2, \cdots, \theta(x)$. The matrix N is nonsingular since it is a Vandermonde matrix, and the proof of the theorem is complete.

We are now in a position to obtain the general solution of (1) where $B(x)$ is a g-circulant, $(g, n) = 1$, having property σ on $[a, b]$. Let M be the $n \times n$ nonsingular constant matrix defined by Theorem 2.1, and let u be the n-component column vector defined by the equation

$$y = Mu. \tag{11}$$

Then $y' = Mu' = B(x)y = B(x)Mu$ so that (1) is transformed into

$$u' = M^{-1}B(x)Mu. \tag{12}$$

Now $M^{-1}B(x)M$ has the block diagonal form described by Theorem 2.1. Partition the column vector u into $k + 1$ blocks having $q(0) + 1 = 1, q(1) + 1, \cdots, q(k) + 1$ components, respectively. Let $u(j)$ denote the jth block of u, $j = 0, 1, \cdots, k$. Then the vector-matrix equation (12) can be written as a system of $k + 1$ vector-matrix equations

$$u'(j) = R_j(x)u(j), \qquad j = 0, 1, \cdots, k. \tag{13}$$

Using Theorems 2.2 and 2.3 and letting

$$v' = R(x)v \tag{14}$$

denote any one of the k equations $u'_j = R_j(x)u(j), j = 1, 2, \cdots, k$, of (13), we define the $[q + 1]$-component column vector z by $v = Nz$, where N is the nonsingular $[q + 1] \times [q + 1]$ constant matrix defined by Theorem 2.3. Equation (14) is then transformed into

$$z' = D(x)z, \tag{15}$$

where $D(x) = \operatorname{diag}[\theta(x)r_q, \theta(x)r_q^2, \cdots, \theta(x)r_q^q, \theta(x)]$, $r_q = \exp(2\pi i/(q + 1))$. Clearly, the vector-matrix equation (15) is equivalent to the $q + 1$ first order equations

$$z'_p = \theta(x)r_q^p z_p, \qquad p = 1, 2, \cdots, q + 1. \tag{16}$$

By the elementary theory of linear differential equations, the general solution of (16) is given by

$$z_p = c_p \exp\left\{r_q^p \int_a^x \theta(t)\,dt\right\}, \qquad p = 1, 2, \cdots, q, \tag{17}$$

where c_p is an arbitrary constant.

We can now use the transformations which define the vectors z and v to solve for the n components of the vector y. We obtain

$$\begin{aligned} y_f(x) &= c_0 \exp\left\{\int_a^x \theta_0(t)\,dt\right\} \\ &\quad + \sum_{j=1}^{k}\left(\sum_{s=0}^{q(j)} \rho^{(f-1)h_{js}}\left[\sum_{p=1}^{q(j)+1} c_p(j) r_{q(j)}^{s[q(j)+1-p]} \exp\left\{r_{q(j)}^p \int_a^x \theta_j(t)\,dt\right\}\right]\right), \end{aligned} \tag{18}$$

$f = 1, 2, \cdots, n$, where $\theta_j(x)$, $j = 0, 1, \cdots, k$, is given by (8) and c_0, $c_p(j)$, $p = 1, 2, \cdots, q(j) + 1, j = 1, 2, \cdots, k$, are arbitrary constants.

We have established the following theorem.

THEOREM 2.4. *Let $B(x)$ be an $n \times n$ g-circulant matrix, $(g, n) = 1$, of Lebesgue integrable functions on $[a, b]$. If $B(x)$ has property σ on $[a, b]$, then the general solution of* (1) *is given by* (18).

Concerning the cyclicly related system (5), we have the following corollary of Theorem 2.4.

COROLLARY. *Given the system of differential equations* (5), *where* $(g, n) = 1$, *let*

$$b_j(x) = \sum_t a_t(x), \qquad g + \alpha(t) \equiv j + 1 \pmod{n},$$

$$j = 0, 1, \cdots, n - 1.$$

If the $n \times n$ *g-circulant matrix* $B(x)$ *has property* σ *on* $[a, b]$, *then the general solution of* (5) *is given by*

$$y_f(x) = c_0 \exp\left\{\int_a^x \psi_0(t)\,dt\right\}$$
$$+ \sum_{j=1}^{k}\left(\sum_{s=0}^{q(j)} \rho^{(f-1)h_{js}}\left[\sum_{p=1}^{q(j)+1} c_p(j) r_{q(j)}^{s[q(j)+1-p]} \exp\left\{r_{q(j)}^{p}\int_a^x \psi_j(t)\,dt\right\}\right]\right), \tag{19}$$

$f = 1, 2, \cdots, n$, *where* $c_0, c_p(j), p = 1, 2, \cdots, q(j) + 1, j = 1, 2, \cdots, k$, *are arbitrary constants and* $\psi_j(x) = \theta_j(x), j = 0, 1, \cdots, k$, $\theta_j(x)$ *defined by* (8).

In the case $g = 1$, equations (19) reduce to equations (12) of [2]. With $g = 1$, $m = n$ and α the function defined by $\alpha(k) = q + hk$, q and h given integers, equations (19) reduce to equations (6) of [5].

3. The case $(g, n) > 1$. Consider the vector-matrix differential equation (1) where the $n \times n$ g-circulant matrix $B(x)$ has the property $(g, n) = d > 1$. Let e and m be the relatively prime integers such that $dm = n$ and $de = g$. Examining the equations of (1), we have

$$y_i' = \sum_{j=0}^{n-1} b_{n-g(i-1)+j}(x) y_{j+1}, \qquad i = 1, 2, \cdots, n, \tag{20}$$

where the subscripts $n - g(i - 1) + j$ are reduced modulo n. It is easily verified that $(g, n) = d > 1$ implies $y_p' = y_{m+p}' = y_{2m+p}' = \cdots = y_{sm+p}'$, where $1 \leqq p \leqq m$ and $s = d - 1$. Thus, we conclude that there exist arbitrary constants $c_{1p}, c_{2p}, \cdots, c_{sp}, p = 1, 2, \cdots, m$, such that $y_{tm+p} = y_p + c_{tp}, 1 \leqq t \leqq s$, and our homogeneous system (20) may be reduced to the nonhomogeneous system

$$y_p' = \sum_{j=0}^{m-1} b_{n-g(p-1)+j}(x) y_{j+1} + \sum_{j=m}^{2m-1} b_{n-g(p-1)+j}(x)[y_{j+1-m} + c_{1(j+1-m)}]$$
$$+ \cdots + \sum_{j=sm}^{n} b_{n-g(p-1)+j}(x)[y_{j+1-sm} + c_{s(j+1-sm)}], \qquad p = 1, 2, \cdots, m. \tag{21}$$

Let $b_j^*(x) = \sum_{k=0}^{s} b_{j+km}(x)$. Then the nonhomogeneous system (21) can be written

$$y_p' = \sum_{j=0}^{m-1} b_{n-g(p-1)+j}^*(x) y_{j+1} + \sum_{j=m}^{n} c_j b_{n-g(p-1)+j}(x), \tag{22}$$

$p = 1, 2, \cdots, m$, where the arbitrary constants c_{ij} have been renumbered c_m, $c_{m+1}, \cdots, c_n$. Writing (22) in vector-matrix form, we have

$$y' = B^*(x)y + f^*(x), \tag{23}$$

where $B^*(x)$ is an $m \times m$ g'-circulant, $g' \equiv g \pmod{m}$ having entries $b_0^*(x)$, $b_1^*(x), \cdots, b_{m-1}^*(x)$ which are Lebesgue integrable on $[a, b]$, y is an m-component column vector and $f^*(x)$ is the m-component column vector having components

$$f_p(x) = \sum_{j=m}^{n} c_j b_{n-g(p-1)+j}(x), \qquad p = 1, 2, \cdots, m.$$

Now, using the elementary theory of linear differential equations, we find the general solution of (23) by first obtaining the general solution of the corresponding homogeneous equation

$$y' = B^*(x)y. \tag{24}$$

To illustrate the procedure which we shall employ in finding the general solution of (1) in the case $(g, n) = d > 1$, we shall assume first that $(g', m) = 1$. With this assumption, let $[h_0] = 0$, $[h_1], \cdots, [h_k]$ be the equivalence classes generated by $\sim$. If $B^*(x)$ has property σ on $[a, b]$, then the general solution of (24) is given by (18) with $b_j(x)$ replaced by $b_j^*(x)$ and with n replaced by m.

Let x_0, $a < x_0 < b$, be any fixed point on $[a, b]$. Using the general solution vector $y(x)$, with components $y_1(x), \cdots, y_m(x)$, obtained above, we solve the m initial value problems

$$\begin{aligned} y' &= B^*(x)y, \\ y(x_0) &= \delta_i, \qquad i = 1, 2, \cdots, n, \end{aligned} \tag{25}$$

where δ_i is the m-component column vector having a 1 in the ith component and zeros elsewhere. Let $W(x)$ be the $m \times m$ matrix whose columns are the solutions of (25). Then $W(x)$ is nonsingular and the general solution of (23) is given by

$$y(x) = W(x)\left[\gamma + \int_a^x W^{-1}(t) f^*(t)\, dt\right], \tag{26}$$

where γ is an m-component column vector whose components are arbitrary constants. We summarize this discussion with the following theorem.

THEOREM 3.1. *Let* $(g, n) = d > 1$ *and assume that* $(g, m) = 1$, *where* $m = n/d$. *Let* $B^*(x)$ *be the* $m \times m$ g'*-circulant*, $g' \equiv g \pmod m$, *with entries* $b_j^*(x) = \sum_{k=0}^{d-1} b_{j+km}(x)$. *If* $B^*(x)$ *has property* σ *on* $[a, b]$, *then the general solution of* (1) *is given by* (26).

The special case $g = 0$ occurs as an interesting corollary of Theorem 3.1. Letting $g = 0$ we have, from (1), $y_1' = y_2' = \cdots = y_n'$. Thus there exist arbitrary constants $c_2, c_3, \cdots, c_n$ such that $y_i = y_1 + c_i$, $i = 2, 3, \cdots, n$. The vector-matrix equation (1) can now be written equivalently as the nonhomogeneous first order equation

$$y_1' = \left[\sum_{j=0}^{n-1} b_j(x)\right] y_1 + \sum_{j=1}^{n-1} c_{j+1} b_j(x). \tag{27}$$

The general solution of (27) is easily seen to be

$$\begin{aligned} y_1 &= \exp\left\{\int_a^x \beta(t)\, dt\right\}\left(c_1 + \int_a^x \left[\exp\left\{-\int_a^t \beta(s)\, ds\right\} \sum_{j=1}^{n-1} c_{j+1} b_j(t)\right] dt\right), \\ y_i &= y_1 + c_i, \qquad i = 2, 3, \cdots, n, \end{aligned} \tag{28}$$

where $\beta(x) = \sum_{j=0}^{n-1} b_j(x)$.

COROLLARY. *Let the* $n \times n$ *matrix* $B(x)$ *of* (1) *be a* 0*-circulant. Then* (1) *can be written equivalently as the nonhomogeneous system* (27) *and the general solution is given by* (28).

We now consider the general situation in the case $(g, n) = d > 1$. Let $m_1 = n/d$ and let $d_1 = (g, m_1) = (g_1, m_1)$, where $g_1 \equiv g \pmod{m_1}$. Let $m_2 = m_1/d_1$ and let $d_2 = (g_1, m_2) = (g_2, m_2), g_2 \equiv g_1 \pmod{m_2}$, and so on. Clearly the sequence $d > d_1 > d_2 > \cdots$ is finite and ends with either $d_p = m_p$ or $d_p = 1$.

Consider equation (1). As indicated in the discussion preceding the statement of Theorem 3.1, (1) can be replaced by the nonhomogeneous system (23), which we now write as

$$y' = B_1(x)y + f_1(x), \tag{29}$$

where $B_1(x)$ is an $m_1 \times m_1$ g_1-circulant matrix, $(g_1, m_1) = d_1 > 1$, and $f_1(x)$ is an m_1-component column vector. We consider the associated homogeneous equation

$$y' = B_1(x)y \tag{30}$$

and in the same manner replace it by the nonhomogeneous equation

$$y' = B_2(x)y + f_2(x), \tag{31}$$

where $B_2(x)$ is an $m_2 \times m_2$ g_2-circulant, $(g_2, m_2) = d_2 > 1$ and $f_2(x)$ is an m_2-component column vector. Continuing we obtain, after p steps, the nonhomogeneous equation

$$y' = B_p(x)y + f_p(x), \tag{32}$$

where $B_p(x)$ is an $m_p \times m_p$ g_p-circulant matrix, $(g_p, m_p) = d_p$, $d_p = 0$ or $d_p = 1$, and $f_p(x)$ is an m_p-component column vector. Now consider the associated homogeneous equation

$$y' = B_p(x)y. \tag{33}$$

If $B_p(x)$ is a 0-circulant, we find the general solution of (33) using the corollary to Theorem 3.1. If $B_p(x)$ is a g-circulant, $(g, m_p) = 1$ and $B_p(x)$ has property σ on $[a, b]$, then we find the general solution of (33) using the results of § 2. Once we have the general solution of (33), we obtain the general solution of (32) as indicated by the discussion preceding Theorem 3.1. Continuing through p applications of this procedure yields the general solution of (1).

4. Extensions. The results and techniques of the preceding two sections depend entirely on the structural properties of g-circulant matrices and not on the entries of the matrix, except in so far as integrations are required. Consequently we have the following immediate generalization of (1). Let $\Gamma(x)$ be an $n \times n$ composite g-circulant matrix whose entries $D_0(x), D_1(x), \cdots, D_{n-1}(x)$ are $m \times m$ matrices of Lebesgue integrable functions on $[a, b]$. Let Y be an n-component composite column vector with components the $m \times m$ matrices $Y_1(x), Y_2(x), \cdots, Y_n(x)$, and consider the "vector-matrix" differential equation

$$Y' = \Gamma(x)Y. \tag{34}$$

DEFINITION 3. Let $Q(x)$ be an $m \times m$ matrix of Lebesgue integrable functions on $[a, b]$. The $m \times m$ matrix $E(\int_a^x Q)$ is defined to be the unique solution of

$$Z' = Q(x)Z, \qquad Z(a) = I. \tag{35}$$

It is well known that $E(\int_a^x Q)$ is the matrix analogue of the exponential function. We now have the following generalization of Theorem 2.4.

THEOREM 4.1. *Let $\Gamma(x)$ be an $n \times n$ composite g-circulant, $(g, n) = 1$, whose entries $D_0(x), D_1(x), \cdots, D_{n-1}(x)$ are $m \times m$ Lebesgue integrable matrices on $[a, b]$. If $\Gamma(x)$ has property σ on $[a, b]$, then the general solution of* (34) *is given by*

$$Y_f(x) = C_0 \cdot E\left(\int_a^x \theta_0\right) + \sum_{j=1}^{k} \left\{ \sum_{s=0}^{q(j)} \rho^{(f-1)h_{js}} \left[\sum_{p=1}^{q(j)+1} C_p(j) r_{q(j)}^{s[q(j)+1-p]} E\left(r_{q(j)}^{p} \int_a^x \theta_j \right) \right] \right\}, \tag{36}$$

$f = 1, 2, \cdots, n$, where

$$\theta_j(x) = D_0(x) + \sum_{s=1}^{k} D_{h_s}(x)[r_n^{h_j h_{s0}} + \cdots + r_n^{h_j h_{sq(s)}}], \qquad j = 0, 1, \cdots, k,$$

and $C_0, C_p(j)$ are $m \times m$ matrices of arbitrary constants.

If $(g, n) = d > 1$, then we can use the procedure outlined in § 3 to obtain the general solution of (34).

Concerning the cyclicly related system (5), we note that there is no restriction on the positive integer m. Consequently, we can also obtain the general solution of

$$y_i' = \sum_{j=1}^{\infty} a_j(x) y_{g \cdot i + \alpha(j)}, \qquad i = 1, 2, \cdots, m, \tag{37}$$

where $\sum_{j=1}^{\infty} a_j(x)$ converges absolutely and uniformly on $[a, b]$ and α is any function whose domain is the set of positive integers and whose range is a subset of $\{1, 2, \cdots, n\}$.

The matrix analogue of (5) and (37) is

$$Y_i' = \sum_{j=1}^{k} A_j(x) Y_{g \cdot i + \alpha(j)}, \qquad 1 \leqq k \leqq \infty, \qquad i = 1, 2, \cdots, n, \tag{38}$$

where $A_j(x), j = 1, 2, \cdots, k$, are $m \times m$ Lebesgue integrable matrices on $[a, b]$ such that the matrix series $\sum_{j=1}^{k} A_j(x)$ converges absolutely and uniformly on $[a, b]$.

Finally, we note that we can apply these techniques to cyclicly related functional equations of the form

$$L(y_i) = \sum_{j=1}^{m} a_j(x) y_{g \cdot i + \alpha(j)}, \qquad 1 \leqq m \leqq \infty, \qquad i = 1, 2, \cdots, n, \tag{39}$$

where L is any linear operator. As indicated by our previous work, system (39) can be converted into the vector-matrix equation

$$L(y) = B(x)y, \tag{40}$$

$B(x)$ an $n \times n$ g-circulant. For a wide class of operators, it will be possible to solve (40) using the techniques presented in § 2 and § 3.

REFERENCES

[1] C. M. ABLOW AND J. L. BRENNER, *Roots and canonical forms for circulant matrices*, Trans. Amer. Math. Soc., 107 (1963), pp. 360–376.

[2] P. BARNHARD AND E. J. PELLICCIARO, *Cyclicly related differential equations and the circulant*, J. Elisha Mitchell Sci. Soc., 77 (1961), pp. 163–167.

[3] W. E. BAXTER AND E. J. PELLICCIARO, *Cyclicly related differential equations*, Duke Math. J., 31 (1964), pp. 229–234.

[4] G. BIRKHOFF AND L. KOTIN, *Third order positive cyclic systems of linear differential systems*, J. Differential Equations, 5 (1969), pp. 182–196.

[5] W. M. WHYBURN, *A set of cyclicly related functional equations*, Bull. Amer. Math. Soc., 36 (1930), pp. 863–868.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

THE EIGENVALUES OF THE BETHE DIFFERENTIAL SYSTEM*

J. ERNEST WILKINS, JR.†

Abstract. The eigenvalues of the Bethe differential system, $d\{(1 - x^2)\,d\psi/dx\}/dx + kx\psi = 0$ $(-1 < x < 1)$, $\psi(\pm 1)$ finite, are approximately determined. The first nine positive eigenvalues are given to 6*S*, and the first three terms of an asymptotic expansion for the large eigenvalues are derived.

1. Introduction. The Bethe differential system consists of the equations

$$\frac{d}{dx}\left\{(1 - x^2)\frac{d\psi}{dx}\right\} + kx\psi = 0, \qquad -1 < x < 1, \tag{1}$$
$$\psi(\pm 1) \quad \text{finite},$$

and we shall be interested in the values k for which there exist nontrivial solutions $\psi(x)$ of this system. This differential system was first studied by Bethe, Rose and Smith [1] and some of its basic properties were proved by Scalettar [2], both of whom were interested in electron scattering and transmission.

We shall show that the eigenvalues are 0 and two infinite sequences k_n, $k_{-n} = -k_n$ such that $0 < k_1 < k_2 < \cdots < k_n \to +\infty$. Each eigenvalue is simple, i.e., there is exactly one linearly independent eigenfunction $\psi_n(x)$ corresponding to the eigenvalue k_n. The eigenvalue $\psi_{-n}(x)$ corresponding to $-k_n$ may be defined as $\psi_n(-x)$.

The smallest positive eigenvalue k_1 is 14.5280, and the first nine positive eigenvalues are furnished in Table 1. We also derive the asymptotic formula

$$k_n = 6.875186(n + \tfrac{1}{2})^2 - 0.91185 - 0.05675(n + \tfrac{1}{2})^{-2} + \cdots,$$

valid for large n, which gives excellent agreement even when $n = 1$. The first term of this expansion had been found by Bethe et al. [1], who also estimated k_1 as 14.476.

2. An equivalent integral equation. If k is an eigenvalue of the system (1), so that $x\psi(x) \in L(-1, 1)$, it follows that the quantities

$$a = \lim_{x \to -1} (1 - x^2)\frac{d\psi}{dx}, \qquad b = \lim_{x \to +1} (1 - x^2)\frac{d\psi}{dx}$$

exist and are such that

$$a - b = k\int_{-1}^{1} t\psi(t)\,dt,$$

$$(1 - x^2)\frac{d\psi}{dx} = a - k\int_{-1}^{x} t\psi(t)\,dt. \tag{2}$$

If we divide (2) by $1 - x$, or by $1 + x$, integrate over $(-1, x)$, or over $(x, 1)$, and

* Received by the editors August 31, 1971.

† Department of Physics and Astronomy, Howard University, Washington, D.C. 20001.

then let x approach $+1$, or -1, we conclude from the finiteness of $\psi(\pm 1)$ that $a = b = 0$, and hence that

$$\int_{-1}^{1} x\psi(x)\,dx = 0 \tag{3}$$

if $k \neq 0$. Equation (3) is also true if $k = 0$ since the only finite solution $\psi(x)$ of (1) when $k = 0$ is constant.

Since $a = 0$, the result of integrating (2) over $(-1, 1)$ is that

$$2\int_{-1}^{1} x\psi(x)\,dx = -k\int_{-1}^{1} dx \int_{-1}^{x} t\psi(t)\,dt = -k\int_{-1}^{1} (1-t)t\psi(t)\,dt.$$

In view of (3), we see that, when $k \neq 0$,

$$\int_{-1}^{1} x^2\psi(x)\,dx = 0. \tag{4}$$

If we divide (2) by $1 - x^2$, and integrate over $(-1, x)$, we find that

$$\psi(x) - \psi(-1) = \frac{k}{2}\int_{-1}^{x} t \ln\left\{\frac{(1-x)(1+t)}{(1+x)(1-t)}\right\}\psi(t)\,dt. \tag{5}$$

If we multiply (5) by x^2, integrate over $(-1, 1)$, we infer from (4) that

$$\psi(-1) = \frac{k}{2}\int_{-1}^{1} t\psi(t)\left\{\frac{1-t^2}{2} + \ln\frac{2}{1+t}\right\} dt.$$

Making use of (3) and (4), it now follows from (5) that any solution $\psi(x)$ of the Bethe differential system (1) for which $k \neq 0$ must satisfy the Fredholm integral equation of the second kind,

$$\psi(x) = k\int_{-1}^{1} H(x, t)t\psi(t)\,dt, \tag{6}$$

in which

$$H(x,t) = \begin{cases} \dfrac{1}{2}\ln\left\{\dfrac{1}{(1-t)(1+x)}\right\} - \dfrac{x^2 + 3xt + t^2}{4} + \dfrac{11}{15}, & -1 \leqq t \leqq x \leqq 1, \\[2ex] \dfrac{1}{2}\ln\left\{\dfrac{1}{(1-x)(1+t)}\right\} - \dfrac{x^2 + 3xt + t^2}{4} + \dfrac{11}{15}, & -1 \leqq x \leqq t \leqq 1. \end{cases} \tag{7}$$

Conversely, suppose that $\psi(x)$ is any function which satisfies the integral equation (6) almost everywhere, and that $k \neq 0$. It is easy to verify that $0 < 11/240 = H(1, 1/2) \leqq H(x, t)$, and it follows from the (Lebesgue) integrability of $H(x, t)t\psi(t)$ for almost all (and hence for some) x that $t\psi(t)$ is integrable over $(-1, 1)$. From this and the analytic form for $H(x, t)$ we infer that, for almost all x,

$$2\psi(x) = k\{-A(x)\ln(1+x) + B(x) - C(x)\ln(1-x)\},$$

in which

$$A(x) = \int_{-1}^{x} t\psi(t)\,dt, \qquad C(x) = \int_{x}^{1} t\psi(t)\,dt,$$

$$B(x) = -\int_{-1}^{x} \ln(1-t)t\psi(t)\,dt - \int_{x}^{1} \ln(1+t)t\psi(t)\,dt$$
$$+ \int_{-1}^{1} \left(\frac{22}{15} - \frac{x^2+3xt+t^2}{2}\right) t\psi(t)\,dt.$$

The functions $A(x)\ln(1+x)$, $B(x)$, and $C(x)\ln(1-x)$ are absolutely continuous on the respective intervals $(-1,1]$, $(-1,1)$, $[-1,1)$. Hence $\psi(x)$ may be redefined on a set of measure zero so that it is absolutely continuous on $(-1,1)$, and $\psi(x) = O(1) + O[\ln(1-|x|)]$. Therefore, $x^2\psi^2(x)$ is integrable, and from Schwarz' inequality we now see that

$$\psi^2(x) \leqq k^2 h(x) \int_{-1}^{1} t^2\psi^2(t)\,dt, \qquad h(x) = \int_{-1}^{1} H^2(x,t)\,dt.$$

A straightforward calculation shows that $h(x)$ is continuous on $[-1,1]$, and hence $\psi(x)$ is bounded. This insures that $A(x)\ln(1+x)$, $B(x)$, and $C(x)\ln(1-x)$, and therefore $\psi(x)$ also, are absolutely continuous on $[-1,1]$ if the first and third of these functions are defined to be zero when $x=-1$ or $x=+1$, respectively.

Differentiation of (6) now shows that

$$\frac{d\psi}{dx} = -\frac{k}{2}\int_{-1}^{x}\left(\frac{1+x+x^2}{1+x}\right)t\psi(t)\,dt + \frac{k}{2}\int_{x}^{1}\left(\frac{1-x+x^2}{1-x}\right)t\psi(t)\,dt$$
$$-\frac{3k}{4}\int_{-1}^{1} t^2\psi(t)\,dt$$

on $[-1,1]$, and hence

$$\frac{d}{dx}\left\{(1-x^2)\frac{d\psi}{dx}\right\} = -kx\psi(x) + \frac{3k}{2}\int_{-1}^{1} xt(x+t)\psi(t)\,dt.$$

Moreover, from (6) and an easily justified inversion of the order of integration,

$$\int_{-1}^{1} x\psi(x)\,dx = k\int_{-1}^{1} t\psi(t)\,dt \int_{-1}^{1} xH(x,t)\,dx = 0,$$

since the integral of $xH(x,t)$ can be calculated in an elementary fashion to be zero.

In addition,

$$\int_{-1}^{1} x^2\psi(x)\,dx = k\int_{-1}^{1} t\psi(t)\,dt \int_{-1}^{1} x^2H(x,t)\,dx = \tfrac{4}{3}(1-\ln 2)k\int_{-1}^{1} t\psi(t)\,dt = 0.$$

We conclude that any function $\psi(x)$ which satisfies the integral equation (6) almost everywhere can be redefined on a set of measure zero so that $\psi(x)$ is a solution of the Bethe differential system (1).

3. Properties of the equivalent integral equation. We have already observed that the integral of $H^2(x, t)$ with respect to t is a continuous function of x. Hence,

$$\int_{-1}^{1}\int_{-1}^{1} H^2(x, t)\, dx\, dt < +\infty. \tag{8}$$

Moreover, it is obvious that

$$H(x, t) = H(t, x). \tag{9}$$

If it were known that the functional

$$J[g] \equiv \int_{-1}^{1}\int_{-1}^{1} g^*(x)xH(x, t)tg(t)\, dx\, dt$$

is nonnegative for an arbitrary complex-valued function $g(x)$ in $L_2(-1, 1)$, then the integral equation (6) would belong to the class of J-definite integral equations of Zimmerberg [3], and certain interesting conclusions could be drawn from his general analysis. (In fact, Zimmerberg requires that $H(x, t)$ be a bounded function with regularly distributed discontinuities. Zaanen [4, pp. 418–422 and p. 565] has shown how to replace this hypothesis with (8).) We shall now establish this property of the kernel $H(x, t)$.

Consider the Legendre differential system

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP}{dx}\right\} + kP = 0, \qquad P(\pm 1) \text{ finite}.$$

Manipulations like those in § 2 show that this differential system is equivalent, when $k \neq 0$, to the Fredholm integral equation

$$P(x) = k\int_{-1}^{1} H_0(x, t)P(t)\, dt, \tag{10}$$

in which

$$H_0(x, t) = \begin{cases} \frac{1}{2}\ln\frac{1}{(1 - t)(1 + x)} + \ln 2 - \frac{1}{2} & \text{if } -1 \leqq t \leqq x \leqq 1, \\ \frac{1}{2}\ln\frac{1}{(1 - x)(1 + t)} + \ln 2 - \frac{1}{2} & \text{if } -1 \leqq x \leqq t \leqq 1. \end{cases}$$

Equation (10) is a Fredholm integral equation of the second kind with a kernel $H_0(x, t)$ which satisfies the relations (8) and (9). The eigenvalues are known to be $k_l = l(l + 1)$, $l = 1, 2, 3, \cdots$, and the corresponding normalized eigenvalues are $(l + 1/2)^{1/2}P_l(x)$. The result that

$$\int_{-1}^{1}\int_{-1}^{1} h^*(x)H_0(x, t)h(t)\, dx\, dt = \sum_{l=1}^{\infty} \frac{2l + 1}{2l(l + 1)}\left|\int_{-1}^{1} P_l(x)h(x)\, dx\right|^2$$

for every $h(x)$ in $L_2(-1, 1)$ is a well-known [5, p. 118] property of kernels $H_0(x, t)$ satisfying (8) and (9). Since

$$H(x, t) = H_0(x, t) - \tfrac{3}{4}P_1(x)P_1(t) - \tfrac{5}{12}P_2(x)P_2(t) + \tfrac{15}{16}(x^2 - \tfrac{3}{5})(t^2 - \tfrac{3}{5}) + 1 - \ln 2,$$

it follows that, for every $g(x)$ in $L_2(-1,1)$,

$$J[g] = \sum_{l=3}^{\infty} \frac{2l+1}{2l(l+1)}\left|\int_{-1}^{1} P_l(x)xg(x)\,dx\right|^2 + \frac{15}{16}\left|\int_{-1}^{1}\left(x^2-\frac{3}{5}\right)xg(x)\,dx\right|^2 + (1-\ln 2)\left|\int_{-1}^{1} xg(x)\,dx\right|^2, \tag{11}$$

and so $J[g] > 0$ unless $xg(x)$ is orthogonal to 1, $x^2 - \frac{3}{5}$, and $P_l(x)$ $(l \geqq 3)$. Since $P_0(x) = 1$ and $P_2(x) = \frac{3}{2}(x^2 - \frac{3}{5}) + \frac{2}{5}$, it follows that $J[g] > 0$ unless $xg(x)$ is orthogonal to $P_l(x)$ when $l \neq 1$, i.e., unless $g(x)$ is constant, and then $J[g] = 0$.

It now follows immediately from the results of Zimmerberg [3] that all eigenvalues k are real, and that the index (i.e., number of linearly independent eigenfunctions) of each eigenvalue is equal to its multiplicity (as a zero of the Fredholm determinant of the integral equation). In fact, each eigenvalue is simple, for it is easy to see that there cannot be two linearly independent eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ both satisfying the differential system (1) for the same value of k.

It is obvious from inspection of the differential equation (1), or from the observation that $H(-x,-t) = H(x,t)$, that $\psi(x)$ is an eigenfunction corresponding to an eigenvalue k if and only if $\psi(-x)$ is an eigenfunction corresponding to an eigenvalue $-k$.

It follows from another result of Zimmerberg [3] that there are at least N positive eigenvalues if

$$\int_{-1}^{1} xf^2(x)\,dx > 0$$

for all nonidentically vanishing $f(x)$ in an N-dimensional linear subspace of the set L of functions $f(x)$ for which there exists a continuous function $g(x)$ such that

$$f(x) = \int_{-1}^{1} H(x,t)tg(t)\,dt. \tag{12}$$

Let the functions $f_n(x)$ be defined when $n = 2, 3, \cdots, N+1$ so that

$$f_n(x) = (3n+2)(3n+3)x^{3n} - 2(3n+3)(3n+4)x^{3n+1} + (3n+4)(3n+5)x^{3n+2}$$

when $x \geqq 0$, and so that $2f_n(x) = f_n(-x)$ if $x < 0$. Since it is easy to see that

$$\int_{-1}^{1} xf_n(x)\,dx = \int_{-1}^{1} x^2 f_n(x)\,dx = 0,$$

it follows from the identity

$$-\int_{-1}^{1} H(x,t)\frac{d}{dt}\left\{(1-t^2)\frac{df}{dt}\right\}dt = f(x) - \frac{3}{2}\int_{-1}^{1}(x+t)tf(t)\,dt,$$

that $f_n(x)$ is in L with a function

$$g(x) = g_n(x) = -\frac{1}{x}\frac{d}{dx}\left\{(1-x^2)\frac{df_n}{dx}\right\},$$

which is continuous on $(-1, 1)$. If $f(x) = \sum a_n f_n$ is any element of the N-dimensional linear space spanned by the linearly independent functions $f_n(x), n = 2, 3, \cdots,$ $N + 1$, then

$$\int_{-1}^{1} xf^2(x)\,dx = \frac{1}{2}\int_0^1 xf^2(x)\,dx > 0$$

unless $f(x) = 0$ on $(0, 1)$, and hence unless $f(x) = 0$ on $(-1, 1)$. Since N is an arbitrary positive integer, we conclude that there exist a denumerable infinity of positive eigenvalues $k_1 < k_2 < k_3 < \cdots$, and a corresponding sequence of real eigenfunctions $\psi_n(x)$.

The totality of eigenfunctions may be orthonormalized so that

$$\int_{-1}^{1} x\psi_p(x)\psi_q(x)\,dx = \delta_{pq}\operatorname{sgn} k_p, \qquad -\infty < p, q < +\infty.$$

In terms of these eigenfunctions another result of Zimmerberg [3] is that

$$J[f] = \sum_{p=-\infty}^{\infty}{}' \frac{1}{|k_p|}\left|\int_{-1}^{1} x\psi_p(x)f(x)\,dx\right|^2, \tag{13}$$

$$\int_{-1}^{1} x|f(x)|^2\,dx = \sum_{p=-\infty}^{\infty}{}' (\operatorname{sgn} k_p)\left|\int_{-1}^{1} x\psi_p(x)f(x)\,dx\right|^2,$$

wherever $f(x)$ is in the set L defined above, or even if $f(x)$ is in the larger set L_0^2 of functions $f(x)$ representable in the form (12) with a function $g(x)$ in $L_2(-1, 1)$. (The prime on the summation sign indicates that the value $p = 0$ is to be omitted.) The identity (13) does not hold for all $f(x)$ in $L_2(-1, 1)$, since the right-hand side vanishes when $f(x) = x$, by virtue of (4), while the left-hand side is $4(1 - \ln 2)/9$, according to (11).

4. An equivalent Jacobi matrix. While we have deduced the existence of the sequence of eigenvalues of the Bethe differential system (1) from properties of the equivalent integral equation, it is more convenient to determine the numerical values of the eigenvalues in a somewhat different manner. An eigenfunction $\psi(x)$ of (1) can be expanded into a series of Legendre polynomials,

$$\psi(x) = \beta_0 + \sum_{l=1}^{\infty}\left\{\frac{2l+1}{2l(l+1)}\right\}^{1/2}\beta_l P_l(x). \tag{14}$$

In view of the orthogonality and the recurrence relations satisfied by the Legendre polynomials, we see that the series (14) satisfies the differential equation (1) if and only if the coefficients β_l satisfy the conditions that

$$\begin{gathered}\beta_0 = -(\tfrac{2}{15})^{1/2}\beta_2, \quad \beta_1 = 0, \quad k^{-1}\beta_l = B_{l-1}\beta_{l-1} + B_l\beta_{l+1}, \quad l \geqq 2, \\ B_l = \{l(l+2)(2l+1)(2l+3)\}^{-1/2}.\end{gathered} \tag{15}$$

Therefore, the nonzero eigenvalues of the Bethe differential system are the reciprocals of the characteristic values of the (Jacobi) matrix

$$B = (B_{lm}) = (B_l\delta_{l+1,m} + B_{l-1}\delta_{l-1,m})_{l,m=2,3,\cdots}.$$

It is a consequence of known results on Jacobi matrices [6, p. 553] that the characteristic values of B may be obtained as limits of the characteristic values of the finite segments of B. B. Forutanpour has used a general purpose computing machine program for finding characteristic values for symmetric matrices and has evaluated the characteristic values of the first few even-ordered segments of B. In this manner he finds that the first nine positive eigenvalues of the Bethe differential system are those given in Table 1. It is possible to get bounds on the value

TABLE 1

Values of the first nine positive eigenvalues of the Bethe differential system (1)

n	k_n
1	14.5280
2	42.0486
3	83.3044
4	138.308
5	207.060
6	289.563
7	385.816
8	495.820
9	619.573

of the smallest eigenvalue k_1 by making use of the general result that the sum of the eigenvalues of a matrix is equal to the trace of the matrix. Since $-1/k$ is a characteristic value of B if $1/k$ is, with a characteristic vector $(-1)^l\beta_l$, it follows that

$$\sum_{n=1}^{\infty} k_n^{-2p} = \tfrac{1}{2}\operatorname{tr}(B^{2p}) \equiv U_{2p}, \qquad p = 1, 2, \cdots,$$

and consequently that

$$k_1 \geqq U_{2p}^{-1/(2p)}, \tag{16}$$

$$k_1 \leqq (U_{2p}/U_{2p+2})^{1/2}. \tag{17}$$

It is easy to see with the help of the partial fraction expansion of B_l^2 that

$$U_2 = \sum_{l=2}^{\infty} B_l^2 = \sum_{l=2}^{\infty}\left\{\frac{1}{6}\left(\frac{1}{l} - \frac{1}{l+2}\right) - \frac{2}{3}\left(\frac{1}{2l+1} - \frac{1}{2l+3}\right)\right\}.$$

The telescoping series can be summed immediately, showing that $U_2 = 1/180$. In a similar manner,

$$U_4 = \sum_{l=2}^{\infty}(B_l^4 + 2B_l^2B_{l+1}^2) = \frac{\pi^2}{108} - \frac{6907}{75600} = 2.279207 \times 10^{-5},$$

$$U_6 = \sum_{l=2}^{\infty}\{B_l^6 + 3B_l^2B_{l+1}^2(B_l^2 + B_{l+1}^2 + B_{l+2}^2)\}$$

$$= \frac{5441}{3572100} - \frac{\pi^2}{6480} = 1.065407 \times 10^{-7}.$$

The inequality (16) becomes $k_1 \geqq 14.523$ when $p = 3$, and the inequality (17) becomes $k_1 \leqq 14.626$ when $p = 2$.

5. Asymptotic properties. The differential equation (1) has singularities at ± 1 and a turning point at 0. It is possible to make use of the results of Langer [7] in order to construct an expansion, asymptotic for large k, of the general solution of the differential equation on the open interval $(-1, 1)$. Similarly, the results of Dorodnicyn [8] enable the construction of an asymptotic expansion of that solution of the differential equation which is finite at -1, or $+1$, which is valid on the half-open interval $[-1, 0)$, or $(0, +1]$. By comparing these asymptotic expansions we can deduce an asymptotic expansion for the large eigenvalues of the differential system (1). The result we obtain is that

$$(18)\quad k_n = \frac{\{\Gamma(1/4)\}^4}{8\pi}(n + 1/2)^2\left[1 - \frac{5/(12\pi)}{(n + 1/2)^2} - \frac{\frac{25}{576} + [35\pi^4/3\{\Gamma(1/4)\}^8]}{(n + 1/2)^4\pi^2} + \cdots\right].$$

We first observe that, if $u = (1 - x^2)^{1/2}\psi(x)$, $\lambda^2 = k$, $r_0(x) = (1 - x^2)^{-2}$, $\varphi(x) = x^{1/2}(1 - x^2)^{-1/2}$ when $x \geqq 0$, $\varphi(x) = e^{\pi i/2}\varphi(-x)$ when $x \leqq 0$, then

$$(19)\qquad \frac{d^2u}{dx^2} + \{\lambda^2\varphi^2(x) + r_0(x)\}u = 0.$$

In view of the results established in § 3, we may assume that $k > 0$, and hence that $\lambda > 0$.

The formal manipulations of [7] used to get an asymptotic expansion of the solution $u(x)$ of (19) are the following. Suppose that $v(x) = \psi(x)V(\xi)$, in which $\xi = \lambda\Phi(x)$, $\psi(x) = \{\varphi(x)\}^{-1/2}\{\Phi(x)\}^{1/6} = \psi(-x)$,

$$\Phi(x) = \begin{cases} \displaystyle\int_0^x \varphi(t)\,dt & \text{when } x \geqq 0, \\ e^{3\pi i/2}\Phi(-x) & \text{when } x < 0, \end{cases}$$

and $V(\xi)$ is any solution of the differential equation

$$(20)\qquad \frac{d^2V}{d\xi^2} + \frac{1}{3\xi}\frac{dV}{d\xi} + V = 0.$$

Now suppose that

$$(21)\qquad y(x) = \left(1 + \frac{\alpha_2}{\lambda^2}\right)v + \left(\beta_0 + \frac{\beta_2}{\lambda^2}\right)\frac{v'}{\lambda^2}, \qquad z = D_1^{-1/2}y,$$

in which

$$\beta_0(x) = \frac{1}{\varphi(x)}\int_0^x \frac{r_0(t) - \theta(t)}{2\varphi(t)}\,dt, \qquad \alpha_2(x) = -\frac{\beta_0'(x) + \beta_0^2(x)\varphi^2(x)}{2},$$

$$\beta_2(x) = \frac{1}{\varphi(x)}\int_0^x \frac{\alpha_2''(t) + \alpha_2(t)\{r_0(t) - \theta(t)\} - 2\beta_0'(t)\theta(t) - \beta_0(t)\theta'(t)}{2\varphi(t)}\,dt,$$

$$\theta(x) = -\frac{\psi''(x)}{\psi(x)} = \frac{\varphi''(x)}{2\varphi(x)} - \frac{3}{4}\left\{\frac{\varphi'(x)}{\varphi(x)}\right\}^2 + \frac{5}{36}\left\{\frac{\varphi(x)}{\Phi(x)}\right\}^2 = \theta(-x),$$

$$D_1 = 1 + \frac{\alpha_2^2 + \alpha_2\beta_0' + \beta_2' - \alpha_2'\beta_0 + 2\beta_0\beta_2\varphi^2 + \beta_0^2\theta}{\lambda^4}$$

$$+ \frac{\alpha_2\beta_2' - \alpha_2'\beta_2 + \beta_2^2\varphi^2 + 2\beta_0\beta_2\theta}{\lambda^6} + \frac{\beta_2^2\theta}{\lambda^8}.$$

It is easy to see that

$$\theta^*(x) \equiv \frac{\varphi''}{2\varphi} - \frac{3}{4}\left(\frac{\varphi'}{\varphi}\right)^2 = -\frac{5}{16x^2} + \frac{1}{4} + x^2 + x^4E(x),$$

$$\Phi(x) = \frac{2x^{3/2}}{3}\left[1 + \frac{3x^2}{14} + \frac{9x^4}{88} + x^6E(x)\right],$$

(22)
$$\theta(x) = \frac{3}{7} + \frac{624x^2}{539} + x^4E(x),$$

$$\frac{r_0(t) - \theta(t)}{2\varphi(t)} = \frac{2t^{-1/2}}{7} + \frac{150t^{3/2}}{539} + t^{7/2}E(t),$$

in which $E(x)$ stands for a convergent power series in x^2 (not necessarily the same function at each occurrence). The integral defining $\beta_0(x)\varphi(x)$ therefore exists, and

$$\beta_0(x)\varphi(x) = \frac{4x^{1/2}}{7}\left[1 + \frac{15x^2}{7} + x^4E(x)\right].$$

Hence $\beta_0(x)$ is even and $\alpha_2(x)$ is odd. Since $\theta(x)$ is even, the numerator of the integrand in the definition of $\beta_2(x)$ is odd, the integrand itself is $t^{1/2}E(t)$, and hence $\beta_2(x)\varphi(x) = x^{3/2}E(x)$ exists and is such that $\beta_2(x)$ is odd.

It follows that

$$\frac{d^2z}{dx^2} + \{\lambda^2\varphi^2(x) + r_0(x) + O(\lambda^{-4})\}z = 0,$$

and hence z and z' are, for a suitable choice of a particular solution $V(\xi)$ of the differential equation (20), asymptotically equal to u and u', with an error which is $O(\lambda^{-4})$. (This last assertion is a rather informal translation of a more carefully worded assertion rigorously demonstrated in [7].) Since $D_1 = 1 + O(\lambda^{-4})$ and $D_1' = O(\lambda^{-4})$, y and y' are also asymptotically equal to u and u' with an error of $O(\lambda^{-4})$ on any closed subinterval of the open interval $(-1, 1)$.

The general solution of the differential equation (20) is

$$V(\xi) = (\xi/\lambda)^{1/3}[a_1J_{1/3}(\xi) + a_2J_{-1/3}(\xi)],$$

in which a_1 and a_2 are arbitrary constants and J_α is the Bessel function of the first kind and order α. Hence

(23)
$$u(x) = \{\Phi(x)/\varphi(x)\}^{1/2}[a_1A_1 + a_2A_2],$$

$$u'(x) = \lambda\{\Phi(x)\varphi(x)\}^{1/2}[a_1B_1 - a_2B_2],$$

in which

$$A_1 = \{1 + \mu_1\lambda^{-2} + O(\lambda^{-4})\}J_{1/3}(\lambda\Phi) + \lambda^{-1}\varphi\{\beta_0 + \beta_2\lambda^{-2} + O(\lambda^{-4})\}J_{-2/3}(\lambda\Phi),$$
$$A_2 = \{1 + \mu_1\lambda^{-2} + O(\lambda^{-4})\}J_{-1/3}(\lambda\Phi) - \lambda^{-1}\varphi\{\beta_0 + \beta_2\lambda^{-2} + O(\lambda^{-4})\}J_{2/3}(\lambda\Phi),$$
$$B_1 = \{1 + \mu_2\lambda^{-2} + O(\lambda^{-4})\}J_{-2/3}(\lambda\Phi) + \lambda^{-1}\{\mu_3 + \mu_4\lambda^{-2} + O(\lambda^{-4})\}J_{1/3}(\lambda\Phi),$$
$$B_2 = \{1 + \mu_2\lambda^{-2} + O(\lambda^{-4})\}J_{2/3}(\lambda\Phi) - \lambda^{-1}\{\mu_3 + \mu_4\lambda^{-2} + O(\lambda^{-4})\}J_{-1/3}(\lambda\Phi),$$
$$\mu_1 = \frac{\beta_0\varphi}{6\Phi} - \frac{\beta_0\varphi'}{2\varphi} - \frac{\beta_0' + \beta_0^2\varphi^2}{2}, \quad \mu_2 = \frac{\beta_0' - \beta_0^2\varphi^2}{2}, \quad \mu_3 = \frac{1}{6\Phi} - \frac{\varphi'}{2\varphi^2} - \beta_0\varphi,$$
$$\mu_4 = \frac{(\beta_0' - \beta_0^2\varphi^2)(\varphi/(6\Phi) - \varphi'/(2\varphi)) - \beta_0'' - \beta_0(r_0 + \theta) - 2\beta_2\varphi^2}{2\varphi}.$$

In view of the relations $J_{\pm 1/3}(e^{3\pi i/2}\lambda\Phi) = e^{\pm\pi i/2}I_{\pm 1/3}(\lambda\Phi)$, $J_{\pm 2/3}(e^{3\pi i/2}\lambda\Phi) = -I_{\pm 2/3}(\lambda\Phi)$, it is also true that, if $x > 0$,

$$\text{(24)} \qquad \begin{aligned} u(-x) &= \{\Phi(x)/\varphi(x)\}^{1/2}[-a_1C_1 + a_2C_2],\\ u'(-x) &= \lambda\{\Phi(x)\varphi(x)\}^{1/2}[a_1D_1 - a_2D_2],\end{aligned}$$

in which

$$C_1 = \{1 - \mu_1\lambda^{-2} + O(\lambda^{-4})\}I_{1/3}(\lambda\Phi) - \lambda^{-1}\varphi\{\beta_0 - \beta_2\lambda^{-2} + O(\lambda^{-4})\}I_{-2/3}(\lambda\Phi),$$
$$C_2 = \{1 - \mu_1\lambda^{-2} + O(\lambda^{-4})\}I_{-1/3}(\lambda\Phi) - \lambda^{-1}\varphi\{\beta_0 - \beta_2\lambda^{-2} + O(\lambda^{-4})\}I_{2/3}(\lambda\Phi),$$
$$D_1 = \{1 - \mu_2\lambda^{-2} + O(\lambda^{-4})\}I_{-2/3}(\lambda\Phi) + \lambda^{-1}\{\mu_3 - \mu_4\lambda^{-2} + O(\lambda^{-4})\}I_{1/3}(\lambda\Phi),$$
$$D_2 = \{1 - \mu_2\lambda^{-2} + O(\lambda^{-4})\}I_{2/3}(\lambda\Phi) + \lambda^{-1}\{\mu_3 - \mu_4\lambda^{-2} + O(\lambda^{-4})\}I_{-1/3}(\lambda\Phi).$$

Having established (23) and (24), let us now consider the neighborhood of the singularity at 1. Suppose that

$$\eta = \lambda\int_x^1 \varphi(t)\,dt = \lambda\{a - \Phi(x)\},$$
$$a = \Phi(1) = \int_0^1 \varphi(t)\,dt = \frac{B(3/4, 1/2)}{2} = \frac{(2\pi)^{3/2}}{\{\Gamma(1/4)\}^2},$$
$$w(x) = \{\eta/\lambda\varphi(x)\}^{1/2}J_0(\eta).$$

Then $(1 - x^2)^{-1/2}w(x)$ is finite when $x = 1$, and we now suppose that

$$Y(x) = \left(1 + \frac{\gamma_2}{\lambda^2}\right)w + \left(\delta_0 + \frac{\delta_2}{\lambda^2}\right)\frac{w'}{\lambda^2},$$

in which

$$\delta_0(x) = \frac{1}{\varphi(x)}\int_x^1 \frac{\theta_1(t) - r_0(t)}{2\varphi(t)}\,dt, \qquad \gamma_2(x) = -\frac{\delta_0'(x) + \delta_0^2(x)\varphi^2(x)}{2},$$
$$\delta_2(x) = \frac{1}{\varphi(x)}\int_x^1 \frac{2\delta_0'\theta_1 + \delta_0\theta_1 - \gamma_2'' + \gamma_2(\theta_1 - r_0)}{2\varphi(t)}\,dt,$$
$$\theta_1(x) = \frac{\varphi''(x)}{2\varphi(x)} - \frac{3}{4}\left\{\frac{\varphi'(x)}{\varphi(x)}\right\}^2 + \frac{1}{4}\left\{\frac{\varphi(x)}{a - \Phi(x)}\right\}^2.$$

If $x = 1 - \varepsilon$, it is easy to see that

$$r_0 = (2\varepsilon)^{-2}\{1 + \varepsilon + O(\varepsilon^2)\}, \qquad \varphi = (2\varepsilon)^{-1/2}\left\{1 - \frac{\varepsilon}{4} + O(\varepsilon^2)\right\},$$

$$\theta^* = \frac{3}{16\varepsilon^2}\left\{1 - \frac{\varepsilon}{3} + O(\varepsilon^2)\right\}, \qquad a - \Phi = (2\varepsilon)^{1/2}\left\{1 - \frac{\varepsilon}{12} + O(\varepsilon^2)\right\}, \tag{25}$$

$$\theta_1 = (2\varepsilon)^{-2}\left\{1 - \frac{\varepsilon}{3} + O(\varepsilon^2)\right\}, \qquad \frac{r_0 - \theta_1}{2\varphi} = \frac{1}{3(2\varepsilon)^{1/2}}\{1 + O(\varepsilon)\}.$$

The integral defining $\delta_0(x)\varphi(x)$ therefore exists, and

$$\delta_0\varphi = -\frac{(2\varepsilon)^{1/2}}{3}\{1 + O(\varepsilon)\}, \qquad \delta_0 = \frac{2\varepsilon}{3}\{1 + O(\varepsilon)\}.$$

The integrand of the integral defining $\delta_2(x)\varphi(x)$ is now seen to be $O(\varepsilon^{-1/2})$, so that $\delta_2(x)\varphi(x)$ exists and is such that $\delta_2(x) = O(\varepsilon)$.

As a consequence of the definitions, it follows by reasoning analogous to that used above for $y(x)$ and $y'(x)$ that u and u' are asymptotically equal to a suitable constant multiple of Y and Y', with an error of $O(\lambda^{-4})$ on any closed subinterval of the half-open interval $(0, 1]$. Therefore,

$$u(x) = a_3\{(a - \Phi)/\varphi\}^{1/2}A_3, \qquad u'(x) = a_3\lambda\{(a - \Phi)\varphi\}^{1/2}B_3, \tag{26}$$

in which a_3 is an arbitrary constant, and

$$A_3 = \{1 - v_1\lambda^{-2} + O(\lambda^{-4})\}J_0(\eta) + \lambda^{-1}\varphi\{\delta_0 + \delta_2\lambda^{-2} + O(\lambda^{-4})\}J_1(\eta),$$

$$B_3 = \{1 + v_2\lambda^{-2} + O(\lambda^{-4})\}J_1(\eta) - \lambda^{-1}\{v_3 + v_4\lambda^{-2} + O(\lambda^{-4})\}J_0(\eta),$$

$$v_1 = \frac{1}{2}\left[\delta_0' + \delta_0^2\varphi^2 + \frac{\delta_0}{a - \Phi} + \frac{\delta_0\varphi'}{\varphi}\right], \qquad v_2 = \frac{\delta_0' - \delta_0^2\varphi^2}{2},$$

$$v_3 = \frac{1}{2(a - \Phi)} + \frac{\varphi'}{2\varphi^2} + \delta_0\varphi,$$

$$v_4 = \frac{(\delta_0' - \delta_0^2\varphi^2)(\varphi/(2(a - \Phi)) + \varphi'/(2\varphi)) + \delta_0'' + \delta_0(r_0 + \theta_1) + 2\delta_2\varphi^2}{2\varphi}.$$

We now observe that the differential equation (19) is unaltered if x and λ are replaced by $-x$ and $i\lambda$. Hence equations analogous to (26) may be derived for negative values of x. Thus we see that, when $x > 0$,

$$u(-x) = a_4\{(a - \Phi)/\varphi\}^{1/2}C_3, \qquad u'(-x) = -a_4\lambda\{(a - \Phi)\varphi\}^{1/2}D_3, \tag{27}$$

in which a_4 is an arbitrary constant, and

$$C_3 = \{1 + v_1\lambda^{-2} + O(\lambda^{-4})\}I_0(\eta) + \lambda^{-1}\varphi\{\delta_0 - \delta_2\lambda^{-2} + O(\lambda^{-4})\}I_1(\eta),$$

$$D_3 = \{1 - v_2\lambda^{-2} + O(\lambda^{-4})\}I_1(\eta) + \lambda^{-1}\{v_3 + v_4\lambda^{-2} + O(\lambda^{-4})\}I_0(\eta).$$

If we now compare (24) and (27) we find that

$$a_1\Phi^{1/2} = a_4(a - \Phi)^{1/2}(C_2D_3 - D_2C_3)/(C_1D_2 - D_1C_2),$$

$$a_2\Phi^{1/2} = a_4(a - \Phi)^{1/2}(C_1D_3 - D_1C_3)/(C_1D_2 - D_1C_2).$$

The denominator $C_1D_2 - D_1C_2$ can be expressed as the product of two determinants,

$$\begin{vmatrix} 1 + O(\lambda^{-2}) & -\beta_0\varphi/\lambda + O(\lambda^{-3}) \\ 1 + O(\lambda^{-2}) & \frac{1}{\lambda}\left\{\frac{1}{6\Phi} - \frac{\varphi'\Phi}{2\varphi^2} - \beta_0\varphi\right\} + O(\lambda^{-3}) \end{vmatrix} \cdot \begin{vmatrix} I_{1/3}(\lambda\Phi) & I_{-1/3}(\lambda\Phi) \\ I_{-2/3}(\lambda\Phi) & I_{2/3}(\lambda\Phi) \end{vmatrix}$$

$$= \frac{1}{\lambda}\left\{\frac{1}{6\Phi} - \frac{\varphi'}{2\varphi^2} + O(\lambda^{-2})\right\}\frac{2\sin(\pi/3)}{\pi\lambda\Phi},$$

and so does not vanish since

$$\frac{1}{6\Phi} - \frac{\varphi'}{2\varphi^2} = \frac{1}{6\int_0^x\{t^{1/2}\,dt/(1-t^2)^{1/2}\}} - \frac{1+x^2}{4x^{3/2}(1-x^2)^{1/2}}$$

$$< \frac{1}{6\int_0^x t^{1/2}\,dt} - \frac{1}{4x^{3/2}} = 0 \quad \text{when } x > 0.$$

Moreover, since $I_{-\alpha} - I_\alpha = (2/\pi)\tan(\alpha\pi)K_\alpha$, and $K_\alpha(z) = O(e^{-z}z^{-1/2})$, it follows that

$$C_1 - C_2 = O(e^{-\lambda\Phi}\lambda^{-1/2}\Phi^{-1/2}), \qquad D_1 - D_2 = O(e^{-\lambda\Phi}\lambda^{-1/2}\Phi^{-1/2}).$$

In addition, $I_\alpha(z) = O(e^z z^{-1/2})$, and hence

$$C_3 = O(e^{\lambda(a-\Phi)}\lambda^{-1/2}(a-\Phi)^{-1/2}), \qquad D_3 = O(e^{\lambda(a-\Phi)}\lambda^{-1/2}(a-\Phi)^{-1/2}),$$

$$a_1 - a_2 = a_4 O(e^{\lambda(a-2\Phi)}\lambda) = a_4 o(\lambda^{-n})$$

for every integer n, if x is sufficiently close to 1 that $a < 2\Phi(x)$.

A comparison of (23) and (26) now shows that for such x,

$$a_1\Phi^{1/2}(A_1 + A_2) = a_3(a-\Phi)^{1/2}A_3 + o(\lambda^{-n}),$$

$$a_1\Phi^{1/2}(B_1 - B_2) = a_3(a-\Phi)^{1/2}B_3 + o(\lambda^{-n}),$$

and consequently, dropping the $o(\lambda^{-n})$ terms,

$$(A_1 + A_2)B_3 = (B_1 - B_2)A_3. \tag{28}$$

With the help of the asymptotic expansions for the Bessel function J_α and some manipulation of the trigonometric functions involved, we see that

$$A_1 + A_2 = \left(\frac{6}{\pi\xi}\right)^{1/2}\left[\{1 - \rho_1\lambda^{-2} + O(\lambda^{-4})\}\sin\left(\xi + \frac{\pi}{4}\right) + \lambda^{-1}\{\rho_2 + \rho_3\lambda^{-2} + O(\lambda^{-4})\}\cos\left(\xi + \frac{\pi}{4}\right)\right],$$

$$B_1 - B_2 = \left(\frac{6}{\pi\xi}\right)^{1/2}\left[\{1 + \rho_4\lambda^{-2} + O(\lambda^{-4})\}\cos\left(\xi + \frac{\pi}{4}\right) + \lambda^{-1}\{\rho_5 + \rho_6\lambda^{-2} + O(\lambda^{-4})\}\sin\left(\xi + \frac{\pi}{4}\right)\right],$$

$$A_3 = \left(\frac{2}{\pi\eta}\right)^{1/2}\left[\{1 - \rho_7\lambda^{-2} + O(\lambda^{-4})\}\cos\left(\eta - \frac{\pi}{4}\right) + \lambda^{-1}\{\rho_8 - \rho_9\lambda^{-2} + O(\lambda^{-4})\}\sin\left(\eta - \frac{\pi}{4}\right)\right],$$

$$B_3 = \left(\frac{2}{\pi\eta}\right)^{1/2}\left[\{1 + \rho_{10}\lambda^{-2} + O(\lambda^{-4})\}\sin\left(\eta - \frac{\pi}{4}\right) - \lambda^{-1}\{\rho_{11} + \rho_{12}\lambda^{-2} + O(\lambda^{-4})\}\cos\left(\eta - \frac{\pi}{4}\right)\right],$$

in which the coefficients $\rho_1, \rho_2, \cdots, \rho_{12}$ are expressible as explicit formulas involving $\beta_0, \beta_2, \varphi, \Phi, r_0, \theta, \delta_0, \delta_2, \theta_1$, and a.

The characteristic equation (28) may now be written in the form

$$\begin{aligned}
&\{1 + (\rho_4 - \rho_7 + \rho_2\rho_{11})\lambda^{-2} + O(\lambda^{-4})\}\cos(\xi + \eta) \\
&\quad + \lambda^{-1}\{\rho_5 + \rho_{11} + \lambda^{-2}(\rho_6 + \rho_{12} - \rho_5\rho_7 - \rho_1\rho_{11}) + O(\lambda^{-4})\} \\
&\qquad \cdot \sin(\xi + \eta) \\
&\quad + \lambda^{-2}\{\rho_4 - \rho_7 + \rho_2\rho_{11} + \rho_1 - \rho_{10} + \rho_5\rho_8 + O(\lambda^{-2})\} \\
&\qquad \cdot \sin\left(\xi + \frac{\pi}{4}\right)\sin\left(\eta - \frac{\pi}{4}\right) \\
&\quad - \lambda^{-1}\{\rho_2 - \rho_8 + \rho_5 + \rho_{11} + \lambda^{-2}(\rho_3 + \rho_9 - \rho_4\rho_8 + \rho_2\rho_{10} + \rho_6 \\
&\qquad + \rho_{12} - \rho_5\rho_7 - \rho_1\rho_{11}) + O(\lambda^{-4})\} \\
&\qquad \cdot \cos\left(\xi + \frac{\pi}{4}\right)\cos\left(\eta - \frac{\pi}{4}\right) = 0.
\end{aligned} \tag{29}$$

Let us introduce the symbol

$$G = \beta_0\varphi - \delta_0\varphi - \frac{1}{8(a - \Phi)} - \frac{5}{72\Phi},$$

so that

$$\frac{dG}{dx} = (\beta_0\varphi)' - (\delta_0\varphi)' - \frac{\varphi}{8(a - \Phi)^2} + \frac{5\varphi}{72\Phi^2},$$

$$\frac{d^2G}{dx^2} = (\beta_0\varphi)'' - (\delta_0\varphi)'' - \frac{\varphi'}{8(a - \Phi)^2} + \frac{5\varphi'}{72\Phi^2} - \frac{\varphi^2}{4(a - \Phi)^3} - \frac{5\varphi^2}{36\varphi^3}.$$

It follows from the definitions of β_0 and δ_0 that $dG/dx = 0$, so that G is a constant and $d^2G/dx^2 = 0$. It is then easy to verify from the explicit formulas for the coefficients ρ_k that

$$\rho_5 + \rho_{11} = \rho_8 - \rho_2 = -G, \qquad \rho_4 - \rho_7 + \rho_2\rho_{11} = -\frac{G^2}{2} + \frac{1}{2\varphi}\frac{dG}{dx} = -\frac{G^2}{2},$$

$$\rho_1 - \rho_{10} + \rho_5\rho_8 = \frac{G^2}{2} + \frac{1}{2\varphi}\frac{dG}{dx} = \frac{G^2}{2},$$

$$\rho_6 + \rho_{12} - \rho_5\rho_7 - \rho_1\rho_{11} = \frac{G^3}{6} + \frac{1}{2\varphi}\left\{\frac{\varphi'}{\varphi^2} + \frac{5}{72\Phi} - \frac{1}{8(a-\Phi)}\right\}\frac{dG}{dx}$$
$$- \frac{1}{2\varphi^2}\frac{d^2G}{dx^2} - H = \frac{G^3}{6} - H,$$
$$\rho_3 + \rho_9 - \rho_4\rho_8 + \rho_2\rho_{10} = -\frac{G^3}{6} - \frac{1}{16\varphi}\left\{\frac{5}{9\Phi} - \frac{1}{a-\Phi}\right\}\frac{dG}{dx} + H$$
$$= -\left(\frac{G^3}{6} - H\right),$$

in which

$$H = H_1 - H_2, \qquad H_1 = \beta_2\varphi + \frac{\beta_0^3\varphi^3}{6} + \frac{\beta_0}{4\varphi}\left\{r_0 - \theta^* + \frac{5\varphi^2}{36\Phi^2}\right\} + \frac{1105}{31104\Phi^3},$$
$$H_2 = \delta_2\varphi + \frac{\delta_0^3\varphi^3}{6} + \frac{\delta_0}{4\varphi}\left\{r_0 - \theta^* + \frac{\varphi^2}{4(a-\Phi)^2}\right\} - \frac{25}{384(a-\Phi)^3}.$$

We shall show in the next section that

$$G = 5\{\Gamma(1/4)\}^2/48(2\pi)^{1/2},$$
$$H = 35a/384 = 35(2\pi)^{3/2}/384\{\Gamma(1/4)\}^2.$$

Assuming these results for the moment, we can write the characteristic equation (29) in the form

$$\left\{1 - \frac{G^2}{2\lambda^2} + O(\lambda^{-4})\right\}\cos\lambda a - \frac{1}{\lambda}\left\{G - \frac{1}{\lambda^2}\left(\frac{G^3}{6} - H\right) + O(\lambda^{-4})\right\}\sin\lambda a = O(\lambda^{-4}).$$

For large values of n, this equation has the solution

$$\lambda a = \left(n + \frac{1}{2}\right)\pi - \frac{Ga}{(n+\frac{1}{2})\pi} - \frac{(Ga)^2 + Ha^3}{(n+\frac{1}{2})^3\pi^3} + O(n^{-5}).$$

Actually, all we can show from the reasoning above is that the error term is $O(n^{-4})$, but it seems reasonable to conjecture that an analysis of the next term in all of the expansions used would result in a term which is $O(n^{-5})$. Therefore,

$$k = \left(n + \frac{1}{2}\right)^2\frac{\pi^2}{a^2} - \frac{2G}{a} - \frac{G^2 + 2Ha}{(n+\frac{1}{2})^2\pi^2} + O(n^{-4})$$
$$= \frac{\{\Gamma(1/4)\}^4}{8\pi}\left(n + \frac{1}{2}\right)^2\left\{1 - \frac{5}{12\pi(n+\frac{1}{2})^2} - \frac{\frac{25}{576} + 35\pi^4/[3\{\Gamma(1/4)\}^8]}{(n+\frac{1}{2})^4\pi^2} + \cdots\right\}$$
$$= 6.87518581\left(n + \frac{1}{2}\right)^2 - 0.9118498 - \frac{0.0567489}{(n+\frac{1}{2})^2} + \cdots.$$

By comparison with the results of Forutanpour recorded in Table 1, it is seen that these three terms of the expansion produce a result which is correct to within one unit in the sixth significant digit when n is as small as 3, and has an error of only 0.0041, or less than 0.03 %, when $n = 1$.

6. The numerical values of G and H. We have already seen that G is constant. In this section we shall evaluate that constant, and also evaluate H, which is itself a constant. From the definitions, we see that

$$\begin{aligned} G &= \int_0^x \frac{r_0 - \theta}{2\varphi}\,dt + \int_x^1 \frac{r_0 - \theta_1}{2\varphi}\,dt - \frac{5}{72\Phi} - \frac{1}{8(a-\Phi)} \\ &= \lim_{s\to 0}\int_s^x \left(\frac{r_0 - \theta^*}{2\varphi} - \frac{5\varphi}{72\Phi^2}\right) dt + \lim_{s'\to 1}\int_x^{s'} \left\{\frac{r_0 - \theta^*}{2\varphi} - \frac{\varphi}{8(a-\Phi)^2}\right\} dt \\ &\quad - \frac{5}{72\Phi} - \frac{1}{8(a-\Phi)} \\ &= \lim_{\substack{s\to 0\\ s'\to 1}} \left[\int_s^{s'} \frac{r_0 - \theta^*}{2\varphi}\,dt - \frac{5}{72\Phi(s)} - \frac{1}{8\{a - \Phi(s')\}}\right]. \end{aligned}$$

In view of the specific analytic form of the functions r_0 and φ,

$$\begin{aligned} G &= \lim_{\substack{s\to 0\\ s'\to 1}} \left[\int_s^{s'} \frac{5 + 2t^2 - 3t^4}{32t^{5/2}(1-t^2)^{3/2}}\,dt - \frac{5}{72\Phi(s)} - \frac{1}{8\{a - \Phi(s')\}}\right] \\ &= \lim_{\substack{s\to 0\\ s'\to 1}} \left[\int_s^{s'} \left\{\frac{5}{24t^{1/2}(1-t^2)^{1/2}} + \frac{d}{dt}\left(\frac{t^{1/2}}{8(1-t^2)^{1/2}} - \frac{5(1-t^2)^{1/2}}{48t^{3/2}}\right)\right\} dt \right. \\ &\quad \left. - \frac{5}{72\Phi(s)} - \frac{1}{8\{a - \Phi(s)\}}\right] \\ &= \frac{5}{24}\int_0^1 \frac{dt}{t^{1/2}(1-t^2)^{1/2}} + \lim_{s\to 0}\left[\frac{5(1-s^2)^{1/2}}{48s^{3/2}} - \frac{5}{72\Phi(s)}\right] \\ &\quad + \frac{1}{8}\lim_{s'\to 1}\left[\varphi(s') - \frac{1}{a - \Phi(s')}\right] \\ &= \frac{5}{48}B(1/4, 1/2) = \frac{5\{\Gamma(1/4)\}^2}{48(2\pi)^{1/2}}, \end{aligned}$$

since it follows from (22) and (25) that the limits of the bracketed quantities are zero.

After some similar, although more complicated, manipulations we find that

$$\begin{aligned} H_1 &= \lim_{s\to 0}\left[\left\{\frac{\beta_0}{4\varphi}\left(r_0 - \theta^* + \frac{5\varphi^2}{36\Phi^2}\right) + \frac{r_0' - \theta^{*\prime}}{8\varphi^3} - \frac{(r_0 - \theta^*)\varphi'}{4\varphi^4} + \frac{1105}{31104\Phi^3}\right\}_s \right. \\ &\quad \left. - \left\{\frac{r_0' - \theta^{*\prime}}{8\varphi^3} - \frac{(r_0 - \theta^*)\varphi'}{4\varphi^4}\right\}_x - \int_s^x \frac{(r_0 - \theta^*)^2}{8\varphi^3}\,dt\right], \\ H_2 &= \lim_{s'\to 1}\left[\left\{\frac{\delta_0}{4\varphi}\left(r_0 - \theta^* + \frac{\varphi^2}{4(a-\Phi)^2}\right) + \frac{r_0' - \theta^{*\prime}}{8\varphi^3} - \frac{(r_0 - \theta^*)\varphi'}{4\varphi^4}\right.\right. \qquad \text{(cont.)} \end{aligned}$$

$$- \frac{25}{384(a - \Phi)^3}\bigg\}_{s'} - \left\{\frac{r_0' - \theta^{*\prime}}{8\varphi^3} - \frac{(r_0 - \theta^*)\varphi'}{4\varphi^4}\right\}_x + \int_x^{s'} \frac{(r_0 - \theta^*)^2}{8\varphi^3}\,dt\Bigg].$$

In view of the specific analytic form of the functions r_0 and φ,

$$\frac{(r_0 - \theta^*)^2}{8\varphi^3} = \frac{(5 + 2t^2 - 3t^4)^2}{2048t^{11/2}(1 - t^2)^{5/2}}$$

$$= \left\{\frac{-50 - 224t^2 - 1974t^4 + 5654t^6 - 3360t^8}{18432t^{9/2}(1 - t^2)^{3/2}}\right\}' - \frac{35t^{1/2}}{384(1 - t^2)^{1/2}},$$

and consequently,

$$H = H_1 - H_2 = \frac{35}{384}\int_0^1 t^{1/2}(1 - t^2)^{-1/2}\,dt$$

$$+ \lim_{s \to 0}\left[\left\{\frac{\beta_0}{4\varphi}\left(r_0 - \theta^* + \frac{5\varphi^2}{36\Phi^2}\right) + \frac{r_0' - \theta^{*\prime}}{8\varphi^3} - \frac{(r_0 - \theta^*)\varphi'}{4\varphi^4} + \frac{1105}{31104\Phi^3}\right\}_s - \frac{50 + 224s^2 + 1974s^4 - 5654s^6 + 3360s^8}{18432s^{9/2}(1 - s^2)^{3/2}}\right]$$

$$- \lim_{s' \to 1}\left[\left\{\frac{\delta_0}{4\varphi}\left(r_0 - \theta^* + \frac{\varphi^2}{4(a - \Phi)^2}\right) + \frac{r_0' - \theta^{*\prime}}{8\varphi^3} - \frac{(r_0 - \theta^*)\varphi'}{4\varphi^4} - \frac{25}{384(a - \Phi)^3}\right\}_{s'} - \frac{50 + 224s'^2 + 1974s'^4 - 5654s'^6 + 3360s'^8}{18432s'^{9/2}(1 - s'^2)^{3/2}}\right]$$

$$= 35a/384,$$

since it is a consequence of (22) and (25) that the limits of the bracketed quantities are zero.

REFERENCES

[1] H. A. Bethe, M. E. Rose and L. P. Smith, *The multiple scattering of electrons*, Proc. Amer. Philos. Soc., 78 (1938), pp. 573–585.

[2] R. Scalettar, unpublished memorandum, 1966.

[3] H. J. Zimmerberg, *Definite integral systems*, Duke Math. J., 15 (1948), pp. 371–388.

[4] A. C. Zaanen, *Linear Analysis*, Interscience, New York, 1953.

[5] F. G. Tricomi, *Integral Equations*, Interscience, New York, 1957.

[6] M. H. Stone, *Linear Transformations in Hilbert Space*, Colloquium Publications, vol. XV, American Mathematical Society, New York, 1932.

[7] R. E. Langer, *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point*, Trans. Amer. Math. Soc., 67 (1949), pp. 461–490.

[8] A. A. Dorodnicyn, *Asymptotic laws of distribution of the characteristic values for certain special forms of differential equations of the second order*, Uspehi Mat. Nauk, 7 (1952), no. 6 (52), pp. 3–96; English transl., Amer. Math. Soc. Transl. (2), 16 (1960), pp. 1–101.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

ADIABATIC INVARIANCE OF A SIMPLE OSCILLATOR*

WOLFGANG WASOW†

Abstract. J. E. Littlewood [5] has derived asymptotic expressions, as $\varepsilon \to 0+$, for the function $r^2 = \phi(\varepsilon\tau)u^2 + \phi^{-1}(\varepsilon\tau)(du/d\tau)^2$, when u is a solution of the differential equation $d^2u/d\tau^2 + \phi^2(\varepsilon\tau)u = 0$. He assumes that $\phi(\tau) > 0$, $\phi(\pm\infty) > 0$, $\phi^{(n)}(\pm\infty) = 0$, and $\phi^{(n)} \in L(-\infty, \infty)$, for all $n > 0$. In the present paper, Littlewood's results are re-proved and strengthened by using the established methods for the solution of differential equations by asymptotic series. A new result is an explicit series construction in powers of ε for the function r^2. Littlewood's asymptotic expression was in terms of the unknown solution of the differential equation.

1. Introduction. If the function $g(\tau)$ in the differential equation

$$\frac{d^2u}{d\tau^2} + g^2(\tau)u = 0 \tag{1.1}$$

for the motion of a simple oscillator is a constant, the energy

$$\frac{1}{2}\left[g^2(\tau)u^2 + \left(\frac{du}{d\tau}\right)^2\right] \tag{1.2}$$

is also a constant. If g is not constant but changes very slowly with the time τ, the function (1.2) can be interpreted as measuring the "local" energy, and it is plausible that it, too, changes slowly.

The statement: "g changes slowly" can be mathematically formulated by setting

$$g(\tau) = \phi(\varepsilon\tau), \tag{1.3}$$

where ε is a small positive parameter. It is then convenient to make a change of time scale by setting

$$t = \varepsilon\tau.$$

This transforms the differential equation (1.1) into

$$\varepsilon^2\ddot{u} + \phi^2(t)u = 0 \qquad \left(\ddot{u} = \frac{d^2}{dt^2}u\right). \tag{1.4}$$

The function $r^2(t, \varepsilon)$, defined by

$$r^2(t, \varepsilon) = \left[g^2u^2 + \left(\frac{du}{d\tau}\right)^2\right]\Big/ g = \phi u^2 + \varepsilon^2\phi^{-1}\dot{u}^2, \tag{1.5}$$

turns out to have simpler and more striking properties, for small ε, than the local energy itself.

In fact, Littlewood [5] proved that, under appropriate hypotheses on ϕ (hypothesis (H), below),

$$r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon) = O(\varepsilon^n) \quad \text{for all } n > 0. \tag{1.6}$$

* Received by the editors November 22, 1971, and in revised form March 23, 1972.

† Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706.

In his proof, Littlewood made no use of the established theory of asymptotic expansions for the solutions of linear differential equations. In this paper it will be shown that Littlewood's results follow almost immediately from a refined version of the standard theorem on asymptotic solution (see, e.g., [7, § 26]), which can be proved by taking into account Littlewood's particular hypotheses on $\phi(t)$.

The method of this paper also yields an asymptotic series for $r^2(t, \varepsilon)$, as $\varepsilon \to 0+$, whose coefficients depend only on $\phi(t)$ and on the initial data. Littlewood's corresponding series involves the unknown solution of the differential equation.

The problem of replacing the right member of (1.6) by an asymptotic expansion, or at least the leading term thereof, requires varying methods depending on more special properties of $\phi(t)$. Some remarks applicable to certain types of analytic differential equations are included in § 6. A more complete account will be published elsewhere. Other results in this direction can be found in [1], [3] and [4].

The term "adiabatic" is commonly applied in physics to phenomena which involve some changes that are much slower than others. An example is wave propagation in a fluid where the compression caused by the wave passes so fast that very little of the heat generated by it can diffuse. This makes it approximately a process without heat exchange, i.e., *adiabatic* in the original sense of the word. The quantities that change slowly in such processes—or, in other words, those nearly invariant—are the adiabatic invariants. Mathematical problems of this sort are sometimes said to involve two time scales: "fast" and "slow" time. An asymptotic analysis from a more general viewpoint can, for example, be found in [2].

The simplest—but by no means the only—physical interpretation of the mathematical problem studied in this paper is that of a pendulum whose length is changed at a rate much slower than the frequency.

DEFINITION 1.1. An indefinitely differentiable real or complex function f of t will be called *gentle* if

$$d^n f/dt^n \in L_1(-\infty, \infty), \qquad n = 0, 1, 2, \cdots.$$

By integrating the derivatives of f one sees that

$$\lim_{t \to \pm\infty} d^n f/dt^n = 0, \qquad n = 0, 1, 2, \cdots.$$

The term "gentle" has occasionally been used in the mathematical literature with a meaning different from the one above.

Hypotheses (H).

(i) ε is a small positive parameter.

(ii) ϕ is a positive function of t in $-\infty < t < \infty$.

(iii) $\lim_{t\to\infty} \phi(t)$ and $\lim_{t\to-\infty} \phi(t)$ exist and are positive.

(iv) $\dot{\phi}$ is gentle.

These are the same assumptions as in Littlewood [5].

2. Reduction to a Riccati equation. The transformation

$$x = \phi^{1/2}u, \quad y = \varepsilon\phi^{-1/2}\dot{u}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$

changes the differential equation (1.4) into the equivalent vectorial system

$$\varepsilon \dot{z} = \begin{pmatrix} \varepsilon\psi & \phi \\ -\phi & -\varepsilon\psi \end{pmatrix} z = Az, \tag{2.1}$$

where

$$\psi = \dot{\phi}/2\phi. \tag{2.2}$$

Observe that ψ is a gentle function. The function r^2 takes on the simple form

$$r^2 = x^2 + y^2. \tag{2.3}$$

The change

$$z = Sv_0, \qquad S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{2.4}$$

of the dependent variable diagonalizes the leading part—with respect to ε—of the coefficient matrix in (2.1) and takes the differential equation into

$$\varepsilon \dot{v}_0 = \begin{pmatrix} i\phi & \varepsilon\psi \\ \varepsilon\psi & -i\phi \end{pmatrix} v_0 = B_0 v_0. \tag{2.5}$$

We shall show, following essentially the method of Sibuya in [6] (see also [7, § 26]), that there exists a matrix $P(t, \varepsilon)$ of the form

$$P(t, \varepsilon) = \begin{pmatrix} 1 & \varepsilon p(t, \varepsilon) \\ \varepsilon \bar{p}(t, \varepsilon) & 1 \end{pmatrix}, \tag{2.6}$$

where the bar represents complex conjugation such that the transformation

$$v_0 = Pv \tag{2.7}$$

reduces (2.5) to

$$\varepsilon \dot{v} = Bv, \tag{2.8}$$

where B is a *diagonal* matrix of the form

$$B(t, \varepsilon) = \begin{pmatrix} i\phi(t) + \varepsilon^2 \bar{b}(t, \varepsilon) & 0 \\ 0 & -i\phi(t) + \varepsilon^2 b(t, \varepsilon) \end{pmatrix}. \tag{2.9}$$

Substituting (2.6), (2.7) into (2.5) and identifying the result with (2.8), (2.9) we are led to the relation

$$B_0 P - PB = \varepsilon \dot{P}, \tag{2.10}$$

and, hence, to

$$\varepsilon \dot{p} = \psi + 2i\phi p - \varepsilon^2 pb, \tag{2.11}$$

$$\psi p = b \tag{2.12}$$

as necessary and sufficient conditions for the existence of such a transformation. By elimination of b the Riccati equation

$$\varepsilon \dot{p} = 2i\phi p + \psi - \varepsilon^2 \psi p^2 \tag{2.13}$$

for p is obtained. The diagonalization of the differential equation (2.1)—and hence its solution—has thus been reduced to the study of the Riccati equation (2.13).

3. Asymptotic solution of the Riccati equation.

LEMMA 3.1. *The differential equation* (2.13) *can be formally satisfied by a series of the form* $\sum_{r=0}^{\infty} p_r(t)\varepsilon^r$ *with gentle coefficients.*

Proof. Substitution of the series for p into (2.13), termwise differentiation and rearrangement according to powers of ε yields the recursion formulas

$$2i\phi p_0 + \psi = 0,$$

$$2i\phi p_r = \dot{p}_{r-1} - \psi \sum_{s=0}^{r-2} p_s p_{r-2-s}, \qquad r > 0,$$

where p_{-2}, p_{-1} are defined as zero. As ψ is gentle, all the p_r so defined are inductively seen to be gentle.

We next prove an adaptation of the so-called Borel–Ritt theorem (see [7, Thm. 9.3]). The theorem below is somewhat stronger than what is needed in the sequel.

THEOREM 3.1. *Let* $a_r(t)$, $r = 0, 1, 2, \cdots$, *be gentle functions. Then there exists a function* $f(t, \varepsilon)$ *defined for* $0 < \varepsilon < 1$, $-\infty < t < \infty$, *such that*

$$\text{(i)} \qquad \frac{d^n}{dt^n} f(t, \varepsilon) \sim \sum_{r=0}^{\infty} a_r^{(n)}(t)\varepsilon^r, \quad \textit{as } \varepsilon \to +0, \quad \textit{for } n = 0, 1, \cdots$$

uniformly on $-\infty < t < \infty$.

(ii) f *is a gentle function for all* ε *in* $0 < \varepsilon < 1$. (See [7, Chap. III] for the definition of the symbol "$\sim$".)

Proof. Let

$$k_r = \max_{0 \leqq j \leqq r} \left\{ \sup_{|t| < \infty} |a_r^{(j)}(t)|, \int_{-\infty}^{\infty} |a_r^{(j)}(t)|\, dt \right\},$$

$$\alpha_r(\varepsilon) = 1 - \exp(-\varepsilon^{-1}k_r^{-1}).$$

Then the series $\sum_{r=0}^{\infty} \alpha_r(\varepsilon)a_r^{(n)}(t)\varepsilon^r$, $n = 0, 1, 2, \cdots$, converge uniformly in $-\infty < t < \infty$, for $0 < \varepsilon < 1$, since $|\alpha_r(\varepsilon)| \leqq \varepsilon^{-1}k_r^{-1}$, and therefore,

$$|\alpha_r(\varepsilon)a_r^{(n)}(t)\varepsilon^r| \leqq \varepsilon^{r-1}$$

for $r \geqq n$. Thus, the function

$$f(t, \varepsilon) = \sum_{r=0}^{\infty} \alpha_r(\varepsilon)a_r(t)\varepsilon^r$$

is in $C^\infty(-\infty, \infty)$ with respect to t, for $0 < \varepsilon < 1$, and it as well as all its derivatives vanish at $\pm\infty$. The integrability of $f^{(n)}(t, \varepsilon)$ follows from the integrability of the right member in the inequality $|f^{(n)}| \leqq \sum_{r=0}^{\infty} |\alpha_r(\varepsilon)||a_r^{(n)}(t)|\varepsilon^r$. The latter function is integrable as a consequence of the definition of k_r, which implies that

$$\int_{-\infty}^{\infty} \sum_{r=n+1}^{\infty} |\alpha_r(\varepsilon)||a_r^{(n)}(t)|\varepsilon^r \, dt = \sum_{r=n+1}^{\infty} |\alpha_r(\varepsilon)| \int_{-\infty}^{\infty} |a_r^{(n)}(t)| \, dt \varepsilon^r$$
$$\leqq \sum_{r=n+1}^{\infty} \varepsilon^{r-1}.$$

This proves assertion (ii).

Next, for all $N > 0$,

$$f^{(n)}(t, \varepsilon) - \sum_{r=0}^{N} \alpha_r(\varepsilon) a_r^{(n)}(t)\varepsilon^r = -\sum_{r=0}^{N-1} \exp(-\varepsilon^{-1}k_r^{-1}) a_r^{(n)}(t)\varepsilon^r + \varepsilon^N \sum_{r=N}^{\infty} \alpha_r(\varepsilon) a_r^{(n)}(t)\varepsilon^{r-N}.$$

The right side is $O(\varepsilon^N)$, uniformly in $-\infty < t < \infty$, i.e., part (i) of the theorem is also proved.

By Lemma 3.1 and Theorem 3.1 there exists a gentle function $\tilde{p}(t, \varepsilon)$ with the asymptotic representation

$$\tilde{p}(t, \varepsilon) \sim \sum_{r=0}^{\infty} p_r(t)\varepsilon^r \quad \text{as } \varepsilon \to 0+, \tag{3.1}$$

uniformly valid in $-\infty < t < \infty$, such that the function $q = q(t, \varepsilon)$, defined by

$$q = -\varepsilon\dot{\tilde{p}} + \psi + 2i\phi\tilde{p} - \varepsilon^2\psi\tilde{p}^2, \tag{3.2}$$

is uniformly asymptotic to zero:

$$q(t, \varepsilon) \sim 0, \quad \varepsilon \to 0+. \tag{3.3}$$

Clearly, q is also gentle.

To show that $\tilde{p}$ is asymptotically equal to an actual solution of equation (2.13) we set

$$p = \tilde{p} + w \tag{3.4}$$

and obtain the differential equation

$$\varepsilon\dot{w} = 2i\phi w - \varepsilon^2\psi(2w\tilde{p} + w^2) + q \quad \text{for } w. \tag{3.5}$$

We now have to construct a solution w of (3.5) that is asymptotic to zero. Set

$$\Phi(t) = \int_0^t \phi(s) \, ds \tag{3.6}$$

and observe that, by the variation of parameters formula, any continuous solution of the integral equation

$$w(t, \varepsilon) = \int_{-\infty}^{t} \exp\left(\frac{2i}{\varepsilon}[\Phi(t) - \Phi(s)]\right) F(w(s, \varepsilon), s, \varepsilon) \, ds, \tag{3.7}$$

where

$$F(w, s, \varepsilon) = -\varepsilon\psi(s)[2w\tilde{p}(s, \varepsilon) + w^2] + \varepsilon^{-1}q(s, \varepsilon),$$

also solves the differential equation (3.5).

The existence of such a solution is a simple consequence of the contraction mapping theorem in the Banach space of bounded continuous functions on the real line with the maximum modulus norm. The operator in this application is defined by

$$(3.8)\qquad Tv = -\int_{-\infty}^{t} \exp\left(\frac{2i}{\varepsilon}[\Phi(t) - \Phi(s)]\right) \varepsilon\psi(s)[2v(s)\tilde{p}(s, \varepsilon) + v^2(s)]\, ds.$$

It is readily verified that this is a contraction operator in the ball $\|v\| \leqq 1$, provided $0 < \varepsilon < \varepsilon_0$, with $\varepsilon_0 < \frac{1}{2}$ and

$$\varepsilon_0 < \left\{\left(2\int_{-\infty}^{\infty} |\psi(s)|\, ds\right) \cdot \left(\sup_{0<\varepsilon<1/2} \|\tilde{p}(\cdot, \varepsilon)\| + 1\right)\right\}^{-1}.$$

The integral equation can also be written

$$(3.9)\qquad w = Tw + \omega,$$

where

$$\omega = \varepsilon^{-1} \int_{-\infty}^{t} \exp\left(\frac{2i}{\varepsilon}[\Phi(t) - \Phi(s)]\right) q(s, \varepsilon)\, ds.$$

The function ω is in the Banach space. Thus, the integral equation has a unique solution. Also ω is uniformly asymptotic to zero. It follows from the contraction property of T that w is also asymptotic to zero. Returning to (3.7), we see, moreover, that $w(-\infty, \varepsilon) = 0$, and that the function

$$(3.10)\qquad \hat{w}(t, \varepsilon) = w(t, \varepsilon) \exp\left(-\frac{2i}{\varepsilon}\Phi(t)\right)$$

has a limit as $t \to +\infty$. However, $\hat{w}(\infty, \varepsilon)$ need not be zero. Thus, $\hat{w}$ is generally not gentle, nor is p.

Now that the differential equation (2.1) has been diagonalized it is readily solved. The result is formulated in the theorem below.

THEOREM 3.2. *The differential equation* (2.1) *possesses a fundamental matrix solution* $Z(t, \varepsilon)$ *with the following properties*:

(i) $$Z = SPV$$

with

$$S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad P = P(t, \varepsilon) = \begin{pmatrix} 1 & \varepsilon p(t, \varepsilon) \\ \varepsilon\bar{p}(t, \varepsilon) & 1 \end{pmatrix};$$

(ii)

$$V = V(t, \varepsilon) = \exp\begin{pmatrix} \frac{i}{\varepsilon}\Phi(t) + \varepsilon\int_{-\infty}^{t} \psi(s)\bar{p}(s, \varepsilon)\, ds & 0 \\ 0 & -\frac{i}{\varepsilon}\Phi(t) + \varepsilon\int_{-\infty}^{t} \psi(s)p(s, \varepsilon)\, ds \end{pmatrix};$$

(iii) $$p(t,\varepsilon) \sim \sum_{r=0}^{\infty} p_r(t)\varepsilon^r \quad \textit{as } \varepsilon \to 0+ \quad (p_r(t) \textit{ gentle}),$$

uniformly in $-\infty < t < \infty$, *and this relation may be indefinitely termwise differentiated*;

(iv) $$p(-\infty, \varepsilon) = 0;$$

(v) *the function*

$$\hat{p}(t,\varepsilon) = p(t,\varepsilon)\exp\left(-\frac{2i}{\varepsilon}\Phi(t)\right)$$

has a limit, as $t \to +\infty$, *and this limit, as a function of* ε, *is asymptotic to zero as* $\varepsilon \to 0+$.

4. Proof of Littlewood's theorem. Let $z = z(t, \varepsilon)$ be the solution of (2.1) with initial values

$$z(0,\varepsilon) = z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4.1}$$

independent of ε, and set

$$c = c(\varepsilon) = (Z(0,\varepsilon))^{-1} z_0. \tag{4.2}$$

The vector $c(\varepsilon)$ has an asymptotic expansion in powers of ε as $\varepsilon \to 0+$. We then have

$$z = Zc \tag{4.3}$$

and

$$r^2 = z^T z = c^T Z^T Z c. \tag{4.4}$$

The matrix

$$M = Z^T Z \tag{4.5}$$

can be calculated in terms of ϕ, ψ and p, by means of Theorem 3.2. Straightforward manipulations lead to

(4.6) $M(t,\varepsilon) =$

$$2\begin{bmatrix} 2\varepsilon\bar{\hat{p}}\exp\left\{2\varepsilon\int_{-\infty}^{t}\psi\bar{p}\,ds\right\} & (1+\varepsilon^2\hat{p}\bar{\hat{p}})\exp\left\{\varepsilon\int_{-\infty}^{t}\psi(p+\bar{p})\,ds\right\} \\ (1+\varepsilon^2\hat{p}\bar{\hat{p}})\exp\left\{\varepsilon\int_{-\infty}^{t}\psi(p+\bar{p})\,ds\right\} & 2\varepsilon\hat{p}\exp\left\{2\varepsilon\int_{-\infty}^{t}\psi p\,ds\right\} \end{bmatrix}.$$

(Observe that $p\bar{p} = \hat{p}\bar{\hat{p}}$.) It follows that, in particular,

$$M(-\infty,\varepsilon) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{4.7}$$

For the asymptotic calculation of $M(\infty, \varepsilon)$ the following lemma is decisive.

LEMMA 4.1.

$$\exp\left\{\varepsilon\int_{-\infty}^{t}\psi(p+\bar{p})\,ds\right\} = (1-\varepsilon^2 p\bar{p})^{-1}. \tag{4.8}$$

Proof. Multiply equation (2.13) by $\bar{p}$ and add the resulting equation to its conjugate. This yields

$$\varepsilon(\dot{p}\bar{p} + \dot{\bar{p}}p) = (1 - \varepsilon^2 p\bar{p})\psi(p + \bar{p})$$

or

$$-\frac{1}{\varepsilon}\frac{d}{dt}\log(1 - \varepsilon^2 p\bar{p}) = \psi(p + \bar{p}).$$

Integrating and forming the exponential function of both sides, we obtain (4.8).

Returning to (4.6) we see that

$$M(\infty, \varepsilon) = 2\begin{bmatrix} 2\varepsilon\bar{\hat{p}}(\infty, \varepsilon)\exp\left\{2\varepsilon\int_{-\infty}^{\infty}\psi\bar{p}\,ds\right\} & 1 \\ 1 & 2\varepsilon\hat{p}(\infty, \varepsilon)\exp\left\{2\varepsilon\int_{-\infty}^{\infty}\psi p\,ds\right\}\end{bmatrix} \tag{4.9}$$
$$+ O(\varepsilon^2|\hat{p}(\infty, \varepsilon)|^2).$$

Littlewood's theorem is now an immediate consequence of (4.4), (4.5), (4.7) and (4.9). Thus we have proved the following theorem.

THEOREM 4.1.

$$r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon) \sim 0 \quad \text{as } \varepsilon \to 0+.$$

Actually, we can obtain more precise information.

THEOREM 4.2.

$$r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon) = 2\varepsilon\,\mathrm{Re}\,\{(x_0 + iy_0)^2\hat{p}(\infty, \varepsilon)\}(1 + O(\varepsilon)). \tag{4.10}$$

Proof. By Theorem 3.2,

$$Z(0, \varepsilon) = S + O(\varepsilon).$$

Insert this into (4.2) and (4.4). The expressions (4.7) and (4.9) for $M(-\infty, \varepsilon)$ and $M(\infty, \varepsilon)$ when substituted into (4.4) lead to (4.10), after a short calculation.

5. The asymptotic form of $r^2(t, \varepsilon)$. With the help of the results of § 4 the uniform asymptotic expansion for $r^2(t, \varepsilon)$ itself, as $\varepsilon \to 0+$, can be explicitly calculated to any number of terms, by means of rational operations, differentiations and quadratures. The computations are tedious and probably of little interest beyond terms of order $O(\varepsilon)$. The approximation to this order is given in the next theorem.

THEOREM 5.1. *Let $(r(t, \varepsilon), \theta(t, \varepsilon))$ be the polar coordinates in the (x, y)-plane of the point $(x(t, \varepsilon), y(t, \varepsilon))$, where $(x(t, \varepsilon), y(t, \varepsilon))^T = z(t, \varepsilon)$ is the solution of the differential equation* (2.1) *with initial values $z(0, \varepsilon) = z_0 = (x_0, y_0)^T$ independent of ε. Set $r(0, \varepsilon) = r_0$, $\theta(0, \varepsilon) = \theta_0$. Then*

$$r^2(t, \varepsilon) = r_0^2\left\{1 - \frac{\varepsilon}{2}\left[\frac{\dot{\phi}(t)}{\phi^2(t)}\sin\left(2\theta_0 - \frac{2}{\varepsilon}\Phi(t)\right) - \frac{\dot{\phi}(0)}{\phi^2(0)}\sin 2\theta_0\right] + O(\varepsilon^2)\right\}. \tag{5.1}$$

Proof. Let the symbol "°" over a letter indicate that the quantity is to be taken at $t = 0$. From the proof of Lemma 3.1 one sees that, in particular,

$$\mathring{p} = p_0(0) + O(\varepsilon) = \frac{i}{4}\phi(0)/\phi^2(0) + O(\varepsilon).$$

Formula (4.2) and Theorem 3.2 imply that

$$c(\varepsilon) = \mathring{V}^{-1}\mathring{P}^{-1}S^{-1}z_0.$$

Hence,

$$r^2(t, \varepsilon) = z_0^T K(t, \varepsilon) z_0,$$

where $K = K(t, \varepsilon)$ is the matrix

$$K = (S^{-1})^T(\mathring{P}^{-1})^T\mathring{V}^{-1}M\mathring{V}^{-1}\mathring{P}^{-1}S^{-1}.$$

If this matrix is calculated to within terms of order $O(\varepsilon)$, and Lemma 4.1 as well as part (v) of Theorem 3.2 are used to simplify the expressions, one is led to formula (5.1) after some calculations, the details of which are omitted.

6. Remarks on the asymptotically leading term of $r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon)$. Theorem 4.2 reduces the asymptotic calculation of $r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon)$ to the determination of $\hat{p}(\infty, \varepsilon)$. This quantity satisfies the relation

(6.1) $$\hat{p}(\infty,, \varepsilon) = \varepsilon^{-1}\int_{-\infty}^{\infty} \exp\left[-\frac{2i}{\varepsilon}\Phi(s)\right]\psi(s)(1 - \varepsilon^2 p^2(s, \varepsilon))\, ds,$$

which follows from the differential (2.13) by the variation of parameters formula. The asymptotic information on $p(s, \varepsilon)$ given in Theorem 3.2 (iii) is probably not sufficient to evaluate that integral asymptotically. For the case that $\phi^2(t)$ is an *analytic* function satisfying certain additional conditions I have obtained more precise results, the details of which will be published elsewhere. A brief account of these methods is given below.

The function

(6.2) $$\xi = \Phi(t)$$

maps a neighborhood of the real axis of the t-plane conformally onto a strip containing the real axis of the ξ-plane. The function $p(t, \varepsilon)$, as a solution of an analytic differential equation, is analytic, and under suitable assumptions on the smallness of $\psi(t)$ at infinity—at least near the real axis—the validity of the expansion in Theorem 3.2 (iii) extends to a complex neighborhood of the real t-axis. Let

$$t = \chi(\xi)$$

be the inverse of the function (6.2). The integral in (6.1) can then be transformed into the integral

(6.3) $$\hat{p}(\infty, \varepsilon) = \frac{1}{2}\varepsilon^{-1}\int_{-\infty}^{\infty} \exp\left(-\frac{2i}{\varepsilon}\xi\right)\dot{\phi}(\chi(\xi))\phi^{-2}(\chi(\xi))[1 - \varepsilon^2 p^2(\chi(\xi), \varepsilon)]\, d\xi$$

along the real ξ-axis. The path of integration may now be replaced by a path $\xi = -ia + \eta$, $a > 0$, a constant, $-\infty < \eta < \infty$, as long as this path lies in the strip where the integrand in (6.3) is holomorphic and falls off sufficiently fast at infinity. In this way one proves that in such cases

$$\hat{p}(\infty, \varepsilon) = o(e^{-2a/\varepsilon}) \quad \text{as } \varepsilon \to 0+. \tag{6.4}$$

Let us now make the stronger assumption that the path can be shifted in this manner until the image in the ξ-plane of a *simple turning point* for the differential equation is met, i.e., a point $\xi = \xi_1 = \Phi(t_1)$, with

$$\phi^2(t_1) = 0, \qquad \frac{d}{dt}\phi^2|_{t=t_1} \neq 0. \tag{6.5}$$

More precisely, the path is replaced by a curve Γ consisting of the line $\operatorname{Im} \xi = \operatorname{Im} \xi_1$, except that the singularity at $\xi = \xi_1$ is avoided by a semicircle described in clockwise direction. The radius of this semicircle can then be shrunk to zero.

Near such a point a short calculation shows that

$$\tfrac{1}{2}\dot{\phi}(\chi(\xi))\phi^{-2}(\chi(\xi)) = \tfrac{1}{6}(\xi - \xi_1)^{-1}[1 + O((\xi - \xi_1)^{2/3})]. \tag{6.6}$$

Formula (6.3) becomes, for the new path of integration,

$$\hat{p}(\infty, \varepsilon) = \varepsilon^{-1} \exp\left(-\frac{2i}{\varepsilon}\xi_1\right) k(\varepsilon), \tag{6.7}$$

where

$$k(\varepsilon) = \int_\Gamma \exp\left[-\frac{2i}{\varepsilon}(\xi - \xi_1)\right] \frac{\dot{\phi}(\chi(\xi))}{2\phi^2(\chi(\xi))}[1 - \varepsilon^2 p^2(\chi(\xi), \varepsilon)]\, d\xi. \tag{6.8}$$

With the help of the theory of simple turning points one can calculate $p^2(\chi(\xi), \varepsilon)$ asymptotically and show that $k(\varepsilon)$ remains bounded, as $\varepsilon \to 0+$. In fact, an explicit expression for $\lim_{\varepsilon \to 0} k(\varepsilon)$ as an integral can be derived from that theory. The integral contains Bessel functions and is somewhat involved.

Formula (4.10) for the adiabatic invariant can now be replaced by the more informative result that

$$r^2(\infty) - r^2(-\infty) = r_0^2 \exp\left[\frac{2}{\varepsilon} \operatorname{Im} \Phi(t_1)\right](c(\varepsilon) + o(1)),$$

as $\varepsilon \to 0+$, with a function $c(\varepsilon)$ which can be calculated explicitly.

A simple example in which all assumptions of the foregoing argument are satisfied is

$$\phi^2(t) = 1 + (1 + 2e^{-t})^{-1}.$$

REFERENCES

[1] G. Backus, A. Lenard and R. Kulsrud, *Bemerkungen über die adiabatische Invarianz des Bahnmomentes geladener Teilchen*, Z. Naturforsch., 15a (1960), pp. 1007–1009.

[2] S. F. Feshchenko, N. I. Shkil' and L. D. Nikolenko, *Asymptotic Methods in the Theory of Linear Differential Equations*, American Elsevier, New York, 1967.

[3] F. HERTWECK AND A. SCHLÜTER, *Die "Adiabatische Invarianz" des magnetischen Bahnmomentes geladener Teilchen*, Z. Naturforsch., 12a (1957), pp. 844–849.

[4] G. KNORR AND D. PFIRSCH, *The variation of the adiabatic invariant of the harmonic oscillator*, Ibid., 21a (1966), pp. 688–693.

[5] J. E. LITTLEWOOD, *Lorentz's pendulum problem*, Ann. Physics, 21 (1963), pp. 233–242.

[6] Y. SIBUYA, *Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contenant un paramètre*, J. Fac. Sci. Univ. Tokyo Sect. I, 7 (1954), pp. 229–241.

[7] W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations*, John Wiley-Interscience, New York, 1965.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

CONNECTION FORMULAS FOR ASYMPTOTIC SOLUTIONS OF SECOND ORDER TURNING POINTS IN UNBOUNDED DOMAINS*

ANTHONY WING-KWOK LEUNG†

Abstract. Asymptotic expansions, as $\varepsilon \to 0^+$ or $x \to \infty$, for fundamental systems of solutions for $\varepsilon^2 u''(x) - p(x)u(x) = 0$ are obtainable by Evgrafov and Fedoryuk's method on unbounded canonical domains with neighborhoods deleted around turning points. When $p(x)$ is a polynomial, they also found a "lateral connection" formula for two fundamental systems of solutions with known asymptotic expansions which are valid in the interior of two different unbounded overlapping canonical regions with a common first order turning point at their boundaries. However, their connecting methods are not applicable to second order turning points. This paper employs techniques of Wasow and of R. Lee to find central connection formulas with a solution having a known asymptotic expansion in a bounded full neighborhood of a second order turning point. With the help of this result, lateral connection formulas are also established.

1. Introduction. In the paper by Evgrafov and Fedoryuk [1], a careful study of the turning point problem is given for the differential equation

$$\varepsilon^2 \frac{d^2 u}{dx^2} - p(x)u = 0.$$

Under certain conditions on $p(x)$, asymptotic expansions for $u(x)$ and $u'(x)$ are given that are valid as $x \to \infty$ and also as $\varepsilon \to 0^+$. These expansions are valid for the functions only in parts of the x-plane. Evgrafov and Fedoryuk also find connection formulas for different solutions with known expansions on different parts of the x-plane around a *simple* turning point x_0, i.e., a point where

$$p(x_0) = 0, \quad p'(x_0) \neq 0.$$

In the case when $p(x)$ has higher order zero, their connection formulas are not complete.

This paper finds such connection formulas for the case when $p(x)$ has a second order zero. It utilizes first the techniques and results of Wasow [2], R. Lee [5], and Hanson and Russell [3] to find connection formulas between solutions that have expansions in the sense of Evgrafov and Fedoryuk and solutions that have uniform expansions with respect to ε in a full bounded neighborhood of x_0 when $p(x)$ is a polynomial with a second order zero at x_0. Subsequently, connection formulas are found between solutions which have asymptotic expansions of Evgrafov and Fedoryuk's type on different unbounded domains of the x-plane. Such formulas are expressed as asymptotic series in terms of ε by making use of Weber (parabolic cylinder) functions. The principal result of the paper is Theorem 2.2 and Corollary 2.2.

2. Connection formulas between different solutions. Consider the differential equation

$$\varepsilon^2 u''(x) - p(x)u(x) = 0, \tag{2.1}$$

* Received by the editors August 30, 1971, and in revised form February 1, 1972.
† Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221.

where $p(x)$ is a polynomial with $p(x_0) = p'(x_0) = 0$ and $p''(x_0) \neq 0$. A Stokes curve for the equation is a curve on the x-plane proceeding from x_0, along which $\operatorname{Re} \int_{x_0}^{x} \sqrt{p(z)}\, dz = 0$. A canonical domain on the x-plane is a domain which is bounded by Stokes curves containing no turning points, i.e., zeros of $p(x)$, in its interior, and which is mapped by the function

$$\mathscr{E}(x) = \int_{x_0}^{x} \sqrt{p(z)}\, dz$$

onto the whole ξ-plane cut by a finite number of vertical rays each of which is unbounded. If all these vertical cuts start from the images of some turning points and extend to infinity in the same direction, then the canonical domain is called consistent; otherwise, it is called inconsistent. (Such terminology was introduced by Wasow [2].)

The purpose of this paper is to find the transition matrix from one fundamental system with known asymptotic expansion on one canonical domain to another such fundamental system on another canonical domain with a common second order zero x_0 for $p(x)$ at the boundary.

Let the four Stokes curves at x_0 be l_1, l_2, l_3, l_4, counting in the counterclockwise direction. By making a suitable choice of roots, the transformation

$$t(x) = \left[2 \int_{x_0}^{x} \sqrt{p(z)}\, dz \right]^{1/2}$$

is uniquely defined near $x = x_0$ and takes the curves l_1, l_2, l_3, l_4 respectively into the four rays $\arg t = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$. The function $t(x)$ is holomorphic and univalent in a neighborhood of $x = x_0$. It maps four subregions of the domains between l_1, l_2; l_2, l_3; l_3, l_4; l_4, l_1 holomorphically and univalently to simply-connected regions in the t-plane bounded by the image rays of the corresponding Stokes curves. Furthermore, the image of these regions and Stokes curves in the t-plane consists of the entire plane, except for a finite number of cuts which are analytic curves tending to infinity and starting at the images of turning points other than x_0. However, there may be choices of unbounded domains in the x-plane, bounded by curves starting at turning points other than x_0 and along which $\operatorname{Re} \mathscr{E}(x) = \text{const.}$, for the domain of definition of $t(x)$. After making a particular choice of four open regions: D_1 between l_1, l_2; D_2 between l_2, l_3; D_3 between l_3, l_4; and D_4 between l_4, l_1, together with the curves l_1, l_2, l_3, l_4 in the x-plane for the domain of $t(x)$, the inverse function $x(t)$ would be holomorphic and univalent on the entire t-plane, except on the cuts.

In matrix form the equation (2.1) is equivalent to

$$\varepsilon \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ p(x) & 0 \end{bmatrix} Y, \tag{2.2}$$

where

$$Y = \begin{bmatrix} u \\ \varepsilon(du/dx) \end{bmatrix}.$$

By the transformation

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & dt/dx \end{bmatrix} Y^*,$$

equation (2.2) is transformed into

$$\varepsilon \frac{dY^*}{dt} = \left\{ \begin{bmatrix} 0 & 1 \\ t^2 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{d^2t}{dx^2}\left(\dfrac{dt}{dx}\right)^{-2} \end{bmatrix} \right\} Y^* \tag{2.3}$$

whose coefficient matrix has a simpler leading part than in (2.2).

Let $q(t) = (dx/dt)^{1/2}$ be an arbitrary but fixed root, for t in the image $t(x)$ of the domain formed by our chosen regions D_1, D_2, D_3, D_4 together with the curves l_1, l_2, l_3, l_4. Let us employ the convention:

$$\arg t^\alpha = \alpha \arg t,$$

whenever we take roots in the t-plane.

THEOREM 2.1 (Evgrafov and Fedoryuk). *There exist solutions $u^\pm(x, \varepsilon)$ for* (2.1) *in $D_1 \cup D_2$ such that for $0 < \varepsilon < \delta_{\varepsilon_0}, x \in D_{\varepsilon_0}$, where*

$$D_{\varepsilon_0} = (D_1 \cup D_2) \cap \left(\bigcup_{i=0}^{k} \{x \mid |x - x_i| < \varepsilon_0\} \right)^C$$

($x_0, x_1, \cdots, x_k$ are all the turning points on the boundary of $D_1 \cup D_2$, and δ_{ε_0} is a constant depending on ε_0), the functions $u^\pm(x, \varepsilon)$ are expressible as follows:

$$u^\pm(x, \varepsilon) = p(x)^{-1/4} \hat{u}^\pm(x, \varepsilon) \exp \left\{ \pm \frac{1}{\varepsilon} \mathscr{E}(x) \right\}. \tag{2.4}$$

The functions $\hat{u}^\pm(x, \varepsilon)$ have asymptotic expansions

$$\hat{u}^\pm(x, \varepsilon) \sim \sum_{r=0}^{\infty} \hat{u}_r^\pm(x) \varepsilon^r, \qquad \hat{u}_0^\pm(x) \equiv 1 \quad \textit{as } \varepsilon \to 0^+ \tag{2.5}$$

or

$$x \to \infty \quad \textit{in } D_{\varepsilon_0} \quad \textit{with} \quad \operatorname{Re} \mathscr{E}(x) \to -\infty \quad \textit{for} \quad \hat{u}^+(x, \varepsilon)$$

or

$$x \to \infty \quad \textit{in } D_{\varepsilon_0} \quad \textit{with} \quad \operatorname{Re} \mathscr{E}(x) \to \infty \quad \textit{for} \quad \hat{u}^-(x, \varepsilon).$$

In a more precise sense, (2.5) means

$$\left| \hat{u}^\pm(x, \varepsilon) - \sum_{r=0}^{N} \hat{u}_r^\pm(x) \varepsilon^r \right| \leqq R_N^\pm(x) \varepsilon^{N+1} \tag{2.6}$$

for $x \in D_{\varepsilon_0}$. The function $R_N^\pm(x)$ is bounded in compact subsets of D_{ε_0}, and is of the order $O(|x|^{-((m+2)/2)(N+1)})$ uniformly in $0 < \varepsilon < \delta_{\varepsilon_0}$ as $x \to \infty$ in D_{ε_0} in such a manner that $\operatorname{Re} \mathscr{E}(x) \to \mp\infty$. The functions $\hat{u}^\pm(x)$ are holomorphic in $D_1 \cup D_2$ and $\hat{u}_r^\pm(x) = O(|x|^{-((m+2)/2)r})$ as $x \to \infty$ in D_{ε_0} in such a manner that $\operatorname{Re} \mathscr{E}(x) \to \mp\infty$. These asymptotic formulas may be formally differentiated. (Here m is the degree of $p(x)$. Each formula above combines two: take the upper or lower sign throughout.) In formula (2.4) we define $u^{\pm}(x, \varepsilon)$ uniquely by specifying

$$(2.7)\quad p^{-1/4}(x) = q(t)t^{-1/2}, \quad \frac{\pi}{4} \leqq \arg t \leqq \frac{7\pi}{4} \quad \text{for } t \in t(D_1 \cup l_2 \cup D_2 \cup l_3 \cup D_3).$$

(We shall make this choice for all subsequent formulas in this paper.)

Suppose $D_1 \cup l_2 \cup D_2$ is a consistent canonical domain. From $\mathscr{E}(D_1 \cup l_2 \cup D_2)$ delete circular neighborhoods of radius δ about the endpoints of the cuts, as well as sectors of central angle δ that have their vertices at the endpoints of the cuts and are bisected by the cuts. The resulting domain in the $\mathscr{E}$-plane may be denoted by $\mathscr{E}(D_1 \cup l_2 \cup D_2)_\delta$; the corresponding domains in the x-plane or t-plane may be denoted by $(D_1 \cup l_2 \cup D_2)_\delta$ or $t(D_1 \cup l_2 \cup D_2)_\delta$, respectively. For a consistent canonical domain, Theorem 2.1 can be improved so that then the asymptotic relations, as $x \to \infty$, are valid in D_1 and in D_2 for u^+ and for u^-.

COROLLARY 2.1 (Evgrafov and Fedoryuk). *Suppose $D_1 \cup l_2 \cup D_2$ is a consistent canonical domain. Then the solutions $u^{\pm}(x, \varepsilon)$ for* (2.1) *of Theorem* 2.1 *are expressible in the form* (2.4), *where*

$$(2.7\text{a})\quad \hat{u}^{\pm}(x, \varepsilon) \sim \sum_{r=0}^{\infty} \hat{u}_r^{\pm}(x)\varepsilon^r, \quad \hat{u}_0^{\pm}(x) \equiv 1 \quad \text{as } \varepsilon \to 0^+ \quad \text{or} \quad x \to \infty \quad \text{in } (D_1 \cup l_2 \cup D_2)_\delta$$

in the sense that

$$(2.7\text{b})\quad \left| \hat{u}^{\pm}(x, \varepsilon) - \sum_{r=0}^{N} \hat{u}_r^{\pm}(x)\varepsilon^r \right| \leqq C(N, \delta)(|x|^{-(m+2)/2}\varepsilon)^{N+1}$$

for $x \in (D_1 \cup l_2 \cup D_2)_\delta$. Here $C(N, \delta)$ is a constant. The functions $\hat{u}^{\pm}(x)$ are holomorphic in $D_1 \cup l_2 \cup D_2$ and

$$(2.7\text{c})\quad \hat{u}_r^{\pm}(x) = O(|x|^{-(m+2)r/2}) \quad \text{as } x \to \infty \quad \text{in } (D_1 \cup l_2 \cup D_2)_\delta.$$

These asymptotic formulas may be formally differentiated.

First, let us not require that $D_1 \cup l_2 \cup D_2$ be consistent. We then have u^+, u^- as stated in Theorem 2.1. When we put (u^+, u^-) in the first row and $(\varepsilon(du^+/dt), \varepsilon(du^-/dt))$ in the second row and express everything as functions of t, we obtain a fundamental matrix solution for (2.3). It has the form

$$(2.8)\quad Y_1^F(t, \varepsilon) = p(x(t))^{-1/4} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \hat{Y}^F(t, \varepsilon) \begin{bmatrix} \exp\left\{\frac{1}{2\varepsilon}t^2\right\} & 0 \\ 0 & \exp\left\{-\frac{1}{2\varepsilon}t^2\right\} \end{bmatrix}$$

for $t \in t(D_{\varepsilon_0})$, $0 < \varepsilon < \delta_{\varepsilon_0}$. The matrix $\hat{Y}^F(t, \varepsilon)$ has the properties:

$$(2.9)\quad \hat{Y}^F(t, \varepsilon) \sim \sum_{r=0}^{\infty} \hat{Y}_r(t)\varepsilon^r, \quad \hat{Y}_0(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

in the sense that, for $0 < \varepsilon < \delta_{\varepsilon_0}$, $t \in t(D_{\varepsilon_0})$,

$$(2.10)\quad \left| \hat{Y}^F(t, \varepsilon) - \sum_{r=0}^{N} \hat{Y}_r(t)\varepsilon^r \right| \leqq C_N(t)\varepsilon^{N+1}.$$

$C_N(t)$ is a 2×2 matrix function of t, and is bounded in compact subsets of $t(D_{\varepsilon 0})$; the $\hat{Y}_r(t)$ are holomorphic in $t(D_{\varepsilon_0})$. The absolute value sign and inequality relation apply to each matrix element. The first columns of $\hat{Y}_r(t)$ and $C_N(t)$ are of the order $O(|t|^{-2r})$ and $O(|t|^{-2(N+1)})$ respectively as $t \to \infty$ in $t(D_1)$ with $\operatorname{Re} \mathscr{E}(x(t)) \to -\infty$, uniformly for $0 < \varepsilon < \delta_{\varepsilon_0}$ (for example, $t \to \infty$ at an angle θ, $\pi/4 < \theta < 3\pi/4$). The second columns of $Y_r(t)$ and $C_N(t)$ are of the order $O(|t|^{-2r})$ and $O(|t|^{-2(N+1)})$ respectively as $t \to \infty$ in $t(D_2)$ with $\operatorname{Re} \mathscr{E}(x(t)) \to +\infty$, uniformly for $0 < \varepsilon < \delta_{\varepsilon_0}$ (for example, $t \to \infty$ at an angle θ, $3\pi/4 < \theta < 5\pi/4$). In case $D_1 \cup l_2 \cup D_2$ is a consistent canonical domain, then we have $Y_1^F(t, \varepsilon)$ expressible in the form (2.8), where the matrix $\hat{Y}^F(t, \varepsilon)$ has an asymptotic expansion of the form (2.9) in the sense that for $\varepsilon > 0$ sufficiently small, $t \in t(D_1 \cup l_2 \cup D_2)_\delta$,

$$\left|\hat{Y}^F(t, \varepsilon) - \sum_{r=0}^{N} \hat{Y}_r(t)\varepsilon^r\right| \leqq C_2(N, \delta)(|t|^{-2}\varepsilon)^{N+1}.$$

Here the $C_2(N, \delta)$ are constants and $\hat{Y}_r(t)$ are holomorphic in $t(D_1 \cup l_2 \cup D_2)_\delta$.

Define

$$\tilde{D}_{\varepsilon 0} = (D_2 \cup D_3) \cap \left(\bigcup_{i=0}^{\bar{k}} \{x \mid |x - \bar{x}_i| < \varepsilon_0\}\right)^C,$$

where $\bar{x}_0 = x_0, \bar{x}_1, \cdots, \bar{x}_{\bar{k}}$ are all the other turning points on the boundary of $D_2 \cup D_3$. Assume δ_{ε_0} has been chosen sufficiently small. Evgrafov and Fedoryuk's theorem still applies in $D_2 \cup D_3$ [1]. Analogously, there exist solutions $\tilde{u}^{\pm}(x, \varepsilon)$ for (2.1) in $D_2 \cup D_3$ such that for $0 < \varepsilon < \delta_{\varepsilon_0}$, $x \in \tilde{D}_{\varepsilon_0}$, $\tilde{u}^{\pm}(x, \varepsilon)$ have representations (2.4), (2.5), (2.6). We only have to replace $u^{\pm}, \hat{u}^{\pm}, R_N^{\pm}, \hat{u}_r^{\pm}, D_{\varepsilon_0}$ and $D_1 \cup D_2$ respectively by $\tilde{u}^{\pm}$, $\hat{\tilde{u}}^{\pm}$, $\tilde{R}_N^{\pm}$, $\hat{\tilde{u}}_r^{\pm}$, $\tilde{D}_{\varepsilon_0}$ and $D_2 \cup D_3$ in the corresponding formulas and statements. However, $\mathscr{E}(x)$ in (2.4) and $p(x)^{-1/4}$ in (2.7) are continued from $D_1 \cup D_2$. Put $(\tilde{u}^-, \tilde{u}^+)$ into the first row and $(\varepsilon(d\tilde{u}^-/dt), \varepsilon(d\tilde{u}^+/dt))$ into the second row of a matrix $Y_2^F(t, \varepsilon)$. Then we see that it is a fundamental matrix solution of (2.3) of the form

$$Y_2^F(t, \varepsilon) = p(x(t))^{-1/4}\begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix} \hat{\tilde{Y}}^F(t, \varepsilon) \begin{bmatrix} \exp\left\{-\dfrac{1}{2\varepsilon}t^2\right\} & 0 \\ 0 & \exp\left\{\dfrac{1}{2\varepsilon}t^2\right\} \end{bmatrix} \tag{2.11}$$

for $t \in t(\tilde{D}_{\varepsilon_0})$, $0 < \varepsilon < \delta_{\varepsilon_0}$. The matrix $\hat{\tilde{Y}}^F$ satisfies (2.9) with " $\hat{\tilde{}}$ " replacing "$\hat{}$". In analogy to (2.10), we have, for $t \in t(\tilde{D}_{\varepsilon_0})$, $0 < \varepsilon < \delta_{\varepsilon_0}$,

$$\left|\hat{\tilde{Y}}^F(t, \varepsilon) - \sum_{r=0}^{N} \hat{\tilde{Y}}_r(t)\varepsilon^r\right| \leqq \tilde{C}_N(t)\varepsilon^{N+1}, \tag{2.12}$$

where the first columns of $\hat{\tilde{Y}}_r$ and $\tilde{C}_N$ are of the corresponding orders as $t \to \infty$ in $t(D_2)$ with $\operatorname{Re} \mathscr{E}(x(t)) \to +\infty$ and second columns similarly as $t \to \infty$ in $t(D_3)$ with $\operatorname{Re} \mathscr{E}(x(t)) \to -\infty$. To see that we should have $-t$ at the lower right corner of the first matrix in (2.11) we check that $\mathscr{E} = \frac{1}{2}t^2$, $-d\mathscr{E}/dt = -t$. Recall from (2.9) that

$$\hat{Y}_0(t) = \hat{\tilde{Y}}_0(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We shall now find the connection between Y_1^F and Y_2^F by first finding their relationship with solutions having known expansions in a full neighborhood of the turning point $t = 0$. This is done by a technique similar to that of Wasow [2].

LEMMA 2.1. *There exists a holomorphic transformation in a neighborhood of zero in the t-plane* $Y^* = P^L(t, \varepsilon)Z$ *where* $P^L(t, \varepsilon)$ *has a uniform asymptotic expansion for* $|t| \leqq t_0$, t_0 *sufficiently small, that takes the differential equation* (2.3) *into the form*

$$\varepsilon \frac{dZ}{dt} = \begin{bmatrix} 0 & 1 \\ t^2 + \mu(\varepsilon)\varepsilon & 0 \end{bmatrix} Z, \tag{2.13}$$

where $\mu(\varepsilon) \sim c\varepsilon$ *for some constant* c, *as* $\varepsilon \to 0^+$.

Proof. Put

$$g(t) = \frac{d^2t}{dx^2}\left(\frac{dt}{dx}\right)^{-2} = -\frac{d^2x}{dt^2}\left(\frac{dx}{dt}\right)^{-1}.$$

Direct calculation shows that

$$\frac{dq}{dt} = -\tfrac{1}{2}g(t)q(t).$$

If we let

$$Y^* = \begin{bmatrix} q(t) & 0 \\ -\frac{1}{2}\varepsilon g q & q(t) \end{bmatrix} Y^{**}, \tag{2.14}$$

then

$$\varepsilon\frac{dY^*}{dt} = \varepsilon \begin{bmatrix} -\frac{1}{2}gq & 0 \\ -\frac{1}{2}\varepsilon q\left(\frac{dg}{dt} - \frac{g^2}{2}\right) & -\frac{1}{2}gq \end{bmatrix} Y^{**} + \varepsilon \begin{bmatrix} q(t) & 0 \\ -\frac{1}{2}\varepsilon g q & q(t) \end{bmatrix} \frac{dY^{**}}{dt}.$$

Using the original equation (2.3), we obtain

$$\begin{aligned} \varepsilon\frac{dY^{**}}{dt} = q^{-1}(t) \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\varepsilon g & 1 \end{bmatrix} \Bigg\{ & \begin{bmatrix} 0 & 1 \\ t^2 & -\varepsilon g \end{bmatrix} q(t) \begin{bmatrix} 1 & 0 \\ -\frac{\varepsilon}{2} g & 1 \end{bmatrix} \\ & - \varepsilon q(t) \begin{bmatrix} -\frac{g}{2} & 0 \\ -\frac{1}{2}\varepsilon\left(\frac{dg}{dt} - \frac{g^2}{2}\right) & -\frac{g}{2} \end{bmatrix} \Bigg\} Y^{**} \\ = & \begin{bmatrix} 0 & 1 \\ t^2 + \frac{\varepsilon^2}{2}\left(\frac{dg}{dt} + \frac{g^2}{2}\right) & 0 \end{bmatrix} Y^{**}. \end{aligned} \tag{2.15}$$

Suppose $dg/dt + g^2/2 = a_0 + a_1 t + a_2 t^2 + \cdots$ in a neighborhood of 0. Then Hanson and Russell [3] show that (2.15) can be formally transformed into

$$\varepsilon\frac{dZ}{dt} = \begin{bmatrix} 0 & 1 \\ t^2 + \frac{\varepsilon^2}{2}a_0 & 0 \end{bmatrix} Z \tag{2.16}$$

by the transformation

$$Y^{**} = \begin{bmatrix} q_1(t,\varepsilon) & q_2(t,\varepsilon) \\ \varepsilon\dfrac{dq_1}{dt} + q_2\left(t + \dfrac{a_0\varepsilon}{2}\right) & q_1 + \varepsilon\dfrac{dq_2}{dt} \end{bmatrix} Z,$$

where $q_1(t, \varepsilon)$, $q_2(t, \varepsilon)$ are formal power series of the forms

$$q_1(t,\varepsilon) = 1 + \sum_{k=1}^{\infty} q_{1k}(t)\varepsilon^k,$$

$$q_2(t,\varepsilon) = \sum_{k=1}^{\infty} q_{2k}(t)\varepsilon^k$$

with q_{1k}, q_{2k} holomorphic for all k for $|t| \leqq t_0$, t_0 sufficiently small. The functions q_{1k}, q_{2k} satisfy some recursive formulas given in [3]. Furthermore, if $\gamma > 0$ is any positive small number, then, by Sibuya [4], there actually exists, in each of the sectors

$$S_j = \left\{ t \mid |t| \leqq t_0, \frac{-3 + 2(j-1)}{4}\pi + \gamma \leqq \arg t \leqq \frac{1 + 2(j-1)}{4}\pi - \gamma \right\}, \qquad j = 1, 2, 3, 4,$$

a holomorphic transformation

$$Y^{**} = P_j(t,\varepsilon)Z$$

that takes (2.15) into (2·16), where

$$P_j(t,\varepsilon) \sim \begin{bmatrix} q_1 & q_2 \\ \varepsilon\dfrac{dq_1}{dt} + q_2\left(t^2 + \dfrac{a_0\varepsilon}{2}\right) & q_1 + \varepsilon\dfrac{dq_2}{dt} \end{bmatrix}$$

as $\varepsilon \to 0^+$, in S_j, and P_j is holomorphic for $|t| \leqq t_0$.

A theorem by Lee [5] shows that the existence of such a P_j in each of these sectors S_j, $j = 1, 2, 3, 4$, suffices to guarantee the existence of a holomorphic transformation

$$Y^{**} = Q(t,\varepsilon)Z \tag{2.17}$$

on a disc with center $t = 0$ that takes (2.15) into (2.13), with $\mu(\varepsilon) \sim (a_0/2)\varepsilon$ as $\varepsilon \to 0^+$, and

$$Q(t,\varepsilon) \sim \begin{bmatrix} q_1 & q_2 \\ \varepsilon\dfrac{dq_1}{dt} + q_2\left(t^2 + \dfrac{a_0\varepsilon}{2}\right) & q_1 + \varepsilon\dfrac{dq_2}{dt} \end{bmatrix}$$

as $\varepsilon \to 0^+$, uniformly for $|t| \leqq t_0$. On combining transformations (2.14) and (2.17) the transformation $Y^* = P^L(t, \varepsilon)Z$ is obtained, where

$$P^L(t,\varepsilon) = \begin{bmatrix} q(t) & 0 \\ -\tfrac{1}{2}\varepsilon q(t) & q(t) \end{bmatrix} Q(t,\varepsilon).$$

It transforms (2.3) into (2.13) with $\mu(\varepsilon) \sim (a_0/2)\varepsilon$. Therefore, $P^L(t, \varepsilon)$ has, for $|t| \leqq t_0$, the uniform asymptotic expansion

$$P^L(t, \varepsilon) = q(t)I + \sum_{r=1}^{\infty} P_r^L(t)\varepsilon^r \tag{2.18}$$

with $P_r^L(t)$ holomorphic for each r. This completes the proof of the lemma.

Next we proceed to investigate the solutions of (2.13). Consider the equation

$$\varepsilon^2 \frac{d^2 z}{dt^2} - (t^2 + \varepsilon\mu(\varepsilon))z = 0. \tag{2.19}$$

By the stretching $s = \sqrt{2}t$, (2.19) is transformed into

$$\varepsilon^2 \frac{d^2 z}{ds^2} - \left(\frac{s^2}{4} + \varepsilon\frac{\mu(\varepsilon)}{2}\right)z = 0. \tag{2.20}$$

From the properties of the parabolic functions (see [5], [6]), it follows that (2.13) has the four vector solutions

$$\begin{bmatrix} z_j \\ \varepsilon \dfrac{dz_j}{ds} \end{bmatrix} = \begin{bmatrix} U\left(\dfrac{\mu}{2} e^{-2i\varphi_j}, \varepsilon^{-1/2} s\, e^{-i\varphi_j}\right) \\ \varepsilon^{1/2} e^{-i\varphi_j} U'\left(\dfrac{\mu}{2} e^{-2i\varphi_j}, \varepsilon^{-1/2} s\, e^{-i\varphi_j}\right) \end{bmatrix}, \tag{2.21}$$

where $\varphi_j = [\pi(j-1)]/2$, $j = 1, 2, 3, 4$. $U(a, x)$ is the standard parabolic cylinder function with the expansion:

$$U(a, x) \sim e^{-x^2/4} x^{-a-1/2}\left\{1 - \frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2x^2} + \frac{(a+\frac{1}{2})(a+\frac{3}{2})(a+\frac{5}{2})(a+\frac{7}{2})}{2\cdot 4x^4} - \cdots\right\} \tag{2.22}$$

for $|\arg x| < \pi/2$ as $x \to \infty$, $|x| \gg |a|$.

Furthermore, the functions z_j satisfy the relation (see [5])

$$z_2(s, \varepsilon, \mu) = \frac{1}{a_1(\mu)b_2(\mu) - a_2(\mu)b_1(\mu)}(-b_1(\mu)z_3(s, \varepsilon, \mu) + a_1(\mu)z_4(s, \varepsilon, \mu)), \tag{2.23}$$

where

$$a_1(\mu) = -\exp[-2\pi i(\tfrac{1}{4}\mu - \tfrac{1}{4})], \qquad a_2(\mu) = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + \mu/2)}\exp[i\pi(\tfrac{1}{4}\mu - \tfrac{1}{4})],$$

$$b_1(\mu) = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - \mu/2)}\exp[i\pi(\tfrac{1}{4}\mu + \tfrac{1}{4})], \qquad b_2(\mu) = -\exp[2i\pi(\tfrac{1}{4}\mu + \tfrac{1}{4})].$$

Expressing z_2, z_3, z_4 as functions of t and ε, we use (2.21), (2.22) to find:

$$\begin{aligned} z_2(t, \varepsilon) &= e^{t^2/(2\varepsilon)} e^{\pi i/4} \varepsilon^{1/4} t^{-1/2} 2^{-1/4}\{1 + O(\varepsilon \log \varepsilon)\} \quad \text{for } 0 < \arg t < \pi, \\ z_3(t, \varepsilon) &= e^{-t^2/(2\varepsilon)} e^{\pi i/2} \varepsilon^{1/4} t^{-1/2} 2^{-1/4}\{1 + O(\varepsilon \log \varepsilon)\} \quad \text{for } \pi/2 < \arg t < \tfrac{3}{2}\pi, \\ z_4(t, \varepsilon) &= e^{t^2/(2\varepsilon)} e^{3\pi i/4} \varepsilon^{1/4} t^{-1/2} 2^{-1/4}\{1 + O(\varepsilon \log \varepsilon)\} \quad \text{for } \pi < \arg t < 2\pi \end{aligned} \tag{2.24}$$

for t bounded away from zero and $\varepsilon \to 0^+$. The restrictions of the arguments of t in (2.24) arise from the restriction $|\arg x| < \pi/2$ for the asymptotic formula

(2.22). The function $O(\varepsilon \log \varepsilon)$ arises from the factor $x^{-a-1/2}$ in (2.22), where in this case a is a function of $\mu(\varepsilon) \sim (a_0/2)\varepsilon$ ($a = \mu e^{-2i\varphi_j}/2$) and $x = \varepsilon^{-1/2} s e^{-i\varphi_j} = \varepsilon^{-1/2}\sqrt{2}t e^{-i\varphi_j}$, $j = 2, 3, 4$.

We are now ready to investigate the structure of a fundamental matrix solution of (2.13) whose first row is a solution of (2.19):

$$V^L(t,\varepsilon) = \begin{bmatrix} e^{\pi i/4} z_2(t,\varepsilon) & z_3 \\ e^{\pi i/4}\varepsilon \dfrac{dz_2}{dt} & \varepsilon \dfrac{dz_3}{dt} \end{bmatrix}. \tag{2.25}$$

It has the asymptotic form

$$\begin{aligned} V^L(t,\varepsilon) &= e^{\pi i/2}\varepsilon^{1/4}2^{-1/4}t^{-1/2} \\ &\cdot \begin{bmatrix} \exp\left\{\dfrac{1}{2\varepsilon}t^2\right\}(1 + O(\varepsilon\log\varepsilon)) & \exp\left\{-\dfrac{1}{2\varepsilon}t^2\right\}(1 + O(\varepsilon\log\varepsilon)) \\ \exp\left\{\dfrac{1}{2\varepsilon}t^2\right\}t(1 + O(\varepsilon\log\varepsilon)) & \exp\left\{-\dfrac{1}{2\varepsilon}t^2\right\}t(-1 + O(\varepsilon\log\varepsilon)) \end{bmatrix} \end{aligned} \tag{2.26}$$

for $\pi/2 < \arg t < \pi$, $|t| > \delta$, $\varepsilon \to 0^+$.

In order to calculate the expansion for $\pi < \arg t < 3\pi/2$ of V^L we have to use formula (2.23) and expand z_3, z_4 to calculate z_2. We find that

$$\begin{aligned} z_2(t,\varepsilon) = \frac{\Gamma(\frac{1}{2})^2\varepsilon^{1/4}}{\Gamma(\frac{1}{2}) - 2\pi}\frac{\varepsilon^{1/4}}{2^{1/4}t^{1/2}}\Bigg[&\left(\frac{e^{\pi i/2}\, e^{-\pi i/4}\sqrt{2\pi}}{\Gamma(\frac{1}{2})} + O(\varepsilon\log\varepsilon)\right)(e^{-t^2/(2\varepsilon)}) \\ &- (e^{3\pi i/4}\, e^{-\pi i/2} + O(\varepsilon\log\varepsilon))\, e^{t^2/(2\varepsilon)}\Bigg] \end{aligned}$$

for $\pi < \arg t < 3\pi/2$. Thus V^L has the form

$$V^L(t,\varepsilon) = \begin{bmatrix} v_{11}(t,\varepsilon) & v_{12}(t,\varepsilon) \\ v_{21}(t,\varepsilon) & v_{22}(t,\varepsilon) \end{bmatrix}, \tag{2.27}$$

where for $\pi < \arg t < 3\pi/2$, $|t| > \delta$, $\varepsilon \to 0^+$,

$$\begin{aligned} v_{11}(t,\varepsilon) &= -\varepsilon^{1/4}2^{-1/4}t^{-1/2}[e^{-t^2/(2\varepsilon)}(\sqrt{2}\, e^{\pi i/2} + O(\varepsilon\log\varepsilon)) \\ &\qquad + e^{t^2/(2\varepsilon)}(e^{\pi i/2} + O(\varepsilon\log\varepsilon))], \\ v_{12}(t,\varepsilon) &= e^{\pi i/2}\varepsilon^{1/4}2^{-1/4}t^{-1/2}e^{-t^2/(2\varepsilon)}(1 + O(\varepsilon\log\varepsilon)), \\ v_{21}(t,\varepsilon) &= -\varepsilon^{1/4}2^{-1/4}t^{-1/2}\cdot t[e^{-t^2/(2\varepsilon)}(-e^{\pi i/2}\sqrt{2} + O(\varepsilon\log\varepsilon)) \\ &\qquad + e^{t^2/(2\varepsilon)}(e^{\pi i/2} + O(\varepsilon\log\varepsilon))], \\ v_{22}(t,\varepsilon) &= e^{\pi i/2}\varepsilon^{1/4}2^{-1/4}t^{-1/2}t\, e^{-t^2/(2\varepsilon)}(-1 + O(\varepsilon\log\varepsilon)). \end{aligned}$$

LEMMA 2.2. *$Y_1^F(t,\varepsilon) = (P^L V^L)(t,\varepsilon)C(\varepsilon)$ for $|t| \leqq t_0$ and $Y_2^F(t,\varepsilon) = (P^L V^L)(t,\varepsilon)\tilde{C}(\varepsilon)$ for $|t| \leqq t_0$, where*

$$C(\varepsilon) = 2^{1/4}e^{-\pi i/2}\varepsilon^{-1/4}\begin{bmatrix} 1 + O(\varepsilon\log\varepsilon) & \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) & 1 + O(\varepsilon\log\varepsilon) \end{bmatrix} \tag{2.28}$$

with $\alpha_1(\varepsilon) \sim 0, \alpha_2(\varepsilon) \sim 0$ *as* $\varepsilon \to 0^+$ *and*

$$(2.29) \quad \tilde{C}(\varepsilon) = 2^{1/4}\varepsilon^{-1/4}\begin{bmatrix} \beta_1(\varepsilon)O(\varepsilon \log \varepsilon) & -i + O(\varepsilon \log \varepsilon) \\ \beta_2(\varepsilon)O(\varepsilon \log \varepsilon) + (-i + O(\varepsilon \log \varepsilon)) & -\sqrt{2}i + O(\varepsilon \log \varepsilon) \end{bmatrix}$$

with $\beta_1(\varepsilon) \sim 0, \beta_2(\varepsilon) \sim 0$ *as* $\varepsilon \to 0^+$.

Proof. Recall the transformation $Y^* = P^L Z$ in Lemma 2.1. We have an expansion for P^L at $|t| \leqq t_0$. As long as $\delta > 0$ is small, $P^L V^L$ is a solution to (2.3) with known expansions in sectors: $\pi/2 < \arg t < \pi$, $\delta < |t| \leqq t_0$; and $\pi < \arg t < 3\pi/2$, $\delta < |t| \leqq t_0$.

We have $C(\varepsilon) = [V^L]^{-1}[P^L]^{-1}Y_1^F$. For the calculations of $C(\varepsilon)$ from that formula we can choose any t with $\pi/2 < \arg t < \pi$. Using (2.18), (2.26) and (2.8) we conclude that

$$C(\varepsilon) = 2^{1/2}e^{-\pi i/2}\varepsilon^{-1/4}\begin{bmatrix} 1 + O(\varepsilon \log \varepsilon) & e^{-t^2/\varepsilon}O(\varepsilon \log \varepsilon) \\ e^{t^2/\varepsilon}O(\varepsilon \log \varepsilon) & 1 + O(\varepsilon \log \varepsilon) \end{bmatrix}.$$

Take $t = t_0 e^{5\pi i/6}$; then $e^{-t^2/\varepsilon} \sim 0$ as $\varepsilon \to 0^+$. If we take $t = t_0 e^{4\pi i/7}$, then $e^{t^2/2} \sim 0$ as $\varepsilon \to 0^+$. Similarly, we use (2.18), (2.27) and (2.11) to evaluate

$$\tilde{C}(\varepsilon) = [V^L]^{-1}[P^L]^{-1}Y_2^F.$$

For this calculation we can choose some t with $\pi < \arg t < 3\pi/2$. We arrive at

$$\tilde{C}(\varepsilon) = \frac{-2^{1/4}\varepsilon^{-1/4}}{2e^{\pi i/2} + O(\varepsilon \log \varepsilon) + e^{-t^2/\varepsilon}O(\varepsilon \log \varepsilon)} \cdot \begin{bmatrix} e^{-t^2/\varepsilon}O(\varepsilon \log \varepsilon) & -2 + O(\varepsilon \log \varepsilon) \\ -e^{-t^2/\varepsilon}O(\varepsilon \log \varepsilon) - 2 + O(\varepsilon \log \varepsilon) & -2\sqrt{2} + O(\varepsilon \log \varepsilon) - e^{t^2/\varepsilon}O(\varepsilon \log \varepsilon) \end{bmatrix}.$$

To evaluate the first row and the term at the first column and second row, take $t = t_0 e^{7\pi i/6}$, then $e^{-t^2/\varepsilon} = \exp\{-(1/\varepsilon)t_0^2 e^{7\pi i/3}\} \sim 0$ as $\varepsilon \to 0^+$. To evaluate the term at the second row and second column, take $t = t_0 e^{5\pi i/4}$. Then $e^{-t^2/\varepsilon} = e^{-(1/\varepsilon)t_0^2 i}$ and $e^{+t^2/\varepsilon} = e^{(1/\varepsilon)t_0^2 i}$, both of which have absolute value 1. Thus we have

$$\tilde{C}(\varepsilon) = \varepsilon^{-1/4}2^{1/4} \cdot \begin{bmatrix} \exp\left\{\dfrac{-t_0^2}{\varepsilon}e^{7\pi i/3}\right\}(O(\varepsilon \log \varepsilon)) & -i + O(\varepsilon \log \varepsilon) \\ \exp\left\{\dfrac{-t_0^2}{\varepsilon}e^{7\pi i/3}\right\}(O(\varepsilon \log \varepsilon)) + (-i + O(\varepsilon \log \varepsilon)) & -\sqrt{2}i + O(\varepsilon \log \varepsilon) \end{bmatrix}$$

which is of the form stated.

THEOREM 2.2. $Y_2^F(t, \varepsilon) = Y_1^F(t, \varepsilon)N(\varepsilon)$, *where*

$$N(\varepsilon) = \begin{bmatrix} c_{11}(\varepsilon) & 1 + c_{12}(\varepsilon) + O(\varepsilon \log \varepsilon) \\ 1 + c_{21}(\varepsilon) + O(\varepsilon \log \varepsilon) & \sqrt{2} + c_{22}(\varepsilon) + O(\varepsilon \log \varepsilon) \end{bmatrix}.$$

The functions $c_{11}(\varepsilon), c_{12}(\varepsilon), c_{21}(\varepsilon), c_{22}(\varepsilon)$ *are all* ~ 0 *as* $\varepsilon \to 0^+$.

Proof. Use Lemma 2.2. We see that $Y_2^F(t, \varepsilon) = Y_1^F(t, \varepsilon)C^{-1}(\varepsilon)\tilde{C}(\varepsilon)$. Then compute $N(\varepsilon)$ directly from the formula $N(\varepsilon) = C^{-1}(\varepsilon)\tilde{C}(\varepsilon)$ together with (2.28) and (2.29).

In order to find series expansions for the functions of order $O(\varepsilon \log \varepsilon)$ as $\varepsilon \to 0^+$ in the matrix $N(\varepsilon)$ we have to make use of the series expansions of $P^L(t, \varepsilon)$, $V^L(t, \varepsilon)$, $Y_1^F(t, \varepsilon)$ and $Y_2^F(t, \varepsilon)$. The asymptotic series for $P^L(t, \varepsilon)$ is given by Hanson and Russell [3]. We can use (2.21), (2.22), (2.23) to find series expansions for $V^L(t, \varepsilon)$, as we have done for the first term. For the expansion of $Y_1^F(t, \varepsilon)$ and $Y_2^F(t, \varepsilon)$ we can use Evgrafov and Fedoryuk's paper [1].

COROLLARY 2.2. *Suppose $D_1 \cup l_2 \cup D_2$ is a consistent canonical domain. Then doubly asymptotic series for $\tilde{u}^+$ and $\tilde{u}^-$ as $\varepsilon \to 0^+$ or $x \to \infty$ in all of $(D_1 \cup l_2 \cup D_2)_\delta$ can be obtained from the formula*

$$[\tilde{u}^-(x, \varepsilon), \tilde{u}^+(x, \varepsilon)] = [u^+(x, \varepsilon), u^-(x, \varepsilon)]N(\varepsilon).$$

Proof. Apply Corollary 2.1 and Theorem 2.2.

The last corollary enables us to find the behavior, in the consistent canonical domain $D_1 \cup l_2 \cup D_2$, of the solutions $\tilde{u}^+(x, \varepsilon)$ and $\tilde{u}^-(x, \varepsilon)$ which are subdominant in D_3 and D_2, respectively.

3. An example. Let $p(x) = x^2(x - 1)^2$ in (2.1). Careful analysis reveals that the patterns of Stokes curves are as illustrated in Fig. 1. The turning points are at 0 and 1 and both are of second order. The lines $l_i, s_i, i = 1, 2, 3, 4$, are Stokes curves; $D_i, R_i, i = 2, 3, 4$, and Ω are open connected unbounded regions with the Stokes curves as boundaries.

To fix the ideas, we choose the turning point $x_0 = 0$. Let $D_1 = R_4 \cup s_1 \cup \Omega$. Consider the region $D_1 \cup l_2 \cup D_2$. We choose

$$\xi(x) = \int_0^x z(z - 1)\, dz = \frac{x^3}{3} - \frac{x^2}{2}.$$

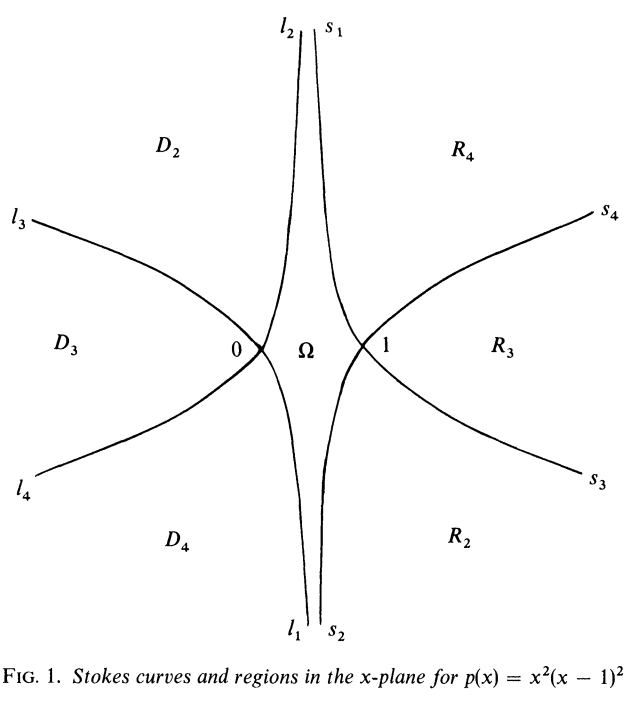

FIG. 1. *Stokes curves and regions in the x-plane for $p(x) = x^2(x - 1)^2$*

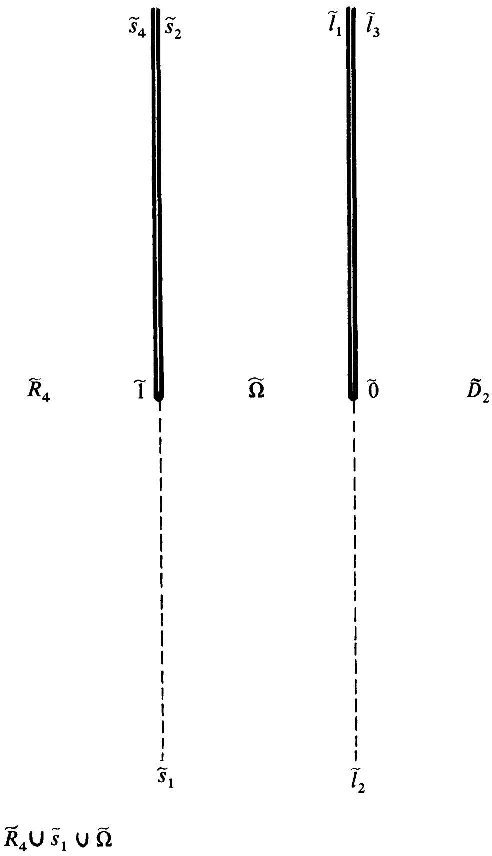

FIG. 2. *The image of* $D_1 \cup l_2 \cup D_2$ *on the* ξ*-plane*

The image of $D_1 \cup l_2 \cup D_2$ and its boundaries in the ξ-plane is shown in Fig. 2. The corresponding parts in the x-plane and ξ-plane are designated by the same letters, with a tilde over the images on the ξ-plane. The solid lines in the ξ-plane are cuts. We see that $D_1 \cup l_2 \cup D_2$ is a consistent canonical domain. The corresponding diagram of Fig. 1 (with $R_2 \cup s_3 \cup R_3$ deleted) on the ($t = [2\xi(x)]^{1/2}$)-plane is shown in Fig. 3.

We apply Corollary 2.1 on the region $D_1 \cup l_2 \cup D_2$. There are two independent solutions of (2.1),

$$u^{\pm}(x, \varepsilon) = [p(x)]^{-1/4}\hat{u}^{\pm}(x, \varepsilon) \exp\left\{ \pm\frac{1}{\varepsilon}\xi(x)\right\},$$

with properties given by formulas (2.7 a, b, c) for $x \in (D_1 \cup l_2 \cup D_2)_\delta$, $0 < \varepsilon < \delta_{\varepsilon 0}$. If we choose the root $q(0) = [(dx/dt)(0)]^{1/2} = e^{3\pi i/4}$, the root $[p(x)]^{-1/4}$ for small $x > 0$ should be chosen to be $|p(x)^{-1/4}|i$, by convention (2.7). Referring to Fig. 2 we see that $u^+(x, \varepsilon)$. $u^-(x, \varepsilon)$ are subdominant in D_1 and D_2 respectively.

Consider the region $D_2 \cup l_3 \cup D_3$. The image of this region in the ξ-plane is shown in Fig. 4. Again, applying Corollary 2.1 on the consistent canonical domain,

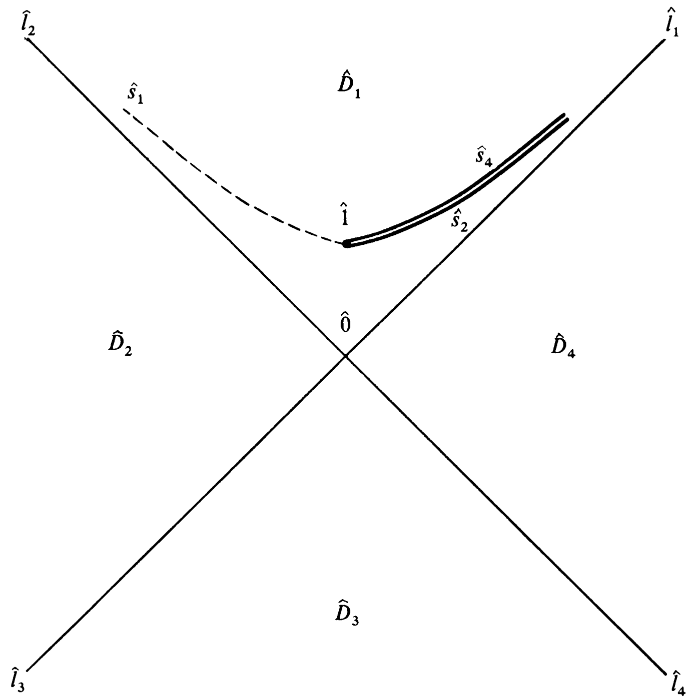

FIG. 3. *The image of Fig. 1 (with $R_2 \cup s_3 \cup R_3$ deleted) on the t-plane. (The corresponding parts are designated with the same letters, with a hat over the images.)*

we have two independent solutions

$$\tilde{u}^{\pm}(x, \varepsilon) = [p(x)]^{-1/4}\hat{\tilde{u}}^{\pm}(x, \varepsilon) \exp\left\{\pm\frac{1}{\varepsilon}\xi(x)\right\}$$

with properties described for $x \in (D_2 \cup l_3 \cup D_3)_\delta$, $0 < \varepsilon < \delta_{\varepsilon 0}$. From Fig. 4 we see that $\tilde{u}^+(x, \varepsilon)$ and $\tilde{u}^-(x, \varepsilon)$ are subdominant in D_3 and D_2 respectively.

To find the behavior of $\tilde{u}^+$ and $\tilde{u}^-$ in the region $(D_1 \cup l_2 \cup D_2)_\delta$, we apply Corollary 2.2. Thus

$$\begin{aligned}\tilde{u}^+(x, \varepsilon) &= [1 + c_{12}(\varepsilon) + O(\varepsilon \log \varepsilon)]u^+(x, \varepsilon) \\ &\quad + [\sqrt{2} + c_{22}(\varepsilon) + O(\varepsilon \log \varepsilon)]u^-(x, \varepsilon), \\ \tilde{u}^-(x, \varepsilon) &= [c_{11}(\varepsilon)]u^+(x, \varepsilon) + [1 + c_{21}(\varepsilon) + O(\varepsilon \log \varepsilon)]u^-(x, \varepsilon),\end{aligned}$$

where $c_{ij}(\varepsilon) \sim 0$, $1 \leqq i, j \leqq 2$, and the terms of order $O(\varepsilon \log \varepsilon)$ have series expansions which can be computed if desired. The inverse relationship can of course be easily found, too.

Further, by the method of Evgrafov and Fedoryuk, we can express $u^+(x, \varepsilon)$, $u^-(x, \varepsilon)$ in terms of two other independent solutions of (2.1), $v^+(x, \varepsilon)$, $v^-(x, \varepsilon)$, with formulas in $(D_1 \cup l_2 \cup D_2)_\delta$. The formulas for $v^{\pm}(x, \varepsilon)$ are analogous to

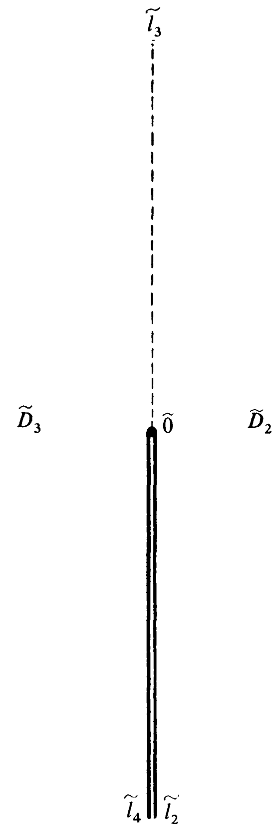

FIG. 4. *The image of $D_2 \cup l_3 \cup D_3$ on the ξ-plane*

(2.4), (2.7 a, b, c), but with $\xi(x)$ replaced by

$$\xi(1, x) = \int_1^x (z^2 - z)\, dz.$$

We can subsequently find the behavior of $v^{\pm}(x, \varepsilon)$ in $R_4 \cup s_4 \cup R_3$, etc., by Corollary 2.2, and "solve" the differential equation "globally."

Acknowledgment. The author wishes to express his gratitude to Professor W. Wasow for his introduction to and advice on the subject matter.

REFERENCES

[1] M. A. EVGRAFOV AND M. V. FEDORYUK, *Asymptotic behavior of solutions of the equation $w''(z) - p(z, \lambda)w(z) = 0$ as $\lambda \to \infty$ in the complex z-plane*, Uspehi Mat. Nauk, 21 (1966), no. 1 (127), pp. 3–50. (In Russian.)

[2] W. WASOW, *Simple turning point problems in unbounded domains*, this Journal, (1970), pp. 153–170.

[3] R. J. HANSON AND D. L. RUSSELL, *Classification and reduction of second order systems at a turning point*, J. Math. and Phys., 46 (1967), pp. 74–92.

[4] Y. SIBUYA, *Asymptotic solutions of a system of linear ordinary differential equations containing a parameter*, Funkcial. Ekvac., 4 (1962), pp. 83–113.

[5] R. LEE, *On uniform simplification of linear differential equations in a full neighborhood of a turning point*, J. Math. Anal. Appl., 27 (1969), pp. 501–510.

[6] M. ABRAMOWITZ AND A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, Washington, D.C., 1965.

[7] J. HEADING, *Phase-Integral Methods*, Methuen's Physical Monographs, London, 1961.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

GENERALIZED GREEN'S MATRICES FOR LINEAR DIFFERENTIAL SYSTEMS*

HOWARD CHITWOOD†

Abstract. This paper investigates the $n \times n$ matrix differential equation $Y' = AY$ together with boundary conditions of the form $\int_a^b dF(t)Y(t) = 0$, where F is an $n \times n$ matrix whose elements are of bounded variation.

It is known that if the above boundary problem is incompatible then the nonhomogeneous boundary problem $Y' = AY + R$, $\int_a^b dF(t)Y(t) = 0$ has a unique solution; here it is shown that if the homogeneous problem is compatible, then the Moore–Penrose generalized inverse of a matrix can be employed to obtain conditions which ensure the existence of a solution to the nonhomogeneous problem.

A generalized Green's matrix is constructed and its properties studied. An adjoint system is defined and properties relating it to the given system and the generalized Green's matrix are explored. A principal generalized Green's matrix is defined and properties analogous to those for the classical case are developed.

1. Introduction. This paper investigates the $n \times n$ matrix differential equation $Y' = AY$ together with boundary conditions of the form $\int_a^b dF(t)Y(t) = 0$, where F is an $n \times n$ matrix function whose elements are of bounded variation.

In the special case of the two-point homogeneous boundary condition $MY(a) + NY(b) = 0$, where M and N are constant matrices such that the rank of the $n \times 2n$ matrix $[M \quad N]$ equals n and where Φ is any fundamental matrix for $Y' = AY$, it is well known that the index of compatibility of the boundary problem is $n - \operatorname{rank}[M\Phi(a) + N\Phi(b)]$. If this boundary problem is incompatible, then the nonhomogeneous boundary problem $Y' = AY + R, MY(a) + NY(b) = 0$ has a unique solution given by

$$Y(t) = \int_a^b G(t, s)R(s)\, ds,$$

where $G(t, s)$ is the Green's matrix for the homogeneous boundary problem. The detailed development of these results can be found in Cole's text [5, Chap. 6] or in the recent book [12, Chap. III] by Reid.

Bradley [2] has generalized the above results to the case where the differential system $Y' = AY$, $MY(a) + NY(b) = 0$ is compatible. His development employs the Moore–Penrose generalized inverse of a matrix; properties of this generalized inverse may be found in Penrose [8] or Reid [12, Appendix B]. Reid, in his 1931 paper [9], discussed such a compatible system and determined a generalized Green's matrix; however, his development did not make use of the Moore–Penrose matrix which allows for considerable simplification in the construction of a generalized Green's matrix. Bradley gives conditions for the nonhomogeneous system $Y' = AY + R$, $MY(a) + NY(b) = 0$ to possess a solution and shows that

* Received by the editors August 31, 1971, and in revised form February 25, 1972.

† Department of Mathematics, Carson-Newman College, Jefferson City, Tennessee 37760. This paper is part of a doctoral thesis written under the direction of Dr. John S. Bradley, University of Tennessee, Knoxville.

a generalized Green's matrix for the homogeneous system exists. While this matrix is not unique, a formula is developed which gives the most general generalized Green's matrix in terms of any particular generalized Green's matrix. Furthermore, the concept of a principal generalized Green's matrix is introduced, and it is shown that with respect to certain orthogonality conditions there exists a unique generalized Green's matrix.

The more general problem with boundary operator $\int_a^b dF(t)Y(t)$ has been studied by Bryan [3] for the incompatible case, and Tucker [13] has dealt with certain aspects of the compatible problem. In the present paper the construction leading to a generalized Green's matrix is more direct than Tucker's and the theory is extended to parallel Bradley's development for the more special classical case.

The boundary operator $\int_a^b dF(t)Y(t)$ arises as a representation of a bounded linear transformation U from the space $\mathscr{C}$ of $n \times n$ matrices whose elements are continuous functions on $[a, b]$ into the space $\mathscr{F}$ of $n \times n$ matrices whose elements are constants. A norm for $K \in \mathscr{F}$ is defined by $|K| = \max\{|k_{ij}|\}$, and $\|Y\| = \max\{|Y(x)| : x \in [a, b]\}$ defines a norm for $Y \in \mathscr{C}$. We shall write

$$U[Y] = \int_a^b dF(t)Y(t).$$

The development of the theory will depend, to a large extent, on the Moore–Penrose generalized inverse of a matrix. Penrose [8] defines such an inverse of a matrix D to be a matrix $D^\dagger$ with the properties $D^\dagger D D^\dagger = D^\dagger$, $DD^\dagger D = D$, and $DD^\dagger$ and $D^\dagger D$ Hermitian. Such a matrix exists and it is unique. Penrose also shows that a necessary and sufficient condition for $AXB = C$ to have a solution is $AA^\dagger CB^\dagger B = C$, in which case the general solution is $X = A^\dagger CB^\dagger + Y - A^\dagger AYBB^\dagger$ for arbitrary Y.

We now proceed to a description of the results in which most of the proofs are omitted.

2. Existence of solutions. The systems to be considered are

(1) $$Y' = AY + R, \quad U[Y] = K,$$

(1a) $$Y' = AY + R, \quad U[Y] = 0,$$

(2) $$Y' = AY, \qquad U[Y] = 0,$$

where $A, R \in \mathscr{C}$, $K \in \mathscr{F}$, and U is a bounded linear transformation from $\mathscr{C}$ into $\mathscr{F}$.

Since U has the representation $U[Y] = \int_a^b dF(t)Y(t)$, it is easily seen that $U[YC] = U[Y]C$ for arbitrary $C \in \mathscr{F}$. Furthermore, we note that if (2) is incompatible and Φ is any fundamental matrix for $Y' = AY$, then $U[\Phi]$ is nonsingular, and if Φ_1 and Φ_2 are fundamental matrices, then rank $U[\Phi_1]$ = rank $U[\Phi_2]$.

THEOREM 2.1. *If m is the index of compatibility for* (2) *and Φ is any fundamental matrix for $Y' = AY$, then $m = n -$ rank $U[\Phi]$.*

The following theorem was first proved by Bryan [3].

THEOREM 2.2. *If the system* (2) *is incompatible, then the system* (1a) *has a unique solution given by*

$$Y(t) = \int_a^b G(t, s)R(s)\, ds,$$

where

$$G(t, s) = \begin{cases} \Phi(t)U[\Phi]^{-1}\int_a^s dF(u)\Phi(u)\Phi^{-1}(s) & \text{for } t > s, \\ -\Phi(t)U[\Phi]^{-1}\int_s^b dF(u)\Phi(u)\Phi^{-1}(s) & \text{for } t < s, \end{cases}$$

and Φ *is a fundamental matrix for* $Y' = AY$. (The matrix $G(t, s)$ is called the Green's matrix.)

The natural question which arises now concerns the possibility of the existence of solutions of systems (1) or (1a) in the event that system (2) is compatible. Since $U[\Phi]^{-1}$ no longer exists, the Moore–Penrose generalized inverse of a matrix is utilized.

The following theorem is of basic importance to the theory.

THEOREM 2.3. *If* D *is an* $n \times n$ *matrix with rank* $n - m$, $m > 0$, S *is an* $n \times m$ *matrix such that* $DS = 0$ *and* $S^*S = I$, *and* T *is an* $n \times m$ *matrix such that* $T^*D = 0$ *and* $T^*T = I$, *then the* $(n + m) \times (n + m)$ *matrix*

$$\begin{bmatrix} D & T \\ S^* & 0 \end{bmatrix}$$

is nonsingular and its inverse is

$$\begin{bmatrix} D^\dagger & S \\ T^* & 0 \end{bmatrix},$$

where $D^\dagger$ *is the Moore–Penrose generalized inverse of* D.

We now give an important application of this theorem, in which the matrix D is taken to be $U[\Phi]$.

THEOREM 2.4. *If the system* (2) *is compatible with index of compatibility* m, *then the system* (1a) *has a solution if and only if*

$$T^* \int_a^b \left[\int_a^s dF(t)\Phi(t) - \int_s^b dF(t)\Phi(t)\right]\Phi^{-1}(s)R(s)\, ds = 0,$$

where Φ *is a fundamental matrix for* $Y' = AY$ *and* T *is the matrix appearing in the preceding theorem.*

The proof of this theorem employs the variation of parameters formula for solutions of $Y' = AY + R$; the operator U is applied to such solutions and then Theorem 2.3 leads to the condition of the theorem.

3. The generalized Green's matrix and an adjoint system.

DEFINITION 3.1. A *generalized Green's matrix for system* (2) is an essentially bounded, measurable $n \times n$ matrix function G defined on the set

$$Q = \{(t, s) : a \leqq t \leqq b, a \leqq s \leqq b\}$$

such that if system (1a) has a solution, then $Y(t) = \int_a^b G(t, s)R(s)\, ds$ is also a solution.

THEOREM 3.1. *A generalized Green's matrix for system* (2) *exists.*

Indeed, we construct the following matrix and show that the conditions of

Definition 3.1 are satisfied:

$$G_0(t,s) = \begin{cases} \frac{1}{2}\Phi(t)SS^*\Phi^{-1}(s) + \Phi(t)U[\Phi]^\dagger \int_a^s dF(u)\Phi(u)\Phi^{-1}(s) & \text{if } t > s, \\ -\frac{1}{2}\Phi(t)SS^*\Phi^{-1}(s) - \Phi(t)U[\Phi]^\dagger \int_s^b dF(u)\Phi(u)\Phi^{-1}(s) & \text{if } t < s. \end{cases}$$

The following theorem displays some properties of $G_0(t, s)$.

THEOREM 3.2. *The generalized Green's matrix $G_0(t, s)$ has the following properties:*

(i) *$G_0(t, s)$ is continuous in t except at $t = s$ and is continuous in s except at $s = t$ and the discontinuities of F;*

(ii) $G_0(s + 0, s) - G_0(s - 0, s) = I$;

(iii) *for each fixed s, $G_0(t, s)$ satisfies $Y' = AY$ in t except at $t = s$;*

(iv) $G_0(t, t + 0) - G_0(t, t - 0) = -I$ *except at the discontinuities of F.*

In conjunction with property (iii), it should be noted that $G_0(t, s)$ fails to satisfy the boundary condition

$$U[G_0(t, s)] = 0 \quad \text{for fixed } s;$$

indeed, it turns out that

$$U[G_0(t, s)] = TT^* \int_s^b dF(u)\Phi(u)\Phi^{-1}(s),$$

where T is the matrix introduced in Theorem 2.3.

Continuing our development of the generalized Green's matrix we discuss uniqueness.

THEOREM 3.3. *The generalized Green's matrix for system* (2) *is not unique.*

In particular, let $G_0(t, s)$ be the generalized Green's matrix constructed above and let $W(s)$ be any essentially bounded measurable $n \times n$ matrix function. Define $G(t, s)$ by

$$G(t, s) = G_0(t, s) + \Phi(t)[W(s) - U[\Phi]^\dagger U[\Phi]W(s)];$$

then $G(t, s)$ is also a generalized Green's matrix.

We can obtain a more precise result relative to the uniqueness and general form of generalized Green's matrices; to do so we must introduce an adjoint system for system (2). The following adjoint system was defined by Bryan [3] and also by Tucker [13].

DEFINITION 3.2. For a parameter matrix $M \in \mathscr{F}$, the system

(3a) $\quad Z' = -A^*Z + F'^*M^*$ on the set where F' exists,

(3b) $\quad Z - F^*M^*$ is absolutely continuous on $[a, b]$,

(3c) $\quad Z(a) = 0,$

(3d) $\quad Z(b) = 0$

is defined as an *adjoint system to system* (2).

DEFINITION 3.3. A matrix Z is a *solution of system* (3) if there exists a matrix $M \in \mathscr{F}$ such that (3a, b, c, d) hold.

We now present a theorem proved first by Bryan [3] and later, in a more general setting, by Tucker [13].

THEOREM 3.4. *For $M \in \mathscr{F}$, the system* (3a, b, c) *has the unique solution*

$$Z(t) = \left[M \int_a^t dF(u)\Phi(u)\Phi^{-1}(t) \right]^*;$$

this Z satisfies (3d) *if and only if* $MU[\Phi] = 0$.

We now give an application of the adjoint system in proving a theorem which generalizes a classical relation that exists between a differential operator and its adjoint.

THEOREM 3.5. *The system $Y' = AY + R$, $U[Y] = 0$ has a solution if and only if $\int_a^b Z^*(t)R(t)\,dt = 0$ for each solution Z of the adjoint system* (3).

We now present a necessary and sufficient condition for a matrix to be a generalized Green's matrix. This generalizes a result of Bradley [2] dealing with the boundary condition $MY(a) + NY(b) = 0$. The proof follows the general procedure of Bradley, and makes use of the fact that system (2) and the adjoint system (3) have the same index of compatibility.

THEOREM 3.6. *Let V_1 be an $n \times n$ matrix of rank m which is a solution of system* (2) *and let V_2 be an $n \times n$ matrix of rank m which is a solution of the adjoint system* (3). *If G_0 is one generalized Green's matrix for* (2), *then G is also a generalized Green's matrix for* (2) *if and only if there exist bounded measurable matrix functions Γ_1 and Γ_2 such that*

$$G(t, s) = G_0(t, s) + V_1(t)\Gamma_1(s) + \Gamma_2(t)V_2^*(s).$$

4. A principal generalized Green's matrix. In [9] Reid introduced the concept of a principal generalized Green's matrix and showed that with respect to certain orthogonality conditions there is a unique generalized Green's matrix. Bradley [2] proved a similar theorem when the number of boundary conditions is different from n. Later, Reid [11] discussed this problem in quite general settings, and we have the following two theorems as specific realizations of Reid's results.

THEOREM 4.1. *Let V_1 and V_2 be as in Theorem* 3.6 *and suppose that Θ and Ω are matrices in $\mathscr{C}$ such that $\int_a^b \Theta^* V_1$ and $\int_a^b V_2^*\Omega$ are nonsingular. Then there exists a unique generalized Green's matrix $G_{\Theta\Omega}$ for* (2) *satisfying*

$$\int_a^b G_{\Theta\Omega}(t, s)\Omega(s)\,ds = 0, \quad t \in [a, b],$$

$$\int_a^b \Theta^*(t)G_{\Theta\Omega}(t, s)\,dt = 0, \quad s \in [a, b].$$

THEOREM 4.2. *If $G_{\Theta\Omega}$ is the unique generalized Green's matrix of Theorem* 4.1, *then*

(i) *$G_{\Theta\Omega}$ is continuous in t except at $t = s$ and is continuous in s except at $s = t$ and the discontinuities of F;*

(ii) $G_{\Theta\Omega}(s + 0, s) - G_{\Theta\Omega}(s - 0, s) = I$;

(iii) *for each s, $G_{\Theta\Omega}(t, s)$ satisfies*

$$Y'(t) = A(t)Y(t) - \Omega(t)V_2^*(s)$$

except at $t = s$ and the discontinuities of F;

(iv) *for almost all s,* $G_{\Theta\Omega}(t, s)$ *satisfies the boundary condition* $U[G_{\Theta\Omega}(t, s)] = 0$ *as a function of* t;

(v) $\int_a^b \Theta^*(t) G_{\Theta\Omega}(t, s)\, dt = 0$.

COROLLARY. *If* $R \in \mathscr{C}$ *and* Y *is defined by* $Y(t) = \int_a^b G_{\Theta\Omega}(t, s) R(s)\, ds$, *then*

(i) $Y'(t) = A(t)Y(t) + R(t) - \Omega(t)\int_a^b V_2^*(s)R(s)\, ds$ *except at* $t = s$ *and the discontinuities of* F;

(ii) $U[Y] = 0$;

(iii) $\int_a^b \Theta^*(t) Y(t)\, dt = 0$.

Remark. This corollary has some interesting implications. Let us suppose that system

$$Y' = AY + R, \qquad U[Y] = 0 \tag{1a}$$

has no solution and let $G(t, s)$ be any generalized Green's matrix. Then G has the form

$$G(t, s) = G_0(t, s) + V_1(t)\Gamma_1(s) + \Gamma_2(t)V_2^*(s)$$

as found in Theorem 3.6. Now if Y is defined by

$$Y(t) = \int_a^b G(t, s)R(s)\, ds,$$

then

$$Y(t) = \int_a^b G_0(t, s)R(s)\, ds + V_1(t)\int_a^b \Gamma_1(s)R(s)\, ds + \Gamma_2(t)\int_a^b V_2^*(s)R(s)\, ds.$$

If $G(t, s) = G_0(t, s)$, then looking back to Theorems 2.4 and 3.1 we see that $Y(t)$ satisfies the system

$$Y'(t) = A(t)Y(t) + R(t), \qquad U[Y(t)] = -TT^*B,$$

where $-T^*B$ is the left member of the condition in Theorem 2.4. On the other hand, if $G(t, s) = G_{\Theta\Omega}(t, s)$, the corollary shows that $Y(t)$ satisfies, almost everywhere, the system

$$Y'(t) = A(t)Y(t) + R(t) - \Omega(t)\int_a^b V_2^*(s)R(s)\, ds, \qquad U[Y(t)] = 0.$$

Thus, we have the choice of satisfying the differential equation or the boundary condition of system 1(a).

REFERENCES

[1] J. S. BRADLEY, *Adjoint quasi-differential operators of Euler type*, Pacific J. Math., 16 (1966), pp. 213–237.

[2] ———, *Generalized Green's matrices for compatible differential systems*, Michigan Math. J., 13 (1966), pp. 97–108.

[3] R. N. BRYAN, *A linear differential system with general linear boundary conditions*, J. Differential Equations, 5 (1969), pp. 38–48.

[4] R. H. COLE, *General boundary conditions for an ordinary linear differential system*, Trans. Amer. Math. Soc., 111 (1964), pp. 521–550.

[5] ———, *Theory of Ordinary Differential Equations*, Appleton-Century-Crofts, New York, 1968.

[6] A. M. KRALL, *Nonhomogeneous differential operators*, Michigan Math. J., 12 (1965), pp. 247–255.

[7] W. L. LOUD, *Generalized inverses and generalized Green's functions*, SIAM J. Appl. Math., 14 (1966), pp. 342–369.

[8] R. PENROSE, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., 51 (1955), pp. 406–413.

[9] W. T. REID, *Generalized Green's matrices for compatible systems of differential equations*, Amer. J. Math., 53 (1931), pp. 443–459.

[10] ———, *Generalized Green's matrices for two-point boundary problems*, SIAM J. Appl. Math., 15 (1967), pp. 856–870.

[11] ———, *Generalized inverses of differential and integral operators*, Proc. Symposium on Theory and Applications of Generalized Inverses of Matrices, Texas Technological College, Lubbock, 1968.

[12] ———, *Ordinary Differential Equations*, John Wiley, New York, 1971.

[13] D. H. TUCKER, *Boundary value problems for linear differential systems*, SIAM J. Appl. Math., 17 (1969), pp. 769–783.

[14] W. M. WHYBURN, *Differential systems with general boundary conditions*, Seminar Reports in Mathematics, University of California Publications in Mathematics, 2 (1944), pp. 45–61.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

NONNEGATIVE AND ALTERNATING EXPANSIONS OF ONE SET OF ORTHOGONAL POLYNOMIALS IN TERMS OF ANOTHER*

WILLIAM F. TRENCH†

Abstract. Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be monic polynomials orthogonal with respect to the distributions $du(x)$ and $dv(x) = w(x)\,du(x)$. Conditions are given on $w(x)$ which imply that, for all n, the coefficients in the expansion of $p_n(x)$ in terms of $q_0(x), \cdots, q_n(x)$ are nonnegative, and those in the expansion of $q_n(x)$ in terms of $p_0(x), \cdots, p_n(x)$ alternate in sign.

1. Introduction. Several recent papers have been concerned with finding conditions under which the constants $c_{0n}, c_{1n}, \cdots, c_{nn}$ in the expansion

$$q_n(x) = \sum_{r=0}^{n} c_{rn} p_r(x), \qquad n = 0, 1, \cdots, \tag{1}$$

are all nonnegative, where $\{p_n(x)\}$ and $\{q_n(x)\}$ are suitably normalized polynomials orthogonal with respect to different distributions. Askey [1], [2], [3], Askey and Gasper [4], and Wilson [7] have obtained results on this question. Askey [3] gives references to areas in which this problem arises.

We shall say that the expansion (1) is *nonnegative* if $c_{rn} \geqq 0$ for $0 \leqq r \leqq n$, or *alternating* if $(-1)^{n-r}c_{rn} \geqq 0$ for $0 \leqq r \leqq n$. An alternating expansion can be transformed into a nonnegative expansion (and vice versa) by the renormalization

$$P_n(x) = (-1)^n p_n(x), \qquad Q_n(x) = (-1)^n q_n(x), \qquad n = 0, 1, 2, \cdots. \tag{2}$$

2. Formulation of the problem. Throughout this paper we assume that $u(x)$ is nondecreasing and $w(x)$ nonnegative on an interval (a, b), that the distributions $du(x)$ and $dv(x) = w(x)\,du(x)$ have finite moments

$$\int_a^b x^r\,du(x) \quad \text{and} \quad \int_a^b x^r\,dv(x)$$

for all nonnegative integers r, and that $\{p_n(x)\}$ and $\{q_n(x)\}$ are the monic polynomials orthogonal over (a, b) with respect to $du(x)$ and $dv(x)$, respectively; i.e.,

$$p_n(x) = x^n + \cdots, \qquad q_n(x) = x^n + \cdots, \tag{3}$$

and

$$\int_a^b p_n(x)p_m(x)\,du(x) = \int_a^b q_n(x)q_m(x)\,dv(x) = 0, \qquad n > m \geqq 0.$$

We shall give conditions under which the expansions

$$q_n(x) = p_n(x) + \sum_{r=0}^{n-1} a_{rn} p_r(x) \tag{4}$$

and

$$p_n(x) = q_n(x) + \sum_{r=0}^{n-1} b_{rn} q_r(x) \tag{5}$$

* Received by the editors January 6, 1972, and in revised form March 15, 1972.

† Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104.

are, respectively, alternating and nonnegative for all n. (If $u(x)$ has only finitely many, say N, points of increase, the phrase "for all n" should be interpreted as "for $n = 0, 1, \cdots, N - 1$.")

3. Results. The following is a known result [6, Thm. 3.1.4, § 3.1].

LEMMA 1. *Suppose x_0 is not in (a, b) and $w(x) = |x - x_0|$. Then* (4) *and* (5) *reduce to*

$$q_n(x) = p_n(x) + \sum_{r=0}^{n-1} \frac{p_r(x_0)}{p_n(x_0)} p_r(x) \tag{6}$$

and

$$p_n(x) = q_n(x) - \frac{p_{n-1}(x_0)}{p_n(x_0)} q_{n-1}(x). \tag{7}$$

LEMMA 2. *If $-\infty < x_0 \leqq a$, then* (6) *is alternating and* (7) *is nonnegative for all n. If $b \leqq x_0 < \infty$, then* (6) *is nonnegative and* (7) *is alternating for all n.*

Proof. The roots of $p_j(x)$ are all in (a, b). Because of the normalization (3), $(-1)^j p_j(x_0) > 0$ if $x_0 \leqq a$, and $p_j(x_0) > 0$ if $x_0 \geqq b$. This yields the conclusion.

Suppose $\{p_n(x)\}$, $\{q_n(x)\}$ and $\{r_n(x)\}$ are sequences of polynomials such that, for all n, the expansion of $p_n(x)$ in terms of $q_0(x), q_1(x), \cdots, q_n(x)$ and the expansion of $q_n(x)$ in terms of $r_0(x), r_1(x), \cdots, r_n(x)$ are both alternating (nonnegative); then the expansion of $p_n(x)$ in terms of $r_0(x), r_1(x), \cdots, r_n(x)$ is also alternating (nonnegative) for all n. This and repeated application of Lemma 2 yield the following theorem.

THEOREM 1. *Let $R(a, b)$ be the set of rational functions with only real zeros and poles, which are positive on (a, b), with finite zeros, if any, confined to $(-\infty, a]$, and finite poles, if any, confined to $[b, \infty)$. If $w(x)$ is in $R(a, b)$, then* (4) *is alternating and* (5) *is nonnegative for all n.*

Example 1. The Jacobi polynomials, defined by

$$P_n^{(\alpha,\beta)}(x) = (1 - x)^{-\alpha}(1 + x)^{-\beta} \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n [(1 - x)^{n+\alpha}(1 + x)^{n+\beta}], \quad \alpha, \beta > -1,$$

are orthogonal with respect to the distribution

$$du(x) = (1 - x)^\alpha (1 + x)^\beta \, dx, \qquad -1 < x < 1,$$

and have positive leading coefficients. From Theorem 1, the expansion

$$P_n^{(\gamma,\delta)}(x) = \sum_{r=0}^{n} A_{rn}(\alpha, \beta; \gamma, \delta) P_r^{(\alpha,\beta)}(x) \tag{8}$$

is alternating for all n if $\gamma = \alpha - r > -1$ and $\delta = \beta + s$, with r and s nonnegative integers, and nonnegative for all n if $\gamma = \alpha + r$ and $\delta = \beta - s > -1$, with r and s nonnegative integers.

For other cases in which (8) is known to be nonnegative for all n, and for a conjecture on this point, see Askey and Gasper [4].

Example 2. Askey [1] has shown that (4) is alternating for all n if $a = 0$ and $w(x) = x^\alpha$, where α is a positive integer, and has conjectured that the result remains valid if α is an arbitrary positive number. (Actually, Askey speaks of nonnegative expansions, but his normalization differs from ours as in (2).) Theorem 1 contains

Askey's result for positive integral α, and also implies that in this case (5) is nonnegative for all n. For this reason it is tempting to extend Askey's conjecture: namely, to conjecture that (4) is alternating and (5) is nonnegative for all n if $a = 0$ and $w(x) = x^\alpha$, with α an arbitrary positive number. However, this extended conjecture is false, as can be seen by taking

$$u(x) = 1, \quad w(x) = x^\alpha, \quad a = 0, \quad b = 1;$$

then straightforward computations yield

$$q_0(x) = 1,$$

$$q_1(x) = x - \frac{\alpha + 1}{\alpha + 2},$$

$$q_2(x) = x^2 - \frac{2(\alpha + 2)}{\alpha + 4}x + \frac{(\alpha + 1)(\alpha + 2)}{(\alpha + 3)(\alpha + 4)},$$

$$p_0(x) = 1,$$

$$p_1(x) = x - \tfrac{1}{2},$$

$$p_2(x) = x^2 - x + \tfrac{1}{6}.$$

Therefore,

$$p_2(x) = q_2(x) + \frac{\alpha}{\alpha + 4}q_1(x) + \frac{\alpha(\alpha - 1)}{6(\alpha + 2)(\alpha + 3)}q_0(x),$$

which is not nonnegative if $0 < \alpha < 1$.

The coefficients of $p_n(x)$ and $q_n(x)$, as well as the coefficients a_{rn} and b_{rn} in (4) and (5), are continuous functions of the moments of $du(x)$ and $dv(x)$. The next lemma follows easily from this.

LEMMA 3. *Suppose $du_m(x)$ and $dv_m(x)$ are sequences of distributions on (a, b) such that*

(9) $$\lim_{m\to\infty} \int_a^b x^r \, du_m(x) = \int_a^b x^r \, du(x), \qquad r = 0, 1, \cdots,$$

(10) $$\lim_{m\to\infty} \int_a^b x^r \, dv_m(x) = \int_a^b x^r \, dv(x), \qquad r = 0, 1, \cdots.$$

Let $\{p_{nm}(x)\}_{n=0}^\infty$ and $\{q_{nm}(x)\}_{n=0}^\infty$ be the sequences of monic polynomials orthogonal over (a, b) with respect to $du_m(x)$ and $dv_m(x)$, respectively. For each m, let the expansions

$$q_{nm}(x) = p_{nm}(x) + \sum_{r=0}^{n-1} a_{rnm}p_{rm}(x)$$

and

$$p_{nm}(x) = q_{nm}(x) + \sum_{r=0}^{n-1} b_{rnm}q_{rm}(x)$$

be, respectively, alternating and nonnegative for all n. Then (4) *is alternating and* (5) *is nonnegative for all n.*

THEOREM 2. *If $\gamma > 0$ and the distribution $dv(x) = e^{\gamma x}\, du(x)$ has moments of all orders on (a, b), then (4) is alternating and (5) is nonnegative for all n.*

Proof. If $a > -\infty$, let $du_m(x) = du(x)$ and $dv_m(x) = w_m(x)\, du(x)$, where

$$w_m(x) = e^{\gamma a}\left(1 + \frac{\gamma(x-a)}{m}\right)^m, \quad x \geqq a.$$

Then (9) is obvious and, since $w_m(x) \leqq e^{\gamma x}$ and $\lim_{m\to\infty} w_m(x) = e^{\gamma x}$, Lebesgue's bounded convergence theorem implies (10). Moreover, $w_m(x)$ is in $R(a, b)$ for every m. Thus, if a is finite, the conclusion follows from Theorem 1 and Lemma 3.

If $a = -\infty$, we again apply Lemma 3, this time with

$$u_m(x) = \begin{cases} u(x), & x \geqq -m, \\ u(-m), & x < -m, \end{cases}$$

and $dv_m(x) = e^{\gamma x}\, du_m(x)$. From the result just proved for finite a, the hypotheses of Lemma 3 are satisfied, and therefore the conclusion follows.

Example 3. Suppose $\alpha > -1$ and

$$du(x) = x^\alpha e^{-x}\, dx, \quad x > 0;$$

then

$$p_n(x) = (-1)^n c_n L_n^{(\alpha)}(x), \tag{11}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial and $c_n > 0$ [6, § 5.1]. If $\rho > 0$, the change of variable $x = \rho y$ transforms the orthogonality condition

$$\int_0^\infty e^{-x} x^\alpha p_n(x) p_m(x)\, dx = 0, \qquad n \neq m,$$

into

$$\int_0^\infty e^{-\rho x} y^\alpha p_n(\rho y) p_m(\rho y)\, dy = 0, \qquad n \neq m;$$

hence, the monic polynomials $q_n(x) = \rho^{-n} p_n(\rho x)$, $n = 0, 1, \cdots$, are orthogonal over $(0, \infty)$ with respect to the distribution

$$dv(x) = e^{-(\rho-1)x}\, du(x).$$

Bearing in mind the difference in normalization indicated in (11), we conclude from Theorem 2 that the expansion

$$L_n^{(\alpha)}(\rho x) = \sum_{r=0}^{n} A_{rn}^{(\alpha)}(\rho) L_r^{(\alpha)}(x)$$

is nonnegative for all n if $0 < \rho < 1$, and alternating for all n if $\rho > 1$. This is a known result; see [5, § 119].

Example 4. If

$$du(x) = e^{-x^2}\, dx, \qquad -\infty < x < \infty,$$

then

$$p_n(x) = d_n H_n(x),$$

where $H_n(x)$ is the nth Hermite polynomial and $d_n > 0$ [6, § 5.5]. The change of variable $x = y - x_0$ transforms the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2} p_n(x) p_m(x)\, dx = 0, \qquad m \neq n,$$

into

$$\int_{-\infty}^{\infty} e^{-(y-x_0)^2} p_n(y - x_0) p_m(y - x_0)\, dy, \qquad m \neq n;$$

hence, the monic polynomials $q_n(x) = p_n(x - x_0)$, $n = 0, 1, \cdots$, are orthogonal over $(-\infty, \infty)$ with respect to the distribution

$$dv(x) = e^{2x_0x}\, du(x).$$

It follows from Theorem 2 that the expansion

$$H_n(x - x_0) = \sum_{r=0}^{n} K_{rn}(x_0) H_r(x)$$

is alternating for all n if $x_0 > 0$, and nonnegative for all n if $x_0 < 0$. This is also a known result; see [6, Prob. 68, p. 385].

We conclude with the following theorem, which can be obtained from Theorem 1, Lemma 3 and Theorem 2.

THEOREM 3. *Suppose* $-\infty < a < b < \infty$, *and let*

$$w(x) = e^{\gamma x} \frac{(x - a)^m \prod_{r=1}^{\infty}[1 + c_r(x - a)]}{(b - x)^n \prod_{s=1}^{\infty}[1 - d_s(x - b)]}, \tag{12}$$

where m *and* n *are nonnegative integers,* $\gamma \geqq 0$, $c_r \geqq 0$, $d_s \geqq 0$, $\sum_1^{\infty} c_r < \infty$, *and* $\sum_1^{\infty} d_s < \infty$. *If the distribution* $dv(x) = w(x)\, du(x)$ *has moments of all orders on* (a, b), *then* (4) *is alternating and* (5) *is nonnegative for all* n.

Remark. If $-\infty = a < b < \infty$, a similar result holds with (12) replaced by

$$w(x) = e^{\gamma x}(b - x)^{-n} \left(\sum_{s=1}^{\infty} [1 - d_s(x - b)] \right)^{-1}.$$

If $-\infty < a < b = \infty$, the appropriate form for $w(x)$ is

$$w(x) = e^{\gamma x}(x - a)^m \sum_{r=1}^{\infty} [1 + c_r(x - a)].$$

REFERENCES

[1] R. ASKEY, *Orthogonal expansions with positive coefficients*, Proc. Amer. Math. Soc., 26 (1965), pp. 1191–1194.

[2] ———, *Jacobi polynomial expansions with positive coefficients and imbeddings of projective spaces*, Bull. Amer. Math. Soc., 74 (1968), pp. 301–304.

[3] ———, *Orthogonal expansions with positive coefficients. II*, this Journal, 2 (1971), pp. 340–346.

[4] R. ASKEY AND G. GASPER, *Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients*, Proc. Cambridge Philos. Soc., 70 (1971), pp. 243–255.

[5] E. D. RAINVILLE, *Special Functions*, Macmillan, New York, 1960.

[6] G. SZEGÖ, *Orthogonal Polynomials*, rev. ed., American Mathematical Society, Providence, 1959.

[7] M. W. WILSON, *Nonnegative expansions of polynomials*, Proc. Amer. Math. Soc., 24 (1970), pp. 100–102.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

ASYMPTOTIC EIGENFUNCTIONS OF A SINGULAR INTEGRO-DIFFERENTIAL EQUATION*

S. E. SHAMMA† AND S. N. KARP‡

Abstract. Asymptotic representations are obtained for the high order eigenfunctions and eigenvalues of the singular integro-differential equation $[py']' + [q + \lambda^2 r]y = \lambda^2 \int K(x, s)y(s)\,ds$, where p is positive in the open interval and has a simple zero at the endpoints, while q may have a pole there. Bounded eigenfunctions are shown to be asymptotic to the corresponding solutions of the differential equation resulting from the absence of the integral term. The basic tool is the use of a generalized Green's function, $\Gamma_{\lambda_n}(x, s)$, for which it is shown that $\int \Gamma^2_{\lambda_n}(x, s)r(s)\,ds$ tends to zero for large λ_n.

1. Introduction. In a recent paper [1], B. I. Aleksendriskii studied the asymptotic solutions of an integro-differential equation whose differential part is a regular Sturm–Liouville operator.

In the present article, we consider the integro-differential equation

$$(1.1)\qquad \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda^2 r(x)]y = \lambda^2 \int_{x_1}^{x_2} K_1(x, s)y(s)\,ds, \qquad x_1 \leqq x \leqq x_2.$$

The coefficients $p(x)$, $r(x)$, and $q(x)$ have the following properties:

(i) $p(x)$ is twice differentiable, positive on $x_1 < x < x_2$, and has a simple zero at the endpoints.

(ii) $r(x)$ is twice differentiable and positive on $x_1 \leqq x \leqq x_2$.

(iii) $q(x) \leqq 0$ for $x_1 \leqq x \leqq x_2$, continuous on $x_1 < x < x_2$, and may have a simple pole at the endpoints. Let

$$q_j = \lim q(x)(x - x_j) \quad \text{as } x \to x_j, \quad j = 1, 2.$$

For convenience, we introduce the two differentiable operators L and A defined by

$$(1.2)\qquad L_x = \left[\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)\right], \qquad A = -\frac{1}{r(x)}L_x.$$

Then (1.1) becomes

$$(1.3)\qquad A[y] - \lambda^2 y(x) = \lambda^2 \int_{x_1}^{x_2} K(x, s)y(s)\,ds, \qquad x_1 \leqq x \leqq x_2.$$

Since the function $p(x)$ vanishes at the endpoints, the proper set of boundary conditions to impose is the boundedness of $y(x)$ at the endpoints.

In § 2, we study the asymptotic behavior of the eigenvalues and the corresponding eigenfunctions of A; that is, solutions of $A[u_n] = \alpha_n^2 u_n$. It is shown that

* Received by the editors April 19, 1971, and in final revised form March 10, 1972.

† The University of West Florida, Pensacola, Florida 32504. The paper was written while the first author was a Visiting Member of the Courant Institute of Mathematical Sciences, New York University. It is based on a part of the doctoral thesis [6].

‡ Courant Institute of Mathematical Sciences, New York University, New York, New York 10012.

$\alpha_n = O(n)$ and $|u_n(x)| = O(\sqrt{\alpha_n})$. Then we investigate the asymptotic properties of a generalized Green's function which will be used in § 3 to show that the bounded eigenfunctions of (1.3) are asymptotic to the corresponding solutions of the differential equation resulting from the absence of the integral term. Namely, they are asymptotic to u_n. The results of this paper, apart from their interest as a generalization of [1] and [3], are also expected to prove useful in extending the study in [7] to three-dimensional problems.

2. Asymptotic eigenvalues and eigenfunctions of A. From the above properties of the operator A it follows, using the first criterion in [2, p. 443], that its spectrum is totally discrete.

LEMMA 2.1. *Let α_n^2 and $u_n(x)$ be the eigenvalues and the corresponding bounded orthonormal eigenfunctions, with respect to the weight function $f(x)$, of the operator A. If the α_n^2's are arranged in a monotonic sequence, then*

$$\alpha_n = \left\{[(n + \tfrac{1}{2}) + \frac{\pi}{2}(\mu_1 + \mu_2)] \int_{x_1}^{x_2} \sqrt{\frac{r(t)}{P(t)}}\, dt\right\} + O\left(\frac{1}{n}\right) \tag{2.1}$$

and

$$|u_n(x)| = O(\sqrt{\alpha_n}) \tag{2.2}$$

uniformly in x as $n \to \infty$, where

$$\mu_j^2 = -4q_j/p'(x_j). \tag{2.3}$$

Proof. It is shown in the Appendix that the uniform asymptotic solutions of the differential equation

$$A[\tau(x, \lambda)] = \lambda^2 \tau(x, \lambda), \tag{2.4}$$

for large positive values of the parameter λ, are

$$\tau_1(x, \lambda) = g(x)\begin{cases} \dfrac{1}{z_1} w_1(z_1), & x_1 \leqq x < \beta, \\[2ex] \dfrac{1}{z_2} w_2(z_2), & x_2 > x > \alpha, \end{cases} \tag{2.5}$$

$$\tau_2(x, \lambda) = g(x)\begin{cases} \dfrac{1}{z_1} w_2(z_1), & x_1 < x < \beta, \\[2ex] \dfrac{1}{z_2} w_1(z_2), & x > x > \alpha, \end{cases} \tag{2.6}$$

where $\alpha < \beta$, $g(x) = [p(x)r(x)]^{-1/4}$,

$$z_j(x) = \frac{(-1)^{j-1}}{2} \int_{x_j}^{x} \sqrt{\frac{r(t)}{p(t)}}\, dt, \qquad j = 1, 2, \tag{2.7}$$

$$w_1(z_j) = z_j J_{\mu_j}(2\lambda z_j)\left[A_0 + O\left(\frac{1}{\lambda^2}\right)\right] + \frac{z_j}{2\lambda} J_{\mu_j+1}(2\lambda z_j)\left[B_0 + \frac{z_j}{1 + |z_j|} O\left(\frac{1}{\lambda^2}\right)\right], \tag{2.8}$$

$$\begin{aligned}\frac{dw_1(z_j)}{dz_j} &= J_{\mu_j}(2\lambda z_j)\left[c_0 + (1+|z_j|)O\left(\frac{1}{\lambda^2}\right)\right]\\ &\quad + 2\lambda z_j J_{\mu_j+1}(2\lambda z_j)\left[D_0 + O\left(\frac{1}{\lambda^2}\right)\right],\end{aligned}\tag{2.9}$$

$$w_2(z_j) = z_j Y_{\mu_j}(2\lambda z_j)\left[A_0 + O\left(\frac{1}{\lambda^2}\right)\right] - \frac{z_j}{2\lambda}Y_{\mu_j+1}(2\lambda z_j)\left[B_0 + \frac{z_j}{1+|z_j|}O\left(\frac{1}{\lambda^2}\right)\right],\tag{2.10}$$

$$\frac{dw_2(z_j)}{dz_j} = Y_{\mu_j}(2\lambda z_j)\left[C_0 + (1+|z_j|)O\left(\frac{1}{\lambda^2}\right)\right] - 2\lambda z_j Y_{\mu_j+1}(2\lambda z_j)\left[D_0 + O\left(\frac{1}{\lambda^2}\right)\right],\tag{2.11}$$

and A_0, B_0, C_0, D_0 are constants.

The above results are derived in a similar way to Olver's work on the asymptotic solutions of linear differential equations [4], as obtained in the Appendix.

The asymptotic behavior of the eigenvalues α_n^2 could be found from the Wronskian of the two solutions $\tau_1(x,\lambda)$ and $\tau_2(x,\lambda)$. The Wronskian vanishes at $\lambda = \alpha_n$. Hence the eigenvalues are roots of the equation

$$p(x)W[\tau_1(x,\alpha_n), \tau_2(x,\alpha_n)] = 0.\tag{2.12}$$

Using the second part of τ_1 and the first part of τ_2 given in (2.5) and (2.6), we get

$$\begin{aligned}W[\tau_1,\tau_2] &= \hat{g}(z_1)\hat{g}(z_2)\left[w_2(z_2)w_2'(z_1)\frac{dz_1}{dx} - w_2(z_1)w_2'(z_2)\frac{dz_2}{dx}\right]\\ &\quad + w_2(z_2)w_2(z_1)[\hat{g}(z_1)\hat{g}'(z_2) - \hat{g}(z_2)\hat{g}'(z_1)],\end{aligned}\tag{2.13}$$

where $w'(z) = dw/dz$, $\hat{g}(z) = g(x)/\sqrt{z}$, and $\hat{g}' = d\hat{g}/dx$. From (2.7), we have

$$\frac{dz_1}{dx} = \frac{-dz_2}{dx} = \frac{1}{2}\sqrt{\frac{r(x)}{p(x)}}\tag{2.14}$$

and

$$z_1 + z_2 = \frac{1}{2}\int_{x_1}^{x_2}\sqrt{\frac{r(t)}{p(x)}}\,dt.\tag{2.15}$$

Using (2.14), (2.15), and the asymptotic values of $w_2(z_j)$ and $w_2'(z_j)$ given by (2.10) and (2.11), we obtain

$$W[\tau_1,\tau_2] = \frac{2\lambda\sqrt{z_1 z_2}}{p(x)}\left[Y_{\mu_2}(2\lambda z_2)Y_{\mu_1+1}(2\lambda z_1) + Y_{\mu_2+1}(2\lambda z_2)Y_{\mu_1}(2\lambda z_1)\right] + O\left(\frac{1}{\lambda}\right).$$

From the asymptotic behavior of $Y_{\mu_j}(x)$, namely,

$$Y_{\mu_j}(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\mu_j\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x\sqrt{x}}\right),$$

we obtain

$$\begin{aligned}W[\tau_1,\tau_2] = \frac{4}{\pi p(x)}\Bigg\{&\sin\left(2\lambda z_2 - \frac{\mu_2\pi}{2} - \frac{\pi}{4}\right)\sin\left(2\lambda z_1 - \frac{(\mu_1+1)\pi}{2} - \frac{\pi}{4}\right)\\ &+ \sin\left(2\lambda z_1 - \frac{\mu_1\pi}{2} - \frac{\pi}{4}\right)\sin\left(2\lambda z_2 - \frac{(\mu_2+1)\pi}{2} - \frac{\pi}{4}\right)\Bigg\} + O\left(\frac{1}{\lambda}\right).\end{aligned}$$

Hence,

$$W[\tau_1, \tau_2] = \frac{-4}{\pi p(x)}\left\{\sin\left[2\lambda(z_1 + z_2) - \frac{(\mu_1 + \mu_2)\pi}{2} - \frac{\pi}{2}\right]\right\} + O\left(\frac{1}{\lambda}\right).$$

Therefore, neglecting the terms of $O(1/\alpha)$, (2.12) gives

$$\sin\left[2\alpha_n(z_1 + z_2) - \frac{(\mu_1 + \mu_2)\pi}{2} - \frac{\pi}{2}\right] = 0.$$

Hence,

$$\alpha_n = \left[(n + \tfrac{1}{2})\pi + \frac{(\mu_1 + \mu_2)\pi}{2}\right] \Big/ [2(z_1 + z_2)] + O\left(\frac{1}{n}\right),$$

since the neglected terms are of order $1/\alpha$. Since the corresponding eigenfunctions are bounded at both ends, their asymptotic behavior is given by

$$\text{(2.16)} \quad u_n(x) = \tau(x, \alpha_n) = g(x)\begin{cases} c_1\sqrt{z_1}J_{\mu_1}(2\alpha_n z_1) + O\left(\dfrac{1}{\alpha_n^{3/2}}\right), & x_1 \leqq x < \beta, \\ c_2\sqrt{z_2}J_{\mu_2}(2\alpha_n z_2) + O\left(\dfrac{1}{\alpha_n^{3/2}}\right), & \alpha < x \leqq x_2. \end{cases}$$

Normalizing these eigenfunctions, we find that $c_{1,2} = O(\sqrt{\alpha_n})$, hence $|u_n(x)| = O(\sqrt{\alpha_n})$, since $J_\mu(t)$ is bounded for all $t \geqq 0$ and $\mu \geqq 0$.

LEMMA 2.2. *Let A be the differential operator in* § 1, *and let $\Gamma_{\lambda_n}(x, s)$ be the bounded solution of*

$$\text{(2.17)} \qquad A[\Gamma_{\lambda_n}(x, s)] - \lambda_n^2\Gamma_{\lambda_n}(x, s) = \frac{\delta(x - s)}{r(x)} - u_n(x)u_n(s),$$

where α_n^2 is a given eigenvalue and $u_n(x)$ is the corresponding normalized eigenfunction of A. Also let λ_n^2 be a given real number such that

$$\text{(2.18)} \qquad \lambda_n^2 = \alpha_n^2 + \varepsilon_n, \quad \varepsilon_n = O(\alpha_n),$$

and

$$\text{(2.19)} \qquad \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)u_n(s)r(s)\,ds = 0.$$

Then

$$\text{(2.20)} \qquad \int_{x_1}^{x_2} \Gamma_{\lambda_n}^2(x, s)r(s)\,ds = O\left(\frac{1}{\alpha_n}\right) \quad (\alpha_n \to \infty).$$

Proof. Expressing the solution of (2.17) in terms of the eigenfunction of the operator A, we find that

$$\Gamma_{\lambda_n}(x, s) = -\sum_{m \neq n} \frac{u_m(x)u_m(s)}{\lambda_n^2 - \alpha_m^2}.$$

Hence

$$\text{(2.21)} \qquad \int_{x_1}^{x_2} \Gamma_{\lambda_n}^2(x, s)r(s)\,ds = \sum_{m \neq n} \frac{u_m^2(x)}{(\lambda_n^2 - \alpha_m^2)^2}.$$

In the remainder of the proof we show that the right-hand side of (2.21) equals $O(1/\alpha_n)$. The method to be used is to break the series into partial sums and show that each partial sum is equal to $O(1/\alpha_n)$. Hence we have

$$(2.22)\qquad \sum_{m\neq n}\frac{u_m^2}{(\lambda_n^2-\alpha_m^2)^2}=\left\{\sum_{1}^{k}+\sum_{k+1}^{m'}+\sum_{m'+1}^{n-1}+\sum_{n+1}^{m''}+\sum_{m''}^{\infty}\right\}\frac{u_m^2}{(\lambda_n^2-\alpha_m^2)^2}.$$

For the first sum in (2.22), we choose $k = n/2$ if n is even and $(n + 1)/2$ if n is odd. And from the asymptotic estimate of $u_n(x)$, it is seen that there exists a constant c such that $u_m^2(x) \leqq c\alpha_m$. Hence

$$(2.23)\quad \sum_{m=1}^{k}\frac{u_m^2}{(\lambda_n^2-\alpha_m^2)^2}\leqq c\sum_{m=1}^{k}\frac{\alpha_m}{(\lambda_n^2-\alpha_m^2)^2}\leqq\frac{k\alpha_k}{(\lambda_n-\alpha_k)^2(\lambda_n-\alpha_1)^2}=O\left(\frac{1}{\alpha_n}\right).$$

For the second sum in (2.22), we choose m' such that

$$\lambda_n^2-\alpha_m^2\geqq\tfrac{1}{2}[(an+b+A/a)^2-(am+b)^2]$$

for all $m < m'$ (m sufficiently large), where a, b are fixed constants such that

$$\alpha_m = am + b + f_m \quad (f_m = O(1/m)),$$

$$\lambda_n^2 = \alpha_n^2 + An + g_n \quad (g_n = O(n^p),\quad p < 1).$$

The last two formulas are clear from (2.1) and (2.18). It is sufficient to take $m' = n - 1 + k_1$, where $k_1 = 0$ if $A \geqq 0$ and the integral part of A/a^2 if $A < 0$. Hence

$$(2.24)\quad \sum_{m=k+1}^{n-1+k_1}\frac{u_m^2}{(\lambda_n^2-\alpha_m^2)^2}\leqq\text{const.}\sum_{m=k+1}^{n-1+k_1}\frac{am+b}{[(an+b+A/a)^2-(am+b)^2]^2}.$$

To estimate the sum on the right of (2.24), we compare it with the integral of the function

$$(2.25)\qquad f(x)=\frac{ax+b}{[(an+b+A/a)^2-(ax+b)^2]^2}.$$

It is clear that $f(x)$ is increasing for $k + 1 \leqq x \leqq n - 1 + k_1$. Hence

$$(2.26)\qquad \sum_{m=k+1}^{n-1+k_1} f(m)\leqq\int_{k+1}^{n-1+k_1} f(x)\,dx+f(n-1+k_1).$$

From (2.24) to (2.26), we find that the second sum on the right of (2.22) is equal to $O(1/\alpha_n)$.

For the third sum on the right of (2.22), we have

$$\sum_{m=n+k_1}^{n-1}\frac{u_m}{(\lambda_n^2-\alpha_m^2)^2}\leqq c\sum_{m=n+k_1}^{n-1}\frac{\alpha_m}{(\lambda_n^2-\alpha_m^2)^2}\leqq\frac{-k_1\alpha_{n-1}}{M^2\lambda_n^2}=O\left(\frac{1}{\alpha_n}\right),$$

where $M = \min|\lambda_n - \alpha_m|$, $n + k_1 \leqq n - 1$.

For the last sum on the right of (2.22), we choose m'' such that

$$\alpha_m^2-\lambda_n^2\geqq\tfrac{1}{2}[(am+b)^2-(an+b+A/a)^2],\qquad m>m''.$$

It is sufficient to take $m'' = n + 1 + k_2$, where $k_2 = 0$ if $A \leqq 0$ and the integral part of $1 + (A/a^2)$ if $A > 0$. Comparing the sum with the integral of the function in (2.25), we find that it is equal to $O(1/\alpha_n)$.

For the fourth sum, we have

$$\sum_{m=n+1}^{n+1+k_2} \frac{u_m^2}{(\lambda_n^2 - \alpha_m^2)^2} \leqq c \sum_{m=n+1}^{n+1+k_2} \frac{\alpha_m}{(\lambda_n^2 - \alpha_m^2)^2} \leqq \frac{(k_2 + 1)\alpha_{n+1+k_2}}{M^2 \lambda_n^2} = O\left(\frac{1}{\alpha_n}\right),$$

where $M = \min |\lambda_n - \alpha_m|$, $n + 1 \leqq m \leqq n + 1 + k_2$.

3. Asymptotic solutions of the integro-differential equation.

THEOREM. (a) *Let L and A be the two differential operators defined in* §1, *and let α_n^2 and $U_n(x)$ be the eigenvalues and the corresponding bounded orthonormal eigenfunctions of the differential operator A.*

(b) *Let $K(x, s)$ be a given continuous kernel on $x_1 \leqq x, s \leqq x_2$ which has continuous partial derivatives of the first order on $x_1 \leqq x, s < x_2$, except possibly at $x = s$, where $K(x, s)$ satisfies*

$$L_z\left[\frac{K}{r(x)}\right] = a_i(x)\delta(x - s) + \tilde{K}_i(x, s) \qquad (z = x, i = 1, \text{or } z = s, i = 2), \tag{3.1}$$

where $a_i(x)$ is bounded and $\tilde{K}_i(x, s)$ is square integrable on $x_1 \leqq s \leqq x_2$.

If the integro-differential equation

$$A[y] = \lambda^2\left[y(x) + \int_{x_1}^{x_2} K(x, s)y(s)\,ds\right], \qquad x_1 \leqq x \leqq x_2, \tag{3.2}$$

has an infinite set of solutions, then the normalized bounded solutions for which

$$\lambda_n^2 = \alpha_n^2 + \varepsilon_n, \qquad \varepsilon_n = O(\alpha_n) \tag{3.3}$$

are given by

$$y_n(x) = d_n U_n(x) + r_n(x), \tag{3.4}$$

where $d_n \to 1$ as $n \to \infty$ and $r_n(x) \to 0$ uniformly in x, as $n \to \infty$.

Proof. It is clear that if $K(x, s) = 0$, then the integro-differential equation reduces to $A[y_n] = \lambda_n^2 y_n$. Hence $\lambda_n^2 = \alpha_n^2$, and $y_n(x) = U_n(x)$ and $r_n(x) = 0$. Therefore we assume that $K(x, s) \neq 0$. Let

$$\Phi_n(x) = \lambda_n^2 \mathscr{H}_n(x), \tag{3.5}$$

where

$$\mathscr{H}_n(x) = \int_{x_1}^{x_2} K(x, s)y_n(s)\,ds. \tag{3.6}$$

Then (3.2) becomes

$$A[y_n] - \lambda_n^2 y_n = \Phi_n(x). \tag{3.7}$$

Let $\Gamma_{\lambda_n}(x, s)$ be the bounded generalized Green's function studied in Lemma 2.2; that is, let $\Gamma_{\lambda_n}(x, s)$ be the bounded solution of

$$A[\Gamma_{\lambda_n}(x, s)] - \lambda_n^2 \Gamma_{\lambda_n}(x, s) = \frac{\delta(x - s)}{r(x)} - U_n(x)U_n(s). \tag{3.8}$$

We now apply Green's identity,

$$\int_{x_1}^{x_2} \{uL_1[v] - vL_1[u]\}\, dx = p(x)W(u,v)\Big|_{x=x_1}^{x=x_2},$$

where $L_1 = -r(x)[A - \lambda_n^2]$ and $W(u,v)$ is the Wronskian of u, v. Taking $u(x) = \Gamma_{\lambda_n}(x,s)$ and $v = y_n(x)$, we obtain

$$y_n(x) = b_n U_n(x) + \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x,s)\Phi_n(s)r(s)\, ds, \tag{3.9}$$

since $p(x)$ vanishes at the endpoints, where

$$b_n = \int_{x_1}^{x_2} y_n(s)U_n(s)r(s)\, ds. \tag{3.10}$$

Using the definition of $\Phi_n(x)$ in (3.5), the last equation becomes

$$y_n(x) = b_n U_n(x) + \lambda_n^2 \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x,s)\mathscr{H}_n(s)r(s)\, ds. \tag{3.11}$$

From the symmetry of $\Gamma_{\lambda_n}(x,s)$ and (3.8) we have

$$\lambda_n^2\Gamma_{\lambda_n}(x,s) = -\frac{\delta(x-s)}{r(s)} + U_n(x)U_n(s) - \frac{1}{r(s)}L_s[\Gamma_{\lambda_n}(x,s)].$$

Using the last equation in the integral on the right of (3.11), we obtain

$$\begin{aligned} &\lambda_n^2 \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x,s)\mathscr{H}_n(s)r(s)\, ds \\ &= \int_{x_1}^{x_2} \left\{\frac{\delta(x-s)}{r(s)} + U_n(x)U_n(s) - \frac{1}{r(s)}L_s[\Gamma_{\lambda_n}(x,s)]\right\}\mathscr{H}_n(s)r(s)\, ds \\ &= -\mathscr{H}_n(x) + c_n U_n(x) - \int_{x_1}^{x_2} L_s[\Gamma_{\lambda_n}(x,s)]\mathscr{H}_n(s)\, ds, \end{aligned} \tag{3.12}$$

where

$$c_n = \int_{x_1}^{x_2} U_n(s)\mathscr{H}_n(s)r(s)\, ds. \tag{3.13}$$

Applying Green's identity to the integral on the right of (3.12), we get

$$\int_{x_1}^{x_2} L_s[\Gamma_{\lambda_n}(x,s)]\mathscr{H}_n(s)\, ds = \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x,s)L_s[\mathscr{H}_n(s)]\, ds, \tag{3.14}$$

since $p(s)$ vanishes at the endpoints. Applying the operator L_x to $\mathscr{H}_n(x)$ in (3.6) and using (3.1), we obtain

$$L_x[\mathscr{H}_n(x)] = a_1(x)r(x)y_n(x) + \int_{x_1}^{x_2} \tilde{K}_1(x,s')y_n(s')r(s')\, ds'. \tag{3.15}$$

From (3.12), (3.14), and (3.15) we have

$$\lambda_n^2 \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)\mathscr{H}_n(s)r(s)\, ds$$
$$= -\mathscr{H}_n(x) + c_n U_n(x) - \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)a_1(s)y_n(s)r(s)\, ds$$
$$- \int_{x_1}^{x_2}\int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)\tilde{K}_1(s, s')y_n(s')r(s')\, ds\, ds'.$$

Substituting the above result in (3.11), we obtain

$$\begin{aligned} y_n(x) = d_n U_n(x) - \mathscr{H}_n(x) - \int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)a_1(s)y_n(s)r(s)\, ds \\ - \int_{x_1}^{x_2}\int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)\tilde{K}_1(s, s')y_n(s')r(s')\, ds\, ds', \end{aligned} \tag{3.16}$$

where

$$d_n = b_n + c_n. \tag{3.17}$$

Using Schwarz' inequality and (2.20), we get

$$\left|\int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)a_1(s)y_n(s)r(s)\, ds\right| = O(\alpha_n^{-1/2}); \tag{3.18}$$

also

$$\left|\int_{x_1}^{x_2}\int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)\tilde{K}_1(s, s')y_n(s)r(s')\, ds'\, ds\right|^2$$
$$\leqq \int_{x_1}^{x_2} \Gamma_{\lambda_n}^2(x, s)\, ds \cdot \int_{x_1}^{x_2}\left[\int_{x_1}^{x_2} \tilde{K}_1(s, s')y_n(s')r(s')\, ds\right]^2 ds$$
$$\leqq \int_{x_1}^{x_2} \Gamma_{\lambda_n}^2(x, s)\, ds \cdot \int_{x_1}^{x_2}\left[\int_{x_1}^{x_2} \tilde{K}_1^2(s, s')r(s')\, ds' \cdot \int_{x_1}^{x_2} y_n^2(s')r(s')\, ds'\right] ds.$$

Hence

$$\int_{x_1}^{x_2}\int_{x_1}^{x_2} \Gamma_{\lambda_n}(x, s)\tilde{K}_1(s, s')y_n(s)r(s')\, ds'\, ds = O(\alpha_n^{-1/2}). \tag{3.19}$$

From (3.16) to (3.19), we obtain

$$y_n(x) = d_n U_n(x) - \mathscr{H}_n(x) + I_n(x), \tag{3.20}$$

where

$$I_n(x) = O(\alpha_n^{-1/2}). \tag{3.21}$$

From the definition of $\mathscr{H}_n(x)$ in (3.6),

$$|\mathscr{H}_n(x)| = O(1). \tag{3.22}$$

From (3.10), (3.13), (3.17), (3.22) and the use of Schwarz' inequality we find that

$$|d_n| \leqq O(1). \tag{3.23}$$

From (3.6) and (3.20), we get the following integral equation for $\mathscr{H}_n(x)$:

$$\mathscr{H}_n(x) = f_n(x) - \int_{x_1}^{x_2} K(x, s)\mathscr{H}_n(s)\,ds, \tag{3.24}$$

where

$$f_n(x) = d_n \int_{x_1}^{x_2} K(x, s)U_n(s)\,ds + \int_{x_1}^{x_2} K(x, s)I_n(s)\,ds. \tag{3.25}$$

Again using Schwarz' inequality and (3.21), we see that the last integral in (3.25) is $O(\alpha_n^{-1/2})$. The first integral is $O(\alpha_n^{-3/2})$, because

$$\begin{aligned}\int_{x_1}^{x_2} K(x, s)U_n(s)\,ds &= -\frac{1}{\alpha_n^2}\int_{x_1}^{x_2} \frac{K(x, s)}{r(s)} L_s[U_n(s)]\,ds \\ &= \frac{-1}{\alpha_n^2}\int_{x_1}^{x_2} U_n(s)L_s\left[\frac{K(x, s)}{r(s)}\right] ds \\ &= \frac{-1}{\alpha_n^2}\int_{x_1}^{x_2} U_n(s)\,[a_2(x)\delta(x - s) + \tilde{K}_2(x, s)]\,ds \\ &= \frac{-a_2(x)U_n(x)}{\alpha_n^2} - \frac{1}{\alpha_n^2}\int_{x_1}^{x_2} \tilde{K}_x(x, s)U_n(s)\,ds.\end{aligned}$$

Hence

$$f_n(x) = O(\alpha_n^{-1/2}). \tag{3.26}$$

If $k(x, s)$ denotes the resolvent kernel corresponding to the kernel $K(x, s)$ in the integral equation (3.24), and if $\phi_1, \phi_2, \cdots, \phi_r$ are the nontrivial orthonormal solutions of the homogeneous part of (3.24), then we get

$$\mathscr{H}_n(x) = \sum_{m=1}^{r} a_{nm}\phi_m(x) + f_n(x) + \int_{x_1}^{x_2} k(x, s)f_n(s)\,ds. \tag{3.27}$$

It is clear that if -1 is not an eigenvalue of the kernel $K(x, s)$, $\phi_m(x) = 0$ and

$$\mathscr{H}_n(x) = f_n(x) + \int_{x_1}^{x_2} k(x, s)f_n(s)\,ds.$$

Hence $\mathscr{H}_n(x) = O(\alpha_n^{-1/2})$. If -1 is an eigenvalue of the kernel $K(x, s)$, then $\mathscr{H}_n(x)$ is given by (3.27). In a similar way to the method used in [2, p. 120], we obtain

$$\lim_{n\to\infty} a_{n\mu} = 0, \tag{3.28}$$

and consequently from (3.27),

$$\lim_{\alpha_n\to\infty} \mathscr{H}_n(x) = 0. \tag{3.29}$$

From (3.20), (3.21), and (3.29), we have

$$y_n(x) = d_nU_n(x) + r_n(x),$$

where $r_n(x) \to 0$ uniformly in x as $n \to \infty$. From the normalization of $y_n(x)$ and $U_n(x)$, $d_n \to 1$ as $n \to \infty$.

Appendix. In this Appendix we obtain the uniform asymptotic solutions of the differential equation (2.4). Let

$$\tau(x, \lambda) = v/\sqrt{p(x)}. \tag{A.1}$$

Then (2.4) reduces to

$$\frac{d^2 v}{dx^2} = (\lambda^2 \hat{p}(x) + \hat{q}(x))v, \tag{A.2}$$

where $\hat{p}(x) = -r(x)/p(x)$ and

$$\hat{q}(x) = -\left[4q(x)p(x) + \left(\frac{dp}{dx}\right)^2 - 2p(x)\frac{d^2 p}{dx^2}\right] \Big/ \{4p^2(x)\}.$$

From the properties of the functions $r(x)$, $p(x)$, and $q(x)$, we obtain the behavior of the functions $\hat{p}(x)$ and $\hat{q}(x)$ near x_1 and x_2, namely,

$$p(x) = -\frac{r(x_j)}{p'(x_j)}(x - x_j)^{-1}\{1 + O(x - x_j)\}, \qquad j = 1, 2,$$

and

$$\hat{q}(x) = -\hat{q}_j(x - x_j)^{-2}\{1 + O(x - x_j)\}, \qquad j = 1, 2,$$

where

$$\hat{q}_j = \frac{q_j}{p'(x_j)} + \frac{1}{4}, \qquad j = 1, 2.$$

Hence the differential equation (A.2) is similar to the differential equation in Case D in Olver's work on the asymptotic solutions of linear differential equations [4]. We follow the same procedure to find the asymptotic solutions. We take new dependent and independent variables w and ξ related by

$$w = \frac{v}{\sqrt{\dot{x}}}, \qquad \dot{x} = \frac{dx}{d\xi},$$

where x and ξ are related by

$$-\dot{x}^2 \hat{p}(x) = 1/\xi. \tag{A.3}$$

Then w satisfies

$$\frac{d^2 w}{d\xi^2} = \left[-\frac{\lambda^2}{\xi} + f(\xi)\right] w, \tag{A.4}$$

where

$$f(\xi) = \dot{x}^2 \hat{q}(x) + \dot{x}^{1/2} \frac{d^2}{d\xi^2}[\dot{x}^{-1/2}]. \tag{A.5}$$

On integrating (A.3), we obtain

$$\xi_j(x) = \left[\frac{1}{2}\int_{x_j}^{x} \sqrt{\frac{r(t)}{p(t)}}\, dt\right]^2, \qquad j = 1, 2.$$

From the properties of $r(x)$ and $p(x)$, we find that

$$\xi_j(x) = \frac{r(x_j)}{p'(x_j)}(x - x_j)\{1 + (x - x_j)O(1)\} \quad \text{as } x \to x_j. \tag{A.6}$$

Hence $\xi_j(x)$ is regular at $x = x_j$; it is also positive and continuous on $x_1 \leqq x \leqq x_2$. Also, $f(\xi_j)$ has a double pole at $\xi_j = 0$ since $x = x_j$ corresponds to $\xi_j = 0$ and $\hat{q}(x)$ has a double pole at $x = x_j$.

Let

$$z_j(x) = \frac{(-)^{j-1}}{2}\int_{x_j}^{x} \sqrt{\frac{r(t)}{p(t)}}\,dt$$

and $\mu_j^2 = 1 - 4\hat{q}_j$, where $j = 1, 2$. Then $\xi_j = z_j^2$ and (A.4) reduces to

$$\frac{d^2w}{dz_j^2} = \frac{1}{z_j}\frac{dw}{dz_j} + \left[-4\lambda^2 + \frac{\mu_j^2 - 1}{z_j^2} + \hat{f}(z_j)\right]w, \tag{A.7}$$

where

$$\hat{f}(z_j) = 4h(z_j^2), \qquad h(z_j^2) = \xi_j[f(\xi_j) + \hat{q}_j/\xi_j^2], \qquad j = 1, 2. \tag{A.8}$$

Using (A.3), (A.5), (A.6), and (A.8), we obtain

$$\hat{f}(z_j) = \frac{p(x)}{r(x)}\left[\hat{q}(x) + \dot{x}^{-3/2}\frac{d^2}{d\xi_j^2}(\dot{x}^{-1/2})\right] + \hat{q}_j\frac{r(x)}{p(x)}, \qquad j = 1, 2.$$

From the properties of $p(x)$, $r(x)$, and $\hat{q}(x)$, there exist constants α, β, and γ such that:

(i) For $x_1 \leqq x \leqq \beta$, $\beta < x_2$, $z_1(x)$ ranges over $0 \leqq z_1(x) \leqq \gamma = z_1(\beta)$, and $f(z_1)$ is regular on $0 \leqq z_1 \leqq \gamma$.

(ii) For $x_2 \geqq x > \alpha$, $\alpha > x_1$, $z_2(x)$ ranges over $0 \leqq z_2(x) \leqq \gamma = z_2(\alpha)$, and $f(z_2)$ is regular on $0 \leqq z_2 \leqq \gamma$.

(iii) There is a common domain $0 \leqq z_j \leqq \gamma, j = 1, 2$, which corresponds to $\alpha \leqq x \leqq \beta$, for which $f(z_j)$ is regular.

The two equations in (A.7) are in the standard form of the differential equation treated in [4, p. 78], except that the basic equation is now the Bessel equation rather than the modified Bessel equation. Appropriate basic solutions are $z_j J_{\mu_j}(2\lambda z_j)$ and $z_j Y_{\mu_j}(2\lambda z_j)$. And, in a similar way to Theorem D in Olver's paper [4], we obtain (2.8) to (2.11). Retracing the transformations, we obtain (2.5) and (2.6).

REFERENCES

[1] B. I. Aleksandriskii, *Asymptotic properties of the solutions of linear integro-differential equations of second order*, Izv. Vyssh. Uchebn. Zaved. Mathematika, 1963, no. 6 (37), pp. 3–14. (In Russian.)

[2] K. O. Friedrichs, *Criteria for discrete spectra*, Symposium on the Theory of Electromagnetic Waves (New York, 1950), Interscience, New York, 1951, pp. 430–449.

[3] A. Hammerstein, *Über die asymptotische Darstellung der Eigenfunktionen linear Integralgleichungen. I*, Math. Ann., 93 (1924), pp. 113–129.

[4] F. W. J. Olver, *The asymptotic solution of linear differential equations of second order in a domain containing one transition point*, Philos. Trans. Roy. Soc. Ser. A, 249 (1957), pp. 65–97.

[5] H. WEYL, *Das asymptotische Verteilungsgeset der Eigenverte-Linear partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohbraumstrahlung)*, Math. Ann., 71 (1910), pp. 441–479.

[6] S. E. SHAMMA, *Asymptotic eigenfunctions of integral equations and certain generalizations of separation of variables*, Doctoral thesis, New York University, New York, 1969.

[7] S. N. KARP AND S. E. SHAMMA, *A generalization of separability in boundary value problems*, SIAM J. Appl. Math., 20 (1971), pp. 536–546.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

COMPLEX ZEROS OF LINEAR COMBINATIONS OF SPHERICAL BESSEL FUNCTIONS AND THEIR DERIVATIVES*

B. DAVIES†

Abstract. We investigate the zeros of the functions $[\cos(\pi s) j_\ell(x) + \sin(\pi s) y_\ell(x)]$ and $[\cos(\pi s) \cdot (x j_\ell(x))' + \sin(\pi s)(x y_\ell(x))']$ for arbitrary real values of s and integer ℓ. In particular, we find the number of complex zeros for each value of s, and give qualitative information about the loci of the complex zeros as s varies. The method depends on elementary considerations from the theory of first order nonlinear differential equations.

1. Introduction. In the course of an investigation which involves the normal modes of electromagnetic radiation in a spherical cavity [4], we have found it necessary to have certain information about the complex zeros of the functions

$$(1)\qquad \begin{aligned} z_\ell(s, x) &= \cos(\pi s) j_\ell(x) + \sin(\pi s) y_\ell(x), \\ \tilde{z}_\ell(s, x) &= \cos(\pi s)(x j_\ell(x))' + \sin(\pi s)(x y_\ell(x))', \end{aligned}$$

where $j_\ell(x)$ and $y_\ell(x)$ are spherical Bessel and Neumann functions [1], and ℓ is an integer. In particular, we have needed to know not only the number of complex zeros of these functions for real values of s, but also the qualitative nature of the curves in the complex plane determined by the functional equations

$$(2a)\qquad z_\ell(s, x(s)) = 0,$$

$$(2b)\qquad \tilde{z}_\ell(s, x(s)) = 0$$

and their relation to the zeros of spherical Hankel functions. This information does not seem to be available among the considerable literature on the zeros of Bessel functions [1], [5], [6] and so we investigate the problem in the remainder of this paper, using an extremely simple method which depends on an analysis of the qualitative behavior of the solutions of a first order nonlinear differential equation.

2. Zeros of $z_\ell(s, x)$. In this section we investigate the zeros of the function $z_\ell(s, x)$ for arbitrary real s. In order to simplify some of the discussions, we restrict ℓ to positive values, neglecting the trivial case of $\ell = 0$. We begin by considering the differential equation

$$(3)\qquad \frac{d}{ds} z_\ell(s, x) = 0.$$

Explicitly, (3) may be written as

$$(4)\qquad \begin{aligned} dx/ds &= N_\ell(s, x)/D_\ell(s, x), \\ N_\ell(s, x) &= -\partial z_\ell/\partial s, \\ D_\ell(s, x) &= \partial z_\ell/\partial x. \end{aligned}$$

Suppose we solve (3), choosing as initial condition any pair x_0, s_0 which satisfy $z_\ell(s_0, x_0) = 0$; then define s_1 and s_2 by the condition that the open interval s_1

* Received by the editors March 2, 1971, and in final revised form January 28, 1972.

† Department of Applied Mathematics, Australian National University, Canberra, Australia.

$< s < s_2$ is the largest open interval containing s_0 such that $D_\ell(s, x(s)) \neq 0$ for $s_1 < s < s_2$. Then the curve defined by $x(s)$ is unique and smooth for $s_1 < s < s_2$ (see [2]), and is a locus in the complex plane of zeros of $z_\ell(s, x)$.

It is well known [1] that the function $z_\ell(s, x)$ has an infinite set of real simple zeros for each value of s, and that for two different values of s these zeros interleave. Therefore, $N_\ell(s, x) = z_\ell(s - \frac{1}{2}, x)$ and $D_\ell(s, x)$ are real and alternate in sign at the zeros of $z_\ell(s, x)$, and dx/ds is always of the same sign at each real zero. By examining asymptotic forms it is easy to conclude that dx/ds is real and positive for real x, and that the real axis is a locus of zeros, which move in a positive direction as s increases.[1]

To investigate other solutions of (3), we need to know what values s_1 and s_2 may assume, and the corresponding values of the limits

$$x_1 = \lim_{s \to s_1} x(s), \qquad x_2 = \lim_{s \to s_2} x(s) \tag{5}$$

if they exist. There are several possibilities; for s_1 they are as follows:

(a) s_1 is finite, x_1 is finite. This gives the conditions

$$\begin{aligned} z_\ell(s_1, x_1) &= \cos(\pi s_1) j_\ell(x_1) + \sin(\pi s_1) y_\ell(x_1) = 0, \\ D_\ell(s_1, x_1) &= \cos(\pi s_1) j'_\ell(x_1) + \sin(\pi s_1) y'_\ell(x_1) = 0. \end{aligned} \tag{6}$$

The determinant of these equations is the Wronskian x_1^{-2} and since it is never zero, the only solution is

$$\begin{aligned} \sin(\pi s_1) &= 0, \\ s_1 &= n, \qquad n = 0, \pm 1, \pm 2, \cdots, \\ x_1 &= 0, \end{aligned} \tag{7}$$

(b) s_1 is finite, $|x_1|$ is infinite. Then we must have $z_\ell(s, x) = 0$ and $D_\ell(s, x) \to 0$ as $s \to s_1$, and using the asymptotic forms for the Bessel functions of large argument, we get the conditions

$$\begin{aligned} \sin(\pi s + x - \ell\pi/2) + O(x^{-1}) &\to 0, \\ \cos(\pi s + x - \ell\pi/2) + O(x^{-1}) &\to 0 \end{aligned} \tag{8}$$

as $s \to s_1$, which are impossible to satisfy simultaneously.

(c) $s_1 = -\infty$. In this case the solution $x(s)$ is unique for all $s \leqq s_0$, and in particular, must be one of the real roots of $j_\ell(x)$ whenever s is an integer. But the uniqueness of the solution implies that the locus is the negative real x-axis.

Similar considerations apply to s_2, so that we find that the only singular point of solutions of (3) subject to given initial conditions is $x = 0$ when $s = 0, \pm 1, \pm 2, \cdots$.

To find the loci of the complex zeros we first note that for $s = n$ there are no complex zeros, since $j_\ell(x)$ has only real zeros. Hence when s is close to an integer, the complex zeros are close to zero, since this is the only singular point of (3).

[1] This result is shown in [3], where the real zeros of z_ℓ and $\tilde{z}_\ell$ are investigated.

Using the asymptotic forms for $j_\ell(x)$ and $y_\ell(x)$ in the neighborhood of $x = 0$, we find that if $s = n + \varepsilon$, then[2]

$$x \approx [\pi(2\ell + 1)!!(2\ell - 1)!!\varepsilon]^{1/(2\ell+1)}. \tag{9}$$

For $\varepsilon > 0$, we have one real positive root and 2ℓ complex roots with arguments

$$\arg(x) = +\frac{2\pi j}{2\ell + 1}, \qquad j = 1, 2, \cdots, 2\ell. \tag{10}$$

When $\varepsilon < 0$, we have one real negative root and 2ℓ complex roots with arguments

$$\arg(x) = -\pi + \frac{2\pi j}{2\ell + 1}, \qquad j = 1, 2, \cdots, 2\ell. \tag{11}$$

As s increases from $n + \varepsilon$ to $n + 1 - \varepsilon$, each of the complex roots must return to the origin. Furthermore, no two loci can intersect (since this would imply a singular point) so that the complex roots travel in 2ℓ closed loops, half in the upper half-plane and half in the lower half-plane.[3] We have therefore proved that the function $z_\ell(s, x)$ has exactly 2ℓ complex roots for every nonintegral value of s. A sketch of the loci for $\ell = 2$ is shown in Fig. 1, with arrows indicating the direction of motion of the zeros for increasing s. We shall show in § 3 that no loop may contain another loop in its interior, so that the arrangement shown in Fig. 1 is the only possible one.

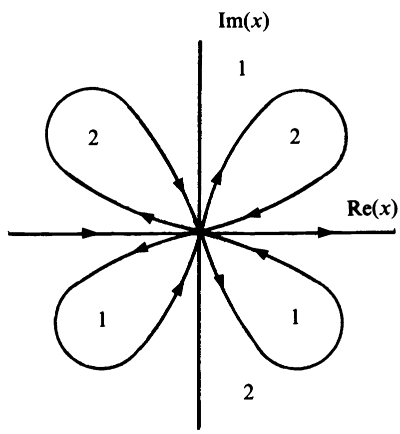

FIG. 1. *Zeros of* $z_2(s, x)$

3. Relation to the zeros of spherical Hankel functions. We shall now show that each closed loop of $z_\ell(s, x) = 0$ circles exactly one complex zero of one of the two spherical Hankel functions defined by [1]

$$\begin{aligned} h_\ell^{(1)}(x) &= j_\ell(x) + iy_\ell(x), \\ h_\ell^{(2)}(x) &= j_\ell(x) - iy_\ell(x). \end{aligned} \tag{12}$$

[2] $(2\ell + 1)!! = (2\ell + 1)(2\ell - 1) \cdots (3)(1)$.

[3] It is trivial to show that the complex zeros occur in complex conjugate pairs.

We first note that the function $z_\ell(s, x)$ may be expressed as

$$z_\ell(s, x) = \tfrac{1}{2}[e^{-i\pi s}h_\ell^{(1)}(x) + e^{i\pi s}h_\ell^{(2)}(x)] \tag{13}$$

so that if $z_\ell(s, x) = 0$ we have

$$|h_\ell^{(1)}(x)| = |h_\ell^{(2)}(x)|, \tag{14a}$$

$$\arg[h_\ell^{(1)}(x)] - \arg[h_\ell^{(2)}(x)] = 2\pi(s - \tfrac{1}{2} + n), \tag{14b}$$

where n is an arbitrary integer. Conversely, if $|h_\ell^{(1)}(x)| = |h_\ell^{(2)}(x)|$, equation (14b) shows that there exist values of s such that $z_\ell(s, x) = 0$. Consequently the curves defined by (2a) are equivalent to those defined by the condition $|h_\ell^{(1)}(x)| = |h_\ell^{(2)}(x)|$. In fact, the curves along which $z_\ell(s, x) = 0$ separate regions where $|h_\ell^{(1)}(x) \lessgtr |h_\ell^{(2)}(x)|$, and we have marked the regions in Fig. 1 with the numbers 1 or 2 as $|h_\ell^{(1)}|$ or $|h_\ell^{(2)}|$ is the smaller. We shall prove this last assertion. In the upper half-plane, using the asymptotic forms of the spherical Hankel functions, we find $|h_\ell^{(1)}(x)| < |h_\ell^{(2)}(x)|$ outside the closed loops. From the facts which are proved below about the loops, particularly that each loop contains one zero of $h_\ell^{(2)}(x)$, we have $|h_\ell^{(1)}(x)| > |h_\ell^{(2)}(x)|$ in the interior of each loop. A similar argument applied in the lower half-plane completes the proof.

Now we consider the integral[4]

$$N = \frac{1}{2\pi i}\int_C \left\{\frac{h_\ell'^{(1)}(x)}{h_\ell^{(1)}(x)} - \frac{h_\ell'^{(2)}(x)}{h_\ell^{(2)}(x)}\right\} dx \tag{15}$$

taken in a positive (anti-clockwise) direction around one of the closed loops. By the principle of argument, its value is equal to the difference between the number of zeros of $h_\ell^{(1)}$ and $h_\ell^{(2)}$ inside the loop. Using (14), we can write (15) as

$$\begin{aligned} N &= \frac{1}{2\pi i}\int_C \frac{d}{dx}\ln\{h_\ell^{(1)}(x)/h_\ell^{(2)}(x)\}\,dx \\ &= \int_C ds = \pm 1. \end{aligned} \tag{16}$$

We get $+1$ if the zeros of $z_\ell(x, s)$ move in a positive (anti-clockwise) direction as s increases, -1 if they move in a negative direction. Now it is well known that $h_\ell^{(1)}$ has ℓ zeros in the lower half-plane and $h_\ell^{(2)}$ has ℓ zeros in the upper half-plane, and since there are ℓ loops in each half-plane, which cannot intersect each other, each loop circles one zero of a spherical Hankel function and no loop contains another. This justifies our statements made at the end of § 2 and following (14).

4. The function $\tilde{z}_\ell(s, x)$. The investigation of the zeros of $\tilde{z}_\ell(s, x)$ proceeds in exactly the same manner as for $z_\ell(s, x)$. The differential equation (3) is the analogue of (4), with

$$\begin{aligned} \tilde{N}(s, x) &= \sin(\pi s)(xj_\ell(x))' - \cos(\pi s)(xy_\ell(x))', \\ \tilde{D}(s, x) &= [\ell(\ell + 1)/x^2 - 1][\cos(\pi s)(xj_\ell(x)) + \sin(\pi s)(xy_\ell(x))], \end{aligned} \tag{17}$$

---

[4] Although the origin is on the contour, the integrand does not have a pole there, on account of cancellations.

and the solutions of this equation with initial conditions $\tilde{z}_\ell(s_0, x_0) = 0$ give a set of smooth curves in the complex plane, except for points where we can simultaneously satisfy the conditions $\tilde{z}_\ell(s, x) = 0$ and $\tilde{D}_\ell(s, x) = 0$. One solution of these equations is again $s = n, x = 0$, but we now have two additional singular points $(x_c^\pm, s_c^\pm)$, given by

$$\begin{aligned} x_c^\pm &= \pm\sqrt{\ell(\ell+1)}, \\ \tan(\pi s_c^\pm) &= -[(xj_\ell(x))'/(xy_\ell(x))']_{x_c^\pm}. \end{aligned} \tag{18}$$

We shall investigate the nature of the loci of zeros in the neighborhood of these critical points below.

Similar arguments to those presented in § 2 show that the real axis is a locus of zeros. Now, however, the presence of the factor $[\ell(\ell + 1)/x^2 - 1]$ in $\tilde{D}_\ell(s, x)$ gives a slightly different result for the sign of dx/ds: it is

$$\frac{dx}{ds}\begin{cases} >0, & |x| > x_c^+, \\ <0, & |x| < x_c^+. \end{cases} \tag{19}$$

Hence the zeros travel in the direction of increasing x if $|x| > x_c^+$, and the opposite way if $|x| < x_c^+$. Let us choose $s = 0$ at $x = 0$, and follow the change in s as x varies along the real axis to x_c^+. We have just noted that s is monotonic decreasing as x increases in this range; we now want to show that $s_c^+ > -1/2$. Inspection shows that if $s_c^+ < -1/2$, then the function $(xy_\ell(x))'$ has a real zero for some $x < x_c^+$. But the first positive real zero of $(xy_\ell(x))'$ is larger than the first zero of $y_\ell(x)$, which in turn is larger than $\ell + 1/2$ (see [1]). Hence we see that $s_c^+ > -1/2$; and by a similar argument that $s_c^- < +1/2$.

Now we shall investigate the nature of the complex curves in the neighborhood of the points $x = 0$ and $x = x_c^\pm$. For x close to zero, we again put $s = n + \varepsilon$, and find that if $\varepsilon > 0$ we have a real negative root and 2ℓ complex roots with arguments

$$\arg(x) = -\pi + \frac{2\pi j}{2\ell + 1}, \qquad j = 1, 2, \cdots, 2\ell. \tag{20}$$

If $\varepsilon < 0$, we have a real positive root and 2ℓ complex roots with arguments

$$\arg(x) = \frac{2\pi j}{2\ell + 1}, \qquad j = 1, 2, \cdots, 2\ell. \tag{21}$$

We must also determine the nature of the curves when x is close to x_c^+ and s to s_c^+. We write $x = x_c^+ + \delta$ and $s = s_c^+ + \varepsilon$ and expand $\tilde{z}_\ell(s, x)$ in a Taylor series about the critical point, keeping only the first nonvanishing terms in ε and δ, to get

$$\varepsilon\left[\frac{\partial}{\partial s}\tilde{z}_\ell(s, x)\right]_{s_c, x_c} + \frac{1}{2}\delta^2\left[\frac{\partial^2}{\partial x^2}\tilde{z}_\ell(s, x)\right]_{s_c, x_c} \approx 0 \tag{22}$$

which gives $\delta^2 \approx A\varepsilon$ for some constant A. We have seen that as s increases from s_c^+, two real roots move away from x_c^+, hence A is a positive constant and for negative values of ε we get the approximation

$$x \approx x_c^+ \pm i\sqrt{A(s_c^+ - s)}. \tag{23}$$

Hence as s approaches s_c^+ from below, two complex roots move in toward the point x_c^+, and then move off along the real axis as s exceeds s_c^+. The behavior near x_c^- is similar; in this case there are two real roots for $s < s_c^-$, and as s increases through this critical value they coalesce, and then move off into the complex plane.

We now have sufficient information to sketch the closed loops in the complex plane, and we have done this for $\ell = 2$ in Fig. 2. From our considerations, we may also say how many complex zeros are possessed by the function $\tilde{z}_\ell(s, x)$ for any particular value of s. When s is an integer, there are no complex zeros. If s lies in the range $n + s_c^+ < s < n + s_c^-$, there are 2ℓ complex zeros, half of them in the upper half-plane and half of them in the lower half-plane. For s in the range $n + s_c^- < s < n + 1 + s_c^+$, there are $2\ell + 2$ complex zeros, and it is interesting to note that this case includes the complex zeros of the function $(xy_\ell(x))'$.

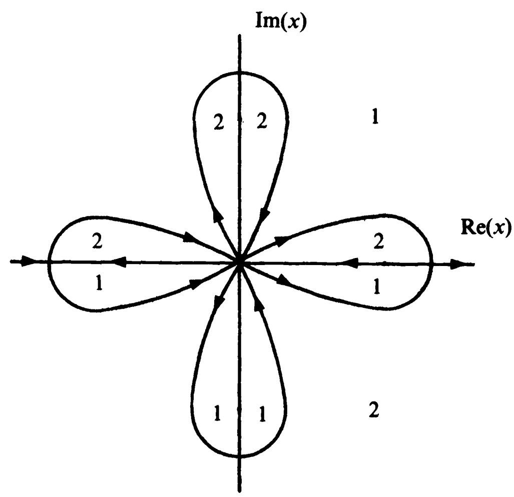

FIG. 2. *Zeros of* $\tilde{z}_2(s, x)$

Finally we remark that the method of § 3 shows that Fig. 2 is the only possible arrangement, and that each closed loop in the upper half-plane encloses exactly one zero of the function $(xh_\ell^{(2)}(x))'$. Since the zeros of this function are the roots of a polynomial of degree $\ell + 1$, this exhausts all the zeros. Similarly the loops in the lower half-plane each enclose one zero of the function $(xh_\ell^{(1)}(x))'$.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1965. (Extensive bibliographies may be found here and in [5], [6].)

[2] G. Birkhoff and G. C. Rota, *Ordinary Differential Equations*, Blaisdell, New York, 1962.

[3] T. H. Boyer, *Concerning the zeros of some functions related to Bessel functions*, J. Mathematical Phys., 10 (1969), pp. 1729–1744.

[4] B. Davies, *Quantum electromagnetic zero-point energy of a conducting spherical shell*, Ibid., 13 (1972), pp. 1324–1328.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.

[6] F. W. J. Olver, ed., *Bessel functions. Part III: Zeros and associated values*, Cambridge University Press, Cambridge, 1960.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

STABILITY THEORY FOR MULTIPLE EQUILIBRIUM STATES OF A NONLINEAR DIFFUSION PROCESS: A SINGULARLY PERTURBED EIGENVALUE PROBLEM*

HERBERT B. KELLER†

Abstract. It has been shown, previously, that the parabolic problem: $Y_t - \varepsilon Y_{xx} - Y_x + g(x, Y) = 0$, $Y_x(t, 0) - aY(t, 0) = A$, $f(Y(t, 1), Y_x(t, 1)) = 0$ has, for sufficiently small $\varepsilon > 0$, a distinct equilibrium state $y_j(x, \varepsilon)$ for each simple root α_j of $F(\alpha) \equiv f(\alpha, g(1, \alpha)) = 0$. Now it is shown that $y_j(x, \varepsilon)$ is stable or unstable provided that $dF(\alpha_j)/d\alpha > 0$ or <0, respectively (with $f_z(y, z) > 0$). This problem leads to a singularly perturbed eigenvalue problem of the form: $\varepsilon\varphi_{xx} + p(x)\varphi_x + [\lambda r(x) - q(x)]\varphi = 0$, $a_0\varphi(0) + a_1\varphi_x(0) = 0$, $b_0\varphi(1) + b_1\varphi_x(1) = 0$. It is shown that as $\varepsilon \downarrow 0 (\varepsilon \uparrow 0)$ the least (greatest) eigenvalue is given by $\hat{\lambda}(\varepsilon) = \lambda_0^+ + O(\varepsilon)(\lambda_0^- + O(\varepsilon))$, where $\lambda_0^+ = [q(1) + b_0 b_1^{-1} p(1)]/r(1)$ and $\lambda_0^- = [q(0) + a_0 a_1^{-1} p(0)]/r(0)$.

1. Introduction. We consider diffusion processes for a quantity $Y(t, x)$ which satisfies a nonlinear initial boundary value problem of the form:

$$Y_t - \varepsilon Y_{xx} - Y_x + g(x, Y) = 0, \qquad 0 \leqq x \leqq 1, \quad t > 0; \tag{1.1a}$$

$$Y_x(t, 0) - aY(t, 0) = A, \quad f(Y(t, 1), Y_x(t, 1)) = 0, \quad t > 0; \tag{1.1b}$$

$$Y(0, x) = Y_0(x), \qquad 0 \leqq x \leqq 1. \tag{1.1c}$$

In particular, the (dimensionless) temperature in various adiabatic chemical reactors with first order irreversible reactions can be shown to satisfy such a system. See R. Aris [1] for a detailed formulation of these applications. The equilibrium states $y(x)$ of such processes are determined by nonlinear two-point boundary value problems of the form:

$$\varepsilon y_{xx} + y_x = g(x, y), \qquad 0 \leqq x \leqq 1; \tag{1.2a}$$

$$y_x(0) - ay(0) = A; \tag{1.2b}$$

$$f(y(1), y_x(1)) = 0. \tag{1.2c}$$

Recent experiments with chemical reactors have shown that multiple equilibrium states can exist and be "stable" and with some experimental finesse it is made rather clear that "unstable" equilibrium states also exist [6]. Theoretical stability analysis for all such problems governed by (1.1) with sufficiently small $\varepsilon > 0$ will be given here.

An existence theory for nonunique solutions of (1.2), for small $\varepsilon > 0$, has just been given by the author [3] under relatively mild assumptions on $g(x, y)$ and $f(y, z)$. In particular, it is shown that if the equation

$$F(\alpha) \equiv f(\alpha, g(1, \alpha)) = 0 \tag{1.3a}$$

* Received by the editors November 19, 1971, and in revised form March 15, 1972.

† Applied Mathematics, Firestone Laboratory, California Institute of Technology, Pasadena, California 91109. This work was supported by the U.S. Army Research Office, Durham, under Contract DAHC 04-68-0007.

has J simple roots α_j in an appropriate interval:

(1.3b) $$M < \alpha_1 < \alpha_2 < \cdots < \alpha_J < N,$$

then (1.2) has at least J distinct solutions, $y = y_j(x, \varepsilon)$, provided $\varepsilon > 0$ is sufficiently small. Under slightly stronger smoothness conditions on $g(x, y)$ and with $f_z(y, z) > 0$, we shall show that

(1.4) $$y_j(x, \varepsilon) \quad \text{is} \begin{Bmatrix} \text{stable} \\ \text{unstable} \end{Bmatrix} \quad \text{if } \frac{dF(\alpha_j)}{d\alpha} \begin{cases} > 0, \\ < 0. \end{cases}$$

Thus the entire existence and stability theory for equilibrium states of (1.1) with small $\varepsilon > 0$ is reduced to a study of some simple properties of the given function $F(\alpha)$.

The stability problem indicated above is formulaed in § 2 and leads to a singularly perturbed eigenvalue problem which is a special case of

(1.5a) $$\varepsilon\varphi_{xx} + p(x)\varphi_x + [\lambda r(x) - q(x)]\varphi = 0,$$

(1.5b) $$a_1\varphi_x(0) + a_0\varphi(0) = 0,$$

(1.5c) $$b_1\varphi_x(1) + b_0\varphi(1) = 0.$$

For stability we need only examine the sign of the least eigenvalue of (1.5) for small $\varepsilon > 0$. However, in § 3 and § 4 we find the limit of $\hat{\lambda}(\varepsilon)$, the least (greatest) eigenvalue of (1.5) as $\varepsilon\downarrow 0$ ($\varepsilon\uparrow 0$). More precisely with

(1.6a) $$p(x), q(x), r(x) \in \mathbb{C}_1[0, 1],$$

(1.6b) $$p(x) \geqq p_0 > 0, \quad r(x) \geqq r_0 > 0,$$

we show that

(1.7) $$\hat{\lambda}(\varepsilon) = \begin{cases} \dfrac{1}{r(1)}\left[q(1) + \dfrac{b_0}{b_1}p(1)\right] + O(\varepsilon) & \text{if } \varepsilon > 0, b_1 \neq 0, \\ \dfrac{1}{r(0)}\left[q(0) + \dfrac{a_0}{a_1}p(0)\right] + O(\varepsilon) & \text{if } \varepsilon < 0, a_1 \neq 0. \end{cases}$$

The leading term in (1.7) is just the eigenvalue of the reduced eigenvalue problem obtained by setting $\varepsilon = 0$ in (1.5a) and retaining (1.5b) if $\varepsilon\uparrow 0$ or (1.5c) if $\varepsilon\downarrow 0$ (see § 3). A variety of results on the convergence of the eigenvalues of singularly perturbed eigenvalue problems to those of the corresponding reduced problems are known [7], [8], [9]. These results are generally of the form that each eigenvalue of the reduced problem is the limit as $\varepsilon\downarrow 0$, say, of *some* eigenvalue of the full problem. Thus no particular information is obtained on the principal eigenvalue of the unreduced problem. More complete information is given by Moser [9] (in the self-adjoint case) where it is shown that *each* eigenvalue of the full problem converges to some eigenvalue of the reduced problem. However, this result requires the eigenfunctions of the reduced problem to be complete. This is clearly not the case in our problem as at most one reduced eigenvalue exists. Thus it appears that none of these rather thorough studies has quite the result we require. The simple proof of (1.7) given in § 4 uses techniques that are completely different from those in [7], [8], [9].

2. The stability problem. An equilibrium state $y_j(x, \varepsilon)$ is asymptotically stable (in the small) if the solution $Y(t, x)$ of (1.1) with the initial data

$$Y_0(x) \equiv y_j(x, \varepsilon) + \delta\psi(x)$$

satisfies, for all sufficiently small $|\delta|$ and smooth bounded perturbations $\psi(x)$,

$$\lim_{t \to \infty} |Y(t, x) - y_j(x, \varepsilon)| = 0. \tag{2.1}$$

The standard (heuristic) examination of stability is to seek a solution of (1.1), with $Y_0(x)$ as above, in the form

$$Y(t, x) = y_j(x, \varepsilon) + \delta\, e^{-\lambda t}\phi(x),$$

and to retain only lowest order terms in δ. This yields

$$\varepsilon\varphi_{xx} + \varphi_x + [\lambda - g_y(x, y_j(x, \varepsilon))]\varphi = 0, \tag{2.2a}$$

$$\varphi_x(0) - a\varphi(0) = 0, \tag{2.2b}$$

$$f_z(y_j(1, \varepsilon), y_j'(1, \varepsilon))\varphi_x(1) + f_y(y_j(1, \varepsilon), y_j'(1, \varepsilon))\varphi(1) = 0. \tag{2.2c}$$

Here we have assumed that $g(x, y)$ and $f(y, z)$ have the appropriate continuous derivatives. If the least eigenvalue of (2.2) is positive, then we expect that $y_j(x, \varepsilon)$ is stable in the sense (2.1). That this holds can be shown in many cases [5] and we assume it to be the case here. Thus we say that an equilibrium state $y_j(x, t)$ is stable, unstable or neutral, respectively, if the least eigenvalue $\hat{\lambda}(\varepsilon)$ of (2.2) is >0, <0 or $=0$.

We shall examine the least eigenvalue of (2.2) for small $\varepsilon > 0$. In this case we can eliminate the dependence of the "coefficients" g_y, f_z and f_y on ε by evaluating them at $\varepsilon = 0$. Indeed, in [3] it is shown that, uniformly on $[0, 1]$ and for all ε in $0 < \varepsilon \leqq \varepsilon_1$,

$$|y_j(x, \varepsilon) - v_j(x)| \leqq C\varepsilon. \tag{2.3}$$

Here $v_j(x)$ is the solution of the reduced problem

$$v'(x) = g(x, v), \quad v(1) = \alpha_j, \tag{2.4}$$

and α_j is the jth root in (1.3). Thus in place of (2.2) we consider

$$\varepsilon\varphi_{xx} + \varphi_x + [\lambda - g_y(x, v_j(x))]\varphi = 0, \tag{2.5a}$$

$$\varphi_x(0) - a\varphi(0) = 0, \tag{2.5b}$$

$$f_z(v_j(1), v_j'(1))\varphi_x(1) + f_y(v_j(1), v_j'(1))\varphi(1) = 0. \tag{2.5c}$$

Now if, as we shall assume, $f_z(v_j(1), v_j'(1)) \neq 0$, it is easy to show that the least eigenvalues of (2.2) and (2.5) differ by $O(\varepsilon)$. Hence the stability or instability (but not neutral stability) for sufficiently small $\varepsilon > 0$ can be determined by examination of the least eigenvalue of (2.5).

Obviously (2.5) is a special case of (1.5), where $p(x) \equiv r(x) \equiv 1$, $q(x) \equiv g_y(x, v_j(x))$, $a_1 = 1$, $a_0 = -a$, $b_1 = f_z(v_j(1), v_j'(1))$ and $b_0 = f_y(v_j(1), v_j'(1))$. So when (1.7) has been established it follows that the least eigenvalue of (2.5) or indeed of (2.2) for $\varepsilon > 0$ is given by

$$\hat{\lambda}(\varepsilon) = \frac{f_z(v_j(1), v_j'(1))g_y(1, v_j(1)) + f_y(v_j(1), v_j'(1))}{f_z(v_j(1), v_j'(1))} + O(\varepsilon).$$

Recalling (1.3a) and using (2.4), this can be written as

$$\hat{\lambda}(\varepsilon) = \frac{dF(\alpha_j)/d\alpha}{f_z(\alpha_j, g(1, \alpha_j))} + O(\varepsilon). \tag{2.6}$$

Thus (1.4) clearly follows if $f_z(y, z) > 0$ and $\varepsilon > 0$ is sufficiently small.

We now turn to the demonstration of (1.7).

3. Reduced eigenvalue problems. The correct limiting values of the least and greatest eigenvalues $\hat{\lambda}(\varepsilon)$ of (1.5) are suggested by an elementary application of singular perturbation theory. That is, if we simply set $\varepsilon = 0$ in (1.5), the problem becomes

$$p(x)\psi_x(x) + [\lambda r(x) - q(x)]\psi(x) = 0, \tag{3.1a}$$

$$a_1\psi_x(0) + a_0\psi(0) = 0, \tag{3.1b}$$

$$b_1\psi_x(1) + b_0\psi(1) = 0. \tag{3.1c}$$

Clearly a solution of the first order equation (3.1a) cannot, in general, satisfy two boundary conditions as in (3.1b, c). Singular perturbation theory (see [2]) usually tells us, or rather suggests, which if either of the boundary conditions is to be retained. Indeed, since $p(x) > 0$ is assumed here, for $\varepsilon > 0$ the theory suggests that (3.1c) is to be imposed and for $\varepsilon < 0$ that (3.1b) is applicable. However, it is a simple matter to consider both possibilities, avoiding any real knowledge of singular perturbation theory, and to show that the indicated behavior is correct. Thus we define the two reduced problems: $RP_- = $ (3.1a, b) and $RP_+ = $ (3.1a, c).

The general solution of (3.1a) is

$$\psi(x) = c \exp\left[-\int_0^x \frac{\lambda r(\xi) - q(\xi)}{p(\xi)} d\xi\right], \tag{3.2}$$

with the constant c arbitrary. If $c = 0$, then $\psi(x) \equiv 0$ and both conditions (3.1b, c) are satisfied for all λ. We seek values of λ for which *nontrivial* solutions of $RP_{\mp}$ exist; such values are eigenvalues of the corresponding reduced problems. Using (3.2) in (3.1b) or (3.1c) we find that the only possible eigenvalues of the reduced problems are

$$\lambda = \begin{cases} \lambda_0^- \equiv \dfrac{1}{r(0)}\left[q(0) + \dfrac{a_0}{a_1}p(0)\right] & \text{if } a_1 \neq 0 \text{ for } RP_-, \\[2ex] \lambda_0^+ \equiv \dfrac{1}{r(1)}\left[q(1) + \dfrac{b_0}{b_1}p(1)\right] & \text{if } b_1 \neq 0 \text{ for } RP_+. \end{cases} \tag{3.3}$$

If $a_1 = 0$ ($b_1 = 0$), then RP_- (RP_+) has no eigenvalue.

4. The limit results. It is of course by no means clear that the eigenvalues (3.3) of the reduced problems are related to any of the eigenvalues of (1.5) as $\varepsilon \uparrow 0$ or $\varepsilon \downarrow 0$. To show the relationship we first transform (1.5a) into the standard Sturmian form:

$$(4.1)\qquad (P(\varepsilon, t; x)\varphi_x)_x + [\lambda R(\varepsilon, t; x) - Q(\varepsilon, t; x)]\varphi = 0,$$

by introducing the quantities:

$$(4.2)\qquad \begin{gathered} E(\varepsilon, t; x) \equiv \exp\left(\frac{1}{\varepsilon}\int_t^x p(\xi)\,d\xi\right), \qquad P(\varepsilon, t; x) \equiv \varepsilon E(\varepsilon, t; x); \\ Q(\varepsilon, t; x) \equiv q(x)E(\varepsilon, t; x), \qquad R(\varepsilon, t; x) \equiv r(x)E(\varepsilon, t; x). \end{gathered}$$

Note from (1.6b) that $R(\varepsilon, t; x) > 0$ and $P(\varepsilon, t; x) > 0$ or < 0 on $0 \leqq x \leqq 1$ if $\varepsilon > 0$ or < 0. The variational characterization of the least eigenvalue can be used in (4.1), (1.5b, c) taking account of this sign change with ε to get:

$$(4.3)\qquad \pm\hat{\lambda}(\varepsilon) = \min_{\varphi \in \mathscr{A}} \pm \left\{ \frac{\int_0^1 [P(\varepsilon, t; x)\varphi_x^2(x) + Q(\varepsilon, t; x)\varphi^2(x)]\,dx - P(\varepsilon, t; 0)\frac{a_0}{a_1}\varphi^2(0) + P(\varepsilon, t; 1)\frac{b_0}{b_1}\varphi^2(1)}{\int_0^1 R(\varepsilon, t; x)\varphi^2(x)\,dx} \right\}.$$

Here $\hat{\lambda}(\varepsilon)$ is the least (greatest) eigenvalue of (1.5) for $\varepsilon > 0$ ($\varepsilon < 0$). Also we have assumed that $a_1 b_1 \neq 0$ and for the admissible functions we can take $\mathscr{A} \equiv \mathbb{C}_1[0, 1]$. If $a_1 = 0$ (or $b_1 = 0$), we must drop the corresponding boundary term in (4.3) and take $\mathscr{A} \equiv \mathbb{C}_1[0, 1] \cap \{\varphi(x) | \varphi(0) = 0\}$ (or $\{\varphi(x) | \varphi(1) = 0\}$). The sign ($\pm$) is that of $\varepsilon \neq 0$.

Now we note some basic properties of the integrating factor $E(\varepsilon, t; x)$. First, by partial integration, it follows that

$$(4.4)\qquad \begin{aligned} \int_0^1 f(x)E(\varepsilon, t; x)\frac{dx}{\varepsilon} &= E(\varepsilon, t; 1)\frac{f(1)}{p(1)} - E(\varepsilon, t; 0)\frac{f(0)}{p(0)} \\ &\quad - \varepsilon\int_0^1 \frac{d}{dx}\left(\frac{f(x)}{p(x)}\right)E(\varepsilon, t; x)\frac{dx}{\varepsilon}, \end{aligned}$$

for all $f(x) \in \mathbb{C}_1[0, 1]$. Further by (1.6b), we get

$$(4.5a)\qquad 0 < E(\varepsilon, 0; x) \leqq e^{p_0 x/\varepsilon} \quad \text{if } \varepsilon < 0, \quad 0 \leqq x \leqq 1,$$

$$(4.5b)\qquad 0 < E(\varepsilon, 1; x) \leqq e^{p_0(x-1)/\varepsilon} \quad \text{if } \varepsilon > 0, \quad 0 \leqq x \leqq 1,$$

$$(4.5c)\qquad E(\varepsilon, 0; 0) = E(\varepsilon, 1; 1) = 1 \quad \text{if } \varepsilon \neq 0.$$

Using (4.2) and (4.4) in (4.3) now yields

$$(4.6a)\qquad \pm\hat{\lambda}(\varepsilon) = \min_{\varphi \in \mathscr{A}} \pm \mathscr{R}\{\varphi; \varepsilon\},$$

where

$$(4.6b)\qquad \mathscr{R}\{\varphi; \varepsilon\} \equiv \left\{ \frac{E(\varepsilon, t; 1)\left[\frac{q(1)}{p(1)} + \frac{b_0}{b_1}\right]\varphi^2(1) - E(\varepsilon, t; 0)\left[\frac{q(0)}{p(0)} + \frac{a_0}{a_1}\right]\varphi^2(0) + \varepsilon\int_0^1 \left[\varphi_x^2 - \frac{d}{dx}\left(\frac{q\varphi^2}{p}\right)\right]E(\varepsilon, t; x)\frac{dx}{\varepsilon}}{E(\varepsilon, t; 1)\frac{r(1)}{p(1)}\varphi^2(1) - E(\varepsilon, t; 0)\frac{r(0)}{p(0)}\varphi^2(0) - \varepsilon\int_0^1 \frac{d}{dx}\left(\frac{r\varphi^2}{p}\right)E(\varepsilon, t; x)\frac{dx}{\varepsilon}} \right\}.$$

Our basic results follow by obtaining upper and lower bounds on $\hat{\lambda}(\varepsilon)$ from (4.6). In particular, we use the trial functions

(4.7a) $\varphi(x) \equiv 1 \qquad \text{if } a_1 b_1 \neq 0,$

(4.7b) $\varphi(x) \equiv x \qquad \text{if } a_1 = 0, b_1 \neq 0,$

(4.7c) $\varphi(x) \equiv 1 - x \quad \text{if } a_1 \neq 0, b_1 = 0.$

Then for $\varepsilon > 0$ take $t = 1$ in (4.6) and recall (4.5b, c) to obtain

$$\text{(4.8a)} \qquad \hat{\lambda}(\varepsilon) \leqq \frac{\left[\dfrac{q(1)}{p(1)} + \dfrac{b_0}{b_1}\right] + O(\varepsilon)}{\dfrac{r(1)}{p(1)} + O(\varepsilon)} = \lambda_0^+ + O(\varepsilon) \quad \text{if } b_1 \neq 0 \text{ and } \varepsilon > 0.$$

Similarly for $\varepsilon < 0$ take $t = 0$ in (4.6) to obtain

$$\text{(4.8b)} \qquad \hat{\lambda}(\varepsilon) \geqq \frac{-\left[\dfrac{q(0)}{p(0)} + \dfrac{a_0}{a_1}\right] + O(\varepsilon)}{-\dfrac{r(0)}{p(0)} + O(\varepsilon)} = \lambda_0^- + O(\varepsilon) \quad \text{if } a_1 \neq 0 \text{ and } \varepsilon < 0.$$

Bounds on the other side are more subtle. We use the fact that the eigenfunctions $\hat{\varphi}(\varepsilon, x)$ can be normalized uniformly in ε by

$$\text{(4.9)} \qquad 0 \leqq \hat{\varphi}(\varepsilon, x) \leqq 1; \quad 0 \leqq x \leqq 1, \quad 0 < |\varepsilon| \leqq \varepsilon_0.$$

Further, without loss in generality we can assume that $\hat{\varphi}(\varepsilon, x) \in \mathbb{C}_2[0, 1]$. These eigenfunctions in (4.6) give $\hat{\lambda}(\varepsilon) = \mathscr{R}\{\hat{\varphi}; \varepsilon\}$. Then we use the fact that

$$\varepsilon \int_0^1 \varphi_x^2 E(\varepsilon, t; x)\, dx \begin{cases} \geqq 0 & \text{if } \varepsilon > 0, \\ \leqq 0 & \text{if } \varepsilon < 0, \end{cases}$$

to drop this term and get

$$\text{(4.10)} \quad \hat{\lambda}(\varepsilon) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} \frac{E(\varepsilon, t; 1)\left[\dfrac{q(1)}{p(1)} + \dfrac{b_0}{b_1}\right]\hat{\varphi}^2(\varepsilon, 1) - E(\varepsilon, t; 0)\left[\dfrac{q(0)}{p(0)} + \dfrac{a_0}{a_1}\right]\hat{\varphi}^2(\varepsilon, 0) - \varepsilon \displaystyle\int_0^1 \frac{d}{dx}\left(\frac{q\hat{\varphi}^2}{p}\right) E(\varepsilon, t; x)\frac{dx}{\varepsilon}}{E(\varepsilon, t; 1)\dfrac{r(1)}{p(1)}\hat{\varphi}^2(\varepsilon, 1) - E(\varepsilon, t; 0)\dfrac{r(0)}{p(0)}\hat{\varphi}^2(\varepsilon, 0) - \varepsilon \displaystyle\int_0^1 \frac{d}{dx}\left(\frac{r\hat{\varphi}^2}{p}\right) E(\varepsilon, t; x)\frac{dx}{\varepsilon}}, \quad \varepsilon \begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix}.$$

To proceed as above and get the opposite inequalities from those in (4.8) we must first show that $|\hat{\varphi}_x(\varepsilon, x)|$ can be bounded independent of ε on $0 < |\varepsilon| \leqq \varepsilon_0$, say.

From (4.1) with $\lambda = \hat{\lambda}(\varepsilon)$, $\varphi = \hat{\varphi}(\varepsilon, x)$ and $\varepsilon > 0$ we integrate over $[0, x]$ and use (1.5b) to get, setting $t = 0$:

$$\hat{\varphi}_x(\varepsilon, x) = \frac{a_0}{a_1}\hat{\varphi}(\varepsilon, 0)E^{-1}(\varepsilon, 0; x) + \int_0^x [q(\xi) - \hat{\lambda}(\varepsilon)r(\xi)]\hat{\varphi}(\varepsilon, \xi)E(\varepsilon, \xi; x)\frac{d\xi}{\varepsilon}.$$

If $|\hat{\lambda}(\varepsilon)| \leqq \mu$ for $0 < \varepsilon \leqq \varepsilon_0$, then, recalling (4.5) and (4.9),

$$\text{(4.11a)} \qquad |\hat{\varphi}_x(\varepsilon, x)| \leqq \frac{a_0}{a_1} e^{-p_0 x/\varepsilon} + \frac{1}{p_0}(\|q\|_\infty + \mu\|r\|_\infty), \qquad \varepsilon > 0.$$

If $a_1 = 0$, then the corresponding term does not enter. For $\varepsilon < 0$ we proceed in an analogous fashion, integrating (4.1) over $[x, 1]$ and setting $t = 1$ to get if $|\hat{\lambda}(\varepsilon)| \leqq \mu$:

$$|\hat{\varphi}_x(\varepsilon, x)| \leqq \frac{b_0}{b_1} e^{p_0(1-x)/\varepsilon} + \frac{1}{p_0}(\|q\|_\infty + \mu\|r\|_\infty), \qquad \varepsilon < 0. \tag{4.11b}$$

It only remains to show that $|\lambda(\varepsilon)| \leqq \mu$ for $0 < |\varepsilon| \leqq \varepsilon_0$ and some μ independent of ε. For $0 < \varepsilon \leqq \varepsilon_0$ we have from (4.8a) that for some constant K_0,

$$\hat{\lambda}(\varepsilon) \leqq \lambda_0^+ + K_0\varepsilon_0.$$

A lower bound is obtained by means of Theorem 15 in Protter and Weinberger [4, p. 38]. In particular, with $w(x) \equiv (x - 1/2)^{2m} + \delta$ for sufficiently large integer $m \geqq 1$ and sufficiently small δ in $0 < \delta < 1/2$, this result implies that

$$\hat{\lambda}(\varepsilon) \geqq -\left(\frac{\|q\|_\infty + 2m\|p\|_\infty + 4m^2\varepsilon_0}{r_0\delta}\right) \quad \text{if } 0 < \varepsilon \leqq \varepsilon_0.$$

Similarly, $|\hat{\lambda}(\varepsilon)|$ is bounded for $0 > \varepsilon > -\varepsilon_0$ and so $|\hat{\varphi}_x(\varepsilon, x)|$ is uniformly bounded for $0 < |\varepsilon| \leqq \varepsilon_0$ as follows from (4.11).

Using (4.5) in (4.10) now implies with (4.8) that:

$$\hat{\lambda}(\varepsilon) = \begin{cases} \lambda_0^+ + O(\varepsilon) & \text{if } \varepsilon > 0, \quad b_1 \neq 0, \\ \lambda_0^- + O(\varepsilon) & \text{if } \varepsilon < 0, \quad a_1 \neq 0. \end{cases} \tag{4.12}$$

We have thus proven under conditions (1.4) that: *when* $b_1 \neq 0$ $(a_1 \neq 0)$ *the least (greatest) eigenvalue* $\hat{\lambda}(\varepsilon)$ *of* (1.3) *converges as* $\varepsilon \downarrow 0$ $(\varepsilon \uparrow 0)$ *to the eigenvalue* λ_0^+ (λ_0^-) *of the reduced problem* RP_+ (RP_-). We stress that this phenomenon occurs only if the boundary condition that is "retained" by singular perturbation theory is *not* of the Dirichlet type. In the case of constant coefficients it is not difficult to show that all of the other eigenvalues (including the principal one if $b_1 = 0$, $\varepsilon < 0$ and $a_1 = 0$, $\varepsilon < 0$) diverge to $+\infty$ $(-\infty)$ like $|\varepsilon|^{-1}$ as $\varepsilon \downarrow 0$ $(\varepsilon \uparrow 0)$.

REFERENCES

[1] R. ARIS, *Introduction to the Analysis of Chemical Reactors*, Prentice-Hall, Englewood Cliffs, N.J., 1956.

[2] J. D. COLE, *Perturbation Methods in Applied Mathematics*, Ginn-Blaisdell, Waltham, Mass., 1968.

[3] H. B. KELLER, *Existence theory for multiple solutions of a singular perturbation problem*, this Journal, 3 (1972), pp. 86–92.

[4] M. PROTTER AND H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

[5] D. SATTINGER, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana U. Math. J., to appear.

[6] R. SCHMITZ, Private communication, 1969.

[7] G. H. HANDELMAN, J. B. KELLER AND R. E. O'MALLEY, *Loss of boundary conditions in the asymptotic solution of linear ordinary differential equations. I: Eigenvalue problems*, Comm. Pure Appl. Math., 21 (1968), pp. 243–261.

[8] R. HARRIS, *Singular perturbations of eigenvalue problems*, Arch. Rational Mech. Anal., 7 (1961), pp. 224–241.

[9] J. MOSER, *Singular perturbation of eigenvalue problems for linear differential equations of even* order, Comm. Pure Appl. Math., 8 (1955), pp. 251–278.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

A STEFAN PROBLEM INVOLVING THE APPEARANCE OF A PHASE*

J. R. CANNON† AND MARIO PRIMICERIO‡

Abstract. The maximum principle is utilized in the demonstration of existence and uniqueness results for the free boundary problem associated with the creation of an additional phase at the boundary; i.e., the free boundary has its point of origin on the boundary of the region under consideration.

1. Introduction. The physical setting of the problem discussed in this paper is a slab of ice of unit thickness which at an initial point in time is uniformly heated at one face while uniformly cooled at the other. This induces the creation of the water phase at one face and a water-ice interface plane which moves into the slab. The mathematical setting of the problem is the determination of $u = u(x, t)$, $v = v(x, t)$ and $s = s(t)$ which satisfy

$$\begin{aligned} &L_1 u \equiv \kappa_1 u_{xx} - u_t = 0, \qquad 0 < x < s(t), && 0 < t \leqq T, \\ (1.1)\qquad &u(0, t) = f(t), \qquad u(s(t), t) = 0, && 0 < t \leqq T, \\ &s(0) = 0, && \end{aligned}$$

$$\begin{aligned} &L_2 v \equiv \kappa_2 v_{xx} - v_t = 0, \qquad s(t) < x < 1, && 0 < t \leqq T, \\ (1.2)\qquad &v(1, t) = g(t), \qquad v(s(t), t) = 0, && 0 < t \leqq T, \\ &v(x, 0) = \psi(x), && 0 \leqq x \leqq 1, \end{aligned}$$

$$(1.3)\qquad \dot{s}(t) = -K_1 u_x(s(t), t) + K_2 v_x(s(t), t), \qquad 0 < t \leqq T,$$

where $\kappa_i = k_i \rho_i^{-1} c_i^{-1}$, $i = 1, 2$, represent the diffusivities; k_i, $i = 1, 2$, the conductivities; ρ_i, $i = 1, 2$, the densities; c_i, $i = 1, 2$, the heat capacities; $K_i = k_i \rho_2^{-1} L^{-1}$, $i = 1, 2$; L is the latent heat of fusion; and all of the preceding constants are positive. T is an arbitrary but fixed positive number, and the functions $f \geqq 0$, $g \leqq 0$ and $\psi \leqq 0$ are the boundary and initial data for (1.1), (1.2) and (1.3).

In this paper we demonstrate the global existence and uniqueness of the solution of (1.1), (1.2) and (1.3). The results are based upon the maximum principle and the results of our paper [11] in which (1.1), (1.2) and (1.3) was considered for the case $s(0) > 0$. The analysis given in [11] depended heavily upon the fact that $s(0) > 0$. The situation here is analogous to the papers of Cannon and Hill [4], [6] in which a substantially different analysis was required for the two cases. As in [11], we shall by necessity restrict ourselves to small data. The major point in our existence argument here is the derivation of a nontrivial lower bound on the free boundary $s = s(t)$. The analysis is similar to that in [4] and [6]. The paper is

* Received by the editors July 20, 1971, and in revised form February 1, 1972.

† Mathematics Department, University of Texas at Austin, Austin, Texas 78712. This author's work was supported in part by the National Science Foundation under Contract GP 15724.

‡ Instituto Matematico U Dini, Universita di Firenze, 50134 Firenze, Italy. This author's work was supported by the NATO Senior Fellowship Program.

concluded with comments on results that can be achieved for the specification of flux on the boundaries.

2. Definitions and hypotheses. We begin with a list of the assumptions needed for the existence theorem.

(A) Let $f = f(t)$ be a bounded piecewise continuous function such that there exist positive constants α_1, β_1, λ, Λ and $\gamma \leqq 0$ such that

$$\min(\alpha_1, \lambda t^{1+\gamma}) \leqq f(t) \leqq \min(\beta_1, \Lambda t^{1+\gamma}) \tag{2.1}$$

for $0 \leqq t \leqq T$.

(B) Let $g = g(t)$ be a bounded piecewise continuous function such that there exist two positive constants α_2 and β_2 such that

$$-\beta_2 < g(t) < -\alpha_2 < 0, \qquad 0 \leqq t \leqq T. \tag{2.2}$$

(C) Let $\psi = \psi(x)$ be a piecewise continuous function such that there exist two positive constants a and η such that

$$0 \geqq \psi(x) \geqq -a(1 - \exp\{-\kappa_2^{-1}\eta x\}). \tag{2.3}$$

(D) With respect to the constants β_i, $i = 1, 2$, and a, we assume that

$$\max(2K_1\beta_1\kappa_1^{-1}, 2K_2\kappa_2^{-1}\max(\beta_2, a)) < 1. \tag{2.4}$$

By a solution (u, v, s) of the Stefan problem (1.1), (1.2) and (1.3), we mean that:

1°. $s = s(t)$ is a continuously differentiable function for $0 < t \leqq T$ and continuous for $0 \leqq t \leqq T$, $s(0) = 0$, and $0 \leqq s(t) < 1$.

2°. u and v are such that:

(a) the derivatives appearing in the equations exist and are continuous in their respective domain of definition;

(b) u and v are continuous in the closure of their respective domain of definition except at points of discontinuity of the data;

(c) for such points of discontinuity, $0 \leqq \liminf u \leqq \limsup u < \infty$ and $-\infty < \liminf v \leqq \limsup v \leqq 0$ as each such point is approached from the interior of the region in question;

(d) u and v satisfy (1.1) and (1.2) respectively.

3°. u, v and s satisfy (1.3).

3. Existence. For all $b > 0$ and sufficiently small, let (u^b, v^b, s^b), where $u^b = u^b(x, t)$, $v^b = v^b(x, t)$, and $s^b = s^b(t)$, denote the solution of (1.1), (1.2) and (1.3), with the alterations $s(0) = b$, $u^b(x, 0) \equiv 0$, $0 < x \leqq b$, f is replaced by $f + b$, and ψ is replaced by ψ^b, where

$$\psi^b(x) = \max(\psi(x), -a(1 - \exp\{-\kappa_2^{-1}\eta(x - b)\}))$$

for $b \leqq x < 1$. The existence and uniqueness of (u^b, v^b, s^b) are discussed in [11] and the assumptions (A), (B), (C) and (D) guarantee the results derived in [11]. Consequently, we have a family of functions s^b for $b > 0$ and sufficiently small. In order to guarantee the existence of a solution of (1.1), (1.2) and (1.3) it suffices to demonstrate that the family s^b is uniformly bounded and equicontinuous. The uniform boundedness of the family is obvious; i.e., $0 \leqq s^b \leqq 1$. The question of equicontinuity of the family is a more complex matter which will occupy most of the remaining paragraphs.

Since

$$\dot{s}^b(t) = -K_1 u_x^b(s^b(t), t) + K_2 v_x^b(s^b(t), t) \tag{3.1}$$

for $0 < t \leqq T$, we begin an estimate of $\dot{s}^b$ by estimating u_x^b and v_x^b. Since v_x^b has already been estimated in [11] we can state the following lemma.

LEMMA 1. *For $0 < b < b_0$, where b_0 is sufficiently small and fixed, there exists a $T_0 = T_0(\Delta)$, $0 < \Delta < 1 - 2b_0$, such that*

$$s^b(t) < 1 - \Delta \tag{3.2}$$

for $0 < t \leqq T_0$ and that

$$\begin{aligned} &v_x^b(s^b(t), t) \geqq -\max(a, \beta_2)\kappa_2^{-1} \\ &\cdot[1 - \exp\{-\kappa_2^{-1}(\|\dot{s}^b\|_{(0,\theta]} + \eta)\Delta\}]^{-1}(\|\dot{s}^b\|_{(0,\theta]} + \eta), \quad 0 < t \leqq \theta \leqq T_0, \end{aligned} \tag{3.3}$$

where for any function $h = h(t)$ defined on $a < t \leqq b$,

$$\|h\|_{(a,b]} = \sup_{a < t \leqq b} |h(t)|. \tag{3.4}$$

Remark. The proof of Lemma 1 follows from the argument given in [11] and the fact that $s^b(t) < s^{b_0}(t)$ which follows from the fact that $f + b \leqq f + b_0$, $\psi_b \leqq \psi_{b_0}$, and $b < b_0$; i.e., the monotone dependence of the free boundary upon the data. Note that $\|\dot{s}^b\|_{(0,\theta]} < \infty$ since it is shown in [11] that s^b is the uniform limit of a sequence of equi-Lipschitz-continuous functions.

In order to derive an estimate of u_x^b we must first derive a lower estimate for s^b. Consider the problem

$$\begin{aligned} &L_1 z = 0, \qquad 0 < x < \sigma(t), && 0 < t \leqq \theta, \\ &z(0, t) = f_1(t), \qquad z(\sigma(t), t) = 0, && 0 < t \leqq \theta, \\ &\dot{\sigma}(t) = -K_1 z_x(\sigma(t), t) - K_2\|v_x^b\|, \qquad \sigma(0) = 0, \end{aligned} \tag{3.5}$$

where $f_1(t) \leqq f(t)$, $\theta \leqq T_0$, and via (3.3) and the remark above,

$$\|v_x^b\| = \sup_{0 < t \leqq \theta} |v_x^b(s^b(t), t)| < \infty. \tag{3.6}$$

Suppose that a solution of (3.5) exists in the sense of a definition analogous to the one given in §2. Then, we can demonstrate the following lemma.

LEMMA 2. *For $0 < t \leqq \theta$,*

$$\sigma(t) < s^b(t), \tag{3.7}$$

Proof. Suppose (3.7) is false. Then since $b > 0$, there exists a first time $t_0 > 0$ such that $s^b(t_0) = \sigma(t_0)$ and $\dot{s}^b(t_0) \leqq \dot{\sigma}(t_0)$. Since $u^b - z \geqq 0$ in $0 \leqq x \leqq \sigma(t)$, $0 \leqq t \leqq t_0$, the parabolic version of Hopf's lemma yields $u_x^b(s^b(t_0), t_0) < z_x(\sigma(t_0), t_0)$. But,

$$-K_1 u_x^b(s^b(t_0), t_0) + K_2 v_x^b(s^b(t_0), t_0) > -K_1 z_x(\sigma(t_0), t_0) - K_2\|v_x^b\|.$$

Hence, $\dot{s}^b(t_0) > \dot{\sigma}(t_0)$, which is a contradiction. Consequently, solutions of (3.5) are of interest to us. However, such problems are generally difficult to solve for specified f_1. Hence, we consider the inverse problem of defining simple boundaries $\sigma(t)$ and investigate the corresponding $z(0, t)$ which arises from the solution of the Cauchy problem

$$\begin{aligned} &L_1 z = 0, \qquad 0 < x < \sigma(t), && 0 < t \leqq \theta, \\ &z(\sigma(t), t) = 0, && 0 < t \leqq \theta, \\ &z_x(\sigma(t), t) = -K_1^{-1}\dot{\sigma}(t) - K_1^{-1}K_2\|v_x^b\|. \end{aligned} \tag{3.8}$$

For smooth data, Hill [18] has shown that the solution of (3.8) may be represented as

$$z(x, t) = \sum_{j=0}^{\infty} \frac{\partial^j}{\partial t^j}\left\{\frac{1}{\kappa_1^j}\frac{(x - \sigma(t))^{2j+1}}{(2j + 1)!} z_x(\sigma(t), t)\right\}. \tag{3.9}$$

From (3.8) we see that

$$\begin{aligned} z(x, t) = &- \sum_{j=0}^{\infty} \frac{\partial^j}{\partial t^j}\left\{\frac{1}{\kappa_1^j}\frac{(x - \sigma(t))^{2j+1}}{(2j + 1)!} K_1^{-1}\dot{\sigma}(t)\right\} \\ &- \sum_{j=0}^{\infty} \frac{\partial^j}{\partial t^j}\left\{\frac{1}{\kappa_1^j}\frac{(x - \sigma(t))^{2j+1}}{(2j + 1)!} K_1^{-1}K_2\|v_x^b\|\right\} \\ = &\; z_1(x, t) + z_2(x, t). \end{aligned} \tag{3.10}$$

Recalling the hypothesis (A) concerning the asymptotic behavior of f as $t \to 0$ and the analysis in [4, p. 13], we consider $\sigma(t) = \mu t^\alpha$, where $\mu > 0$, $\alpha = 1 + \gamma$ and $\beta = 1 + 2\gamma$. Then for $x = 0$, the first series z_1 in (3.10) satisfies

$$\kappa_1 K_1^{-1}(\exp\{\alpha\mu^2\kappa_1^{-1}t^\beta\} - 1) \leqq z_1(0, t) \leqq \kappa_1 K_1^{-1}(\exp\{\beta\mu^2\kappa_1^{-1}t^\beta\} - 1) \tag{3.11}$$

which can be verified by an argument and analysis similar to that of Cannon and Hill [4, p. 18]. Turning to $z_2(x, t)$ which is the second series in (3.10), we see that

$$\begin{aligned} z_2(0, t) &= K_1^{-1}K_2\|v_x^b\| \sum_{j=0}^{\infty} \frac{\mu^{2j+1}}{\kappa_1^j} \frac{1}{(j + 1)!} \cdot \prod_{k=0}^{j-1} \frac{[(2j + 1)\alpha - k]}{[(2j + 1) - k]} t^{(2j+1)\alpha - j} \\ &= \kappa_1 K_1^{-1}K_2\|v_b^x\|\mu^{-1}t^{\alpha-\beta} \sum_{j=0}^{\infty} \frac{1}{(j + 1)!}(\mu^2\kappa_1^{-1}t^\beta)^{j+1} \cdot \prod_{k=0}^{j-1} \frac{[(2j + 1)\alpha - k]}{[(2j + 1) - k]}. \end{aligned} \tag{3.12}$$

Hence,

$$\begin{aligned} \alpha^{-1}\kappa_1 K_1^{-1}K_2\|v_x^b\|\mu^{-1}t^{\alpha-\beta}(\exp\{\alpha\mu^2\kappa_1^{-1}t^\beta\} - 1) &\leqq z_2(0, t) \\ \leqq \beta^{-1}\kappa_1 K_1^{-1}K_2\|v_x^b\|\mu^{-1}t^{\alpha-\beta}(\exp\{\beta\mu^2\kappa_1^{-1}t^\beta\} - 1) & \end{aligned} \tag{3.13}$$

since

$$\alpha \leqq \frac{(2j + 1)\alpha - k}{(2j + 1) - k} \leqq \beta \tag{3.14}$$

for $k = 0, \cdots, j - 1$. Combining the respective upper estimates of the $z_i(0, t)$, $i = 1, 2$, we obtain an upper estimate for $z(0, t)$ which is

$$\begin{aligned} z(0, t) \leqq &[\kappa_1 K_1^{-1} + \beta^{-1}\kappa_1 K_1^{-1}K_2\|v_x^b\|\mu^{-1}t^{\alpha-\beta}] \\ &\cdot(\exp\{\beta\mu^2\kappa_1^{-1}t^\beta\} - 1), \qquad 0 \leqq t \leqq \theta. \end{aligned} \tag{3.15}$$

Let y_0 be the solution of

(3.16) $$\exp\{y\} - 1 = 2y$$

and let

(3.17) $$T_\mu = (y_0 \kappa_1 \beta^{-1} \mu^{-2})^{1/\beta}.$$

From the convexity of $\exp\{y\} - 1$, it follows that for $0 \leqq t \leqq \theta \leqq T_\mu$,

(3.18) $$z(0,t) \leqq 2\beta\mu^2 K_1^{-1} t^\beta + 2\mu K_1^{-1} K_2 \|v_x^b\| t^\alpha.$$

In order to apply Lemma 2, we must have $z(0,t) \leqq \lambda t^\alpha$. This is accomplished by selecting $\mu > 0$ such that

$$2\beta\mu^2 K_1^{-1} t^\beta + 2\mu K_1^{-1} K_2 \|v_x^b\| t^\alpha \leqq \lambda t^\alpha.$$

For $t \leqq 1$, set

(3.19) $$\mu = \lambda K_1 (2K_2 \|v_x^b\| + 2\sqrt{2\lambda\beta K_1})^{-1},$$

where this value for μ was derived from the best possible choice of $\mu > 0$. From Lemma 2, we see that

(3.20) $$s^b(t) \geqq \mu t^\alpha$$

for $0 \leqq t \leqq \theta \leqq \min(1, T_\mu, T_0, (\alpha_1 \Lambda^{-1})^{1/\alpha})$. We can now estimate $|u_x^b(s^b(t), t)|$ via the maximum principle by considering the function

(3.21) $$Y(x,t) = \frac{2\Lambda\kappa_1}{\mu\|\dot{s}^b\|_{(0,\theta]}}\{1 - \exp\{\kappa_1^{-1}\|\dot{s}^b\|_{(0,\theta]}(x - s^b(t))\}\} - u^b(x,t),$$
$$0 \leqq x \leqq s^b(t), \qquad 0 < t \leqq \theta.$$

Observe that

(3.22) $$\begin{aligned} &L_1 Y \leqq 0, \quad 0 < x < s^b(t), \quad 0 < t \leqq \theta,\\ &Y(x,0) \geqq 0, \quad 0 \leqq x \leqq b,\\ &Y(s^b(t),t) = 0, \quad 0 < t \leqq \theta, \end{aligned}$$

and that

(3.23) $$\begin{aligned} Y(0,t) &= \frac{2\Lambda\kappa_1}{\mu\|\dot{s}^b\|_{(0,\theta]}}\{1 - \exp\{-\kappa_1^{-1}\|\dot{s}^b\|_{(0,\theta]} s^b(t)\}\} - f(t)\\ &\geqq \frac{2\Lambda\kappa_1}{\mu\|\dot{s}^b\|_{(0,\theta]}}\{1 - \exp\{-\kappa_1^{-1}\|\dot{s}^b\|_{(0,\theta]}\mu t^\alpha\}\} - \Lambda t^\alpha \geqq 0 \end{aligned}$$

provided that $\theta \leqq \min(1, T_\mu, T_0, (\alpha_1\Lambda^{-1})^{1/\alpha})$ and

(3.24) $$\theta \leqq \left(\frac{\kappa_1 \log 2}{\mu\|\dot{s}^b\|_{(0,\theta]}}\right)^{1/\alpha}.$$

Remark. Note that the limitation on θ involves $\|\dot{s}^b\|_{(0,\theta]}$ as explicitly shown in (3.24) and implicitly contained in $\|v_x^b\|$ in (3.17) using the definition in (3.19). Consequently, the range of t (i.e., θ) depends upon b. If for each b and all admissible θ associated with it we can derive an upper estimate of $\|\dot{s}^b\|_{(0,\theta]}$ which is independent of b, then (3.24), (3.17) and (3.19) will define a positive θ over which the $\dot{s}^b$ can be estimated independent of b.

Considering $Y_x(s^b(t), t)$, we see that

$$|u_x^b(s^b(t), t)| \leqq 2\Lambda/\mu \quad \text{for } 0 < t \leqq \theta. \tag{3.25}$$

Recalling (3.1), (3.3) and (3.25) we can write

$$\begin{aligned} \|\dot{s}^b\|_{(0,\theta]} \leqq\ & 2K_1\Lambda/\mu + \max(a, \beta_2)\kappa_2^{-1}K_2 \\ & \cdot [1 - \exp\{-\kappa_2^{-1}(\|\dot{s}^b\|_{(0,\theta]} + \eta)\Delta\}]^{-1}(\|\dot{s}^b\|_{(0,\theta]} + \eta). \end{aligned} \tag{3.26}$$

From (3.19) we see that (3.26) can be written in the form

$$\|\dot{s}^b\|_{(0,\theta]} \leqq \Gamma(1 - \exp\{-\kappa_2^{-1}\Delta\|\dot{s}^b\|_{(0,\theta]}\})^{-1} \cdot (\|\dot{s}^b\|_{(0,\theta]} + \eta_1), \tag{3.27}$$

where

$$\Gamma = 8K_2\Lambda\lambda^{-1} + \max(a, \beta_2)\kappa_2^{-1}K_2 \tag{3.28}$$

and

$$\eta_1 = \eta + 4\Lambda\sqrt{K_1\lambda^{-1}\beta}. \tag{3.29}$$

Employing the argument of Lemma 2 in [11], we can state the following lemma.

LEMMA 3. *If* $\Gamma < 1$, *then*

$$\|\dot{s}^b\|_{(0,\theta]} \leqq \max\left(-\frac{\kappa_2}{\Delta}\log\left\{\frac{1-\Gamma}{2}\right\}; \frac{2\Gamma\eta_1}{1-\Gamma}\right). \tag{3.30}$$

Consequently, $\|\dot{s}^b\|_{(0,\theta]}$ is bounded uniformly with respect to b in each admissible interval and thus from (3.24), (3.17) and (3.19) we see that there exists a $\theta_0 > 0$ such that (3.30) holds for each $b > 0$ and sufficiently small with θ replaced by θ_0 which is independent of b. Utilizing the Ascoli–Arzela theorem, the maximum principle, and arguments similar to those used in [4], [5], [6], [9], [10], [11], [12], we can state the following theorem.

THEOREM 1. *Under the assumptions* (A), (B), (C), (D) *and*

$$8K_2\Lambda\lambda^{-1} + \max(a, \beta_2)\kappa_2^{-1}K_2 < 1,$$

there exists a solution to (1.1), (1.2) *and* (1.3) *for each* $T > 0$ *in the sense defined in* § 2.

4. Stability, monotone dependence and uniqueness. The results of this section require the additional assumption that the initial data ψ be continuously differentiable in a neighborhood of $x = 1$. Specifically we assume that

(E) there exists a $\delta > 0$ such that ψ is continuously differentiable in $1 - \delta < x < 1$.

Using the arguments, techniques and analysis of [3], [4], [6], [10], [11], [12] we can state Theorem 2.

THEOREM 2. *Let* (u_i, v_i, s_i) *denote the solution of the Stefan problem* (1.1), (1.2) *and* (1.3) *for the respective data* f_i, g_i *and* ψ_i, $i = 1, 2$, *which satisfy the assumptions*

(A), (B), (C), (D) *and* (E). *Then there exists a constant* C *which depends upon* $T, \kappa_i, K_i, a, \alpha_i, \beta_i$ $(i = 1, 2), \eta, \lambda, \Lambda$ and γ *such that for* $0 \leqq t \leqq T$,

$$|s_1(t) - s_2(t)| \leqq C\{\|f_1 - f_2\|_{[0,t]} + \|g_1 - g_2\|_{[0,t]} + \|\psi_1 - \psi_2\|_{[0,1]} + \|\psi_1' - \psi_2'\|_{(1-\delta,1)}\}, \tag{4.1}$$

where the various norms used in (4.1) *are the obvious modification of the definition given in* (3.4).

Proof. The proof is omitted. See [3], [4], [6], [11], [12] for the necessary arguments and techniques. From Theorem 2 we have the following theorem.

THEOREM 3. *Using the notation and assumptions of Theorem* 2, *if* $f_1 \leqq f_2$, $g_1 \leqq g_2$, *and* $\psi_1 \leqq \psi_2$, *then* $s_1(t) \leqq s_2(t)$ *for all* $t > 0$.

Proof. The proof is omitted.

As a corollary of Theorem 2 and Theorem 3 we have the following theorem.

THEOREM 4. *Under the assumptions* (A), (B), (C), (D) *and* (E), *on the data* f, g *and* ψ, *there exists at most one solution to the Stefan problem* (1.1), (1.2) *and* (1.3).

5. Case of prescribed boundary flux. The specification of temperature at the boundaries can be replaced by the specification of the heat flux. The existence, uniqueness, stability and monotone dependence results hold for restrictions on the size of the fluxes analogous to those for the temperatures stated above; i.e., for small data classical solutions exist, are unique, and depend continuously upon the data. The technique involves the maximum principle and the right choice of barrier function. The details and exact hypotheses are left to the reader.

REFERENCES

[1] B. M. BUDAK AND M. Z. MOSKAL, *Classical solution of the multidimensional multifront Stefan problem*, Soviet Math. Dokl., 10 (1969), no. 5, pp. 1043–1046.

[2] J. R. CANNON, *A priori estimate for continuation of the solution of the heat equation in the space variable*, Ann. Mat. Pura Appl., 65 (1964), pp. 377–388.

[3] J. R. CANNON AND JIM DOUGLAS, JR., *The stability of the boundary in a Stefan problem*, Ann. Scuola Norm. Sup. Pisa, Vol. XXI, Fasc I (1967), pp. 83–91.

[4] J. R. CANNON AND C. DENSON HILL, *Existence, uniqueness, stability, and monotone dependence in a Stefan problem for the heat equation*, J. Math. and Mech., 17 (1967), pp. 1–20.

[5] J. R. CANNON, JIM DOUGLAS, JR. AND C. DENSON HILL, *A multi-boundary Stefan problem and the disappearance of phases*, Ibid., 17 (1967), pp. 21–34.

[6] J. R. CANNON AND C. DENSON HILL, *Remarks on a Stefan problem*, Ibid., 17 (1967), pp. 433–442.

[7] ———, *On the infinite differentiability of the free boundary in a Stefan problem*, J. Math. Anal. Appl., 22 (1968), pp. 385–397.

[8] ———, *On the movement of a chemical reaction interface*, Indiana Univ. Math. J., 20 (1970), pp. 429–454.

[9] J. R. CANNON, C. DENSON HILL AND MARIO PRIMICERIO, *The one-phase Stefan problem for the heat equation with boundary temperature specification*, Arch. Rational Mech. Anal., 39 (1970), pp. 270–274.

[10] J. R. CANNON AND MARIO PRIMICERIO, *Remarks on the one-phase Stefan problem for the heat equation with flux prescribed on the fixed boundary*, J. Math. Anal. Appl., to appear.

[11] ———, *A two phase Stefan problem with temperature boundary conditions*, Ann. Mat. Pura Appl., 88 (1971), pp. 177–192.

[12] ———, *A two phase Stefan problem with flux boundary conditions*, Ibid., 88 (1971), pp. 193–206.

[13] ———, *A two phase Stefan problem: regularity of the free boundary*, Ibid., 88 (1971), pp. 217–228.

[14] AVNER FRIEDMAN, *The Stefan problem in several space variables*, Trans. Amer. Math. Soc., 133 (1968), pp. 51–87.

[15] AVNER FRIEDMAN, *One dimensional Stefan problems with nonmonotone free boundary*, Ibid., 133 (1968), pp. 89–114.

[16] ———, *Correction to "The Stefan problem in several space variables"*, Ibid., 142 (1969), p. 557.

[17] M. GEVREY, *Sur les equations aux derivées partielles du type parabolique*, J. Math. (6), 9 (1913), pp. 305–471.

[18] C. DENSON HILL, *Parabolic equations in one space variable and the non-characteristic Cauchy problem*, Comm. Pure Appl. Math., 20 (1967), pp. 619–633.

[19] S. L. KAMENOMOSTSKAJA, *On Stefan's problem*, Mat. Sb., 53 (1965), no. 95, pp. 485–514.

[20] JIANG LI-SHANG, *Existence and differentiability of the solution of a two-phase Stefan problem for quasi-linear parabolic equations*, Chinese Math., 7 (1965), pp. 481–496.

[21] D. QUILGHINI, *Una analisi fisico-matematica del processo del cambiamento di fase*, Ann. Mat. Pura Appl., 67 (1965), pp. 33–74.

[22] L. I. RUBINSTEIN, *Two phase Stefan problem on a segment with one-phase initial state of thermoconductive medium*, Učen. Zap. Lat. Gos. Univ. Stučki, 58 (1964), pp. 111–148.

[23] G. SESTINI, *Esistenza ed unicita nel problema di Stefan relativo a campi dotati di simmetría*, Riv. Mat. Univ. Parma, 3 (1952), pp. 103–113.

[24] B. SHERMAN, *A free boundary problem for the heat equation with heat input at a melting interface*, Quart. Appl. Math., 23 (1966), pp. 337–347.

[25] ———, *A free boundary problem for the heat equation with prescribed flux at both fixed face and the melting interface*, Ibid., 25 (1967), pp. 53–63.

[26] ———, *Some generalizations in the theory of free boundary problems for the heat equation*, Res. Rep. 68-7, Rocket Dyne, Canoga Park, Calif., 1968.

[27] ———, *A general one-phase Stefan problem*, Quart. Appl. Math., 28 (1970), pp. 377–382.

[28] ———, *Continuous dependence and differentiability properties of the solution of a free boundary problem for the heat equation*, Ibid., 27 (1970), pp. 427–439.

[29] ———, *Free boundary problems for the heat equation in which the moving interface coincides initially with the fixed face*, J. Math. Anal. Appl., 33 (1971), pp. 449–466.

[30] ———, *A general one-phase Stefan problem and free boundary problems for the heat equation with Cauchy data prescribed on the free boundary*, SIAM J. Appl. Math., 20 (1971), pp. 555–570.

[31] ———, *Limiting behavior in some Stefan problems as the latent heat goes to zero*, Ibid., 20 (1971), pp. 319–327.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

A GENERAL ADDITION THEOREM FOR SPHEROIDAL WAVE FUNCTIONS*

B. J. KING AND A. L. VAN BUREN†

Abstract. A general addition theorem has been obtained for the spheroidal wave functions $R_{ml}^{(j)}(h, \xi)S_{ml}^{(1)}(h, \eta)\exp(im\varphi)$, $j = 1, 2, 3, 4$. This theorem gives the expansion of a spheroidal wave function with reference to one coordinate frame in terms of spheroidal wave functions with reference to a second coordinate frame with arbitrary relative position and orientation. The expressions are applicable whether the two spheroidal coordinate frames are both prolate, both oblate, or one prolate and one oblate.

Introduction. The Helmholtz scalar wave equation

$$(\nabla^2 + k^2)\psi = 0 \tag{1}$$

is separable in both prolate and oblate spheroidal coordinates ξ, η, φ. Thus, solutions to boundary value problems in radiation and scattering from spheroid-shaped objects can be constructed from the eigenfunctions $\psi_{ml}^{(j)}(h; \xi, \eta, \varphi) = R_{ml}^{(j)}(h, \xi)S_{ml}^{(1)}(h, \eta)\exp(im\varphi)$. Here $R_{ml}^{(j)}$ is the spheroidal radial wave function of the jth kind, $j = 1, 3, 4$, $S_{ml}^{(1)}$ is the spheroidal angle wave function of the first kind, and h is equal to $kd/2$, where d is the interfocal distance of the elliptical cross section of the spheroid, and k is the wave number. The radial functions $R_{ml}^{(3)}$ and $R_{ml}^{(4)}$ are given by $R_{ml}^{(1)} + iR_{ml}^{(2)}$ and $R_{ml}^{(1)} - iR_{ml}^{(2)}$, respectively, where $R_{ml}^{(1)}$ and $R_{ml}^{(2)}$ are the two independent solutions to the separated ordinary differential equation in ξ.

The spheroidal wave functions $\psi_{ml}^{(1)}$, $\psi_{ml}^{(3)}$ and $\psi_{ml}^{(4)}$ play the same role in describing spheroidal waves as the spherical wave functions $j_l(kr)P_l^m(\cos\theta)\exp(im\varphi)$, $h_l^{(1)}(kr)P_l^m(\cos\theta)\exp(im\varphi)$, and $h_l^{(2)}(kr)P_l^m(\cos\theta)\exp(im\varphi)$, respectively, do in describing spherical waves. Therefore, standing spheroidal waves, outgoing spheroidal waves, and incoming spheroidal waves are represented by $\psi_{ml}^{(1)}$, $\psi_{ml}^{(3)}$ and $\psi_{ml}^{(4)}$, respectively.

When a single spheroidal surface is involved in the radiation and scattering process, every contribution to the resultant field can be expressed as a series of spheroidal wave functions referenced to the coordinate frame in which the spheroid is a natural surface. The unknown expansion coefficients can then be obtained by application of the boundary conditions on the spheroid. When two or more spheroidal surfaces are involved, the resultant field includes contributions from each spheroid. The contributions from a given spheroid are expressed as a series of spheroidal wave functions with reference to the coordinate frame in which that spheroid is a natural surface. In order to apply the boundary conditions in this case, one must be able to express spheroidal wave functions with reference to one coordinate frame in terms of spheroidal wave functions with reference to a second coordinate frame. Analogous transformations or addition theorems are

* Received by the editors September 21, 1971, and in revised form February 3, 1972.
† Naval Research Laboratory, Washington, D.C. 20390.

well known for both cylindrical wave functions [17] and spherical wave functions [4]. Meixner [9] develops an addition theorem for spheroidal wave functions for the limited case where the two spheroids overlap and share a common axis of symmetry. In this paper an addition theorem is developed for the spheroidal wave functions $\psi_{ml}^{(1)}$, $\psi_{ml}^{(3)}$ and $\psi_{ml}^{(4)}$ for two spheroidal coordinate frames with completely arbitrary relative positions and orientations. The addition theorem is also valid for $\psi_{ml}^{(2)}$, although these wave functions do not correspond to spheroidal waves and therefore are not normally used. The expressions are applicable whether the two spheroidal coordinate frames are both prolate, both oblate, or one prolate and one oblate.

A brief introduction to spheroidal wave functions is given in § 1. The development of the addition theorem follows in § 2. The relation between coefficients appearing in the expansions developed in this report and both the Wigner 3-j symbols [18] and the Clebsch–Gordon coefficients [2] is described in the Appendix.

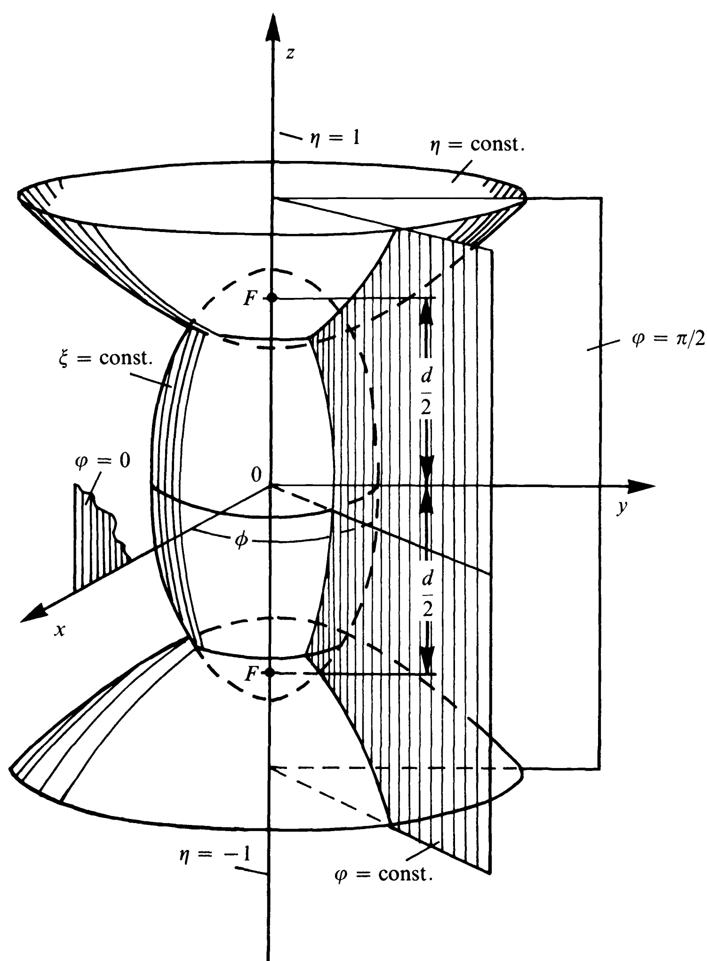

FIG. 1. *The prolate spheroidal coordinate system*

1. Spheroidal geometry and wave functions. The prolate spheroidal coordinates ξ, η, φ are related to rectangular coordinates by the transformation

$$
\begin{aligned}
x &= (d/2)(1 - \eta^2)^{1/2}(\xi^2 - 1)^{1/2} \cos \varphi, \\
y &= (d/2)(1 - \eta^2)^{1/2}(\xi^2 - 1)^{1/2} \sin \varphi. \\
z &= (d/2)\xi\eta,
\end{aligned} \tag{2}
$$

where $-1 \leqq \eta \leqq 1$, $1 \leqq \xi < \infty$, $0 \leqq \varphi < 2\pi$, and where d is the interfocal distance. This geometry is shown in Fig. 1. The corresponding transformation for the oblate spheroidal coordinates is

$$
\begin{aligned}
x &= (d/2)(1 - \eta^2)^{1/2}(\xi^2 + 1)^{1/2} \cos \varphi, \\
y &= (d/2)(1 - \eta^2)^{1/2}(\xi^2 + 1)^{1/2} \sin \varphi, \\
z &= (d/2)\xi\eta,
\end{aligned} \tag{3}
$$

where now $-1 \leqq \eta \leqq 1, 0 \leqq \xi < \infty, 0 \leqq \varphi < 2\pi$. The oblate geometry is shown in Fig. 2. The oblate spheroidal coordinate system can be obtained from the prolate spheroidal coordinate system by use of the interchange $\xi \to i\xi$ and $d \to -id$. Expressions developed for prolate spheroidal geometry can be converted into analogous expressions for oblate spheroidal geometry by use of the same interchange. Consequently, although the following discussion of spheroidal wave functions is restricted to the prolate system, the corresponding oblate expressions are also valid.

The spheroidal angle wave function of the first kind can be expanded in terms of the corresponding spherical functions:

$$
S_{ml}^{(1)}(h, \eta) = \sum_{n=0,1}^{\infty}{}' \, d_n(h|ml) P_{m+n}^m(\eta), \tag{4}
$$

where the $P_{m+n}^m(\eta)$ are associated Legendre functions of the first kind, and where the prime indicates that $n = 0, 2, 4, \cdots$ if $l - m$ is even, and $n = 1, 3, 5, \cdots$ if $l - m$ is odd. A recursion relation for the expansion coefficients $d_n(h|ml)$ is obtained by substitution of this expression into the separated ordinary differential equation for $S_{ml}^{(1)}(h, \eta)$ and by use of known recursion relations for $P_n^m(\eta)$:

$$
\begin{aligned}
&\frac{(2m + n + 2)(2m + n + 1)}{(2m + 2n + 3)(2m + 2n + 5)} h^2 d_{n+2} \\
&+ \left[(m + n)(m + n + 1) - A_{ml}(h) + \frac{2(m + n)(m + n + 1) - 2m^2 - 1}{(2m + 2n + 3)(2m + 2n - 1)} h^2\right] d_n \\
&+ \frac{n(n - 1)}{(2m + 2n - 3)(2m + 2n - 1)} h^2 d_{n-2} = 0.
\end{aligned} \tag{5}
$$

Here $A_{ml}(h)$ is one of the two separation constants. The other separation constant is m, required to be an integer for single-valuedness and chosen, without loss of generality, to be nonnegative. For fixed m and h, the allowable values for $A_{ml}(h)$ are ordered numerically in an ascending sequence and labeled with the integers $l = m$, $l = m + 1$, etc. Unnormalized values for $d_n(h|ml)$ are obtained by use of

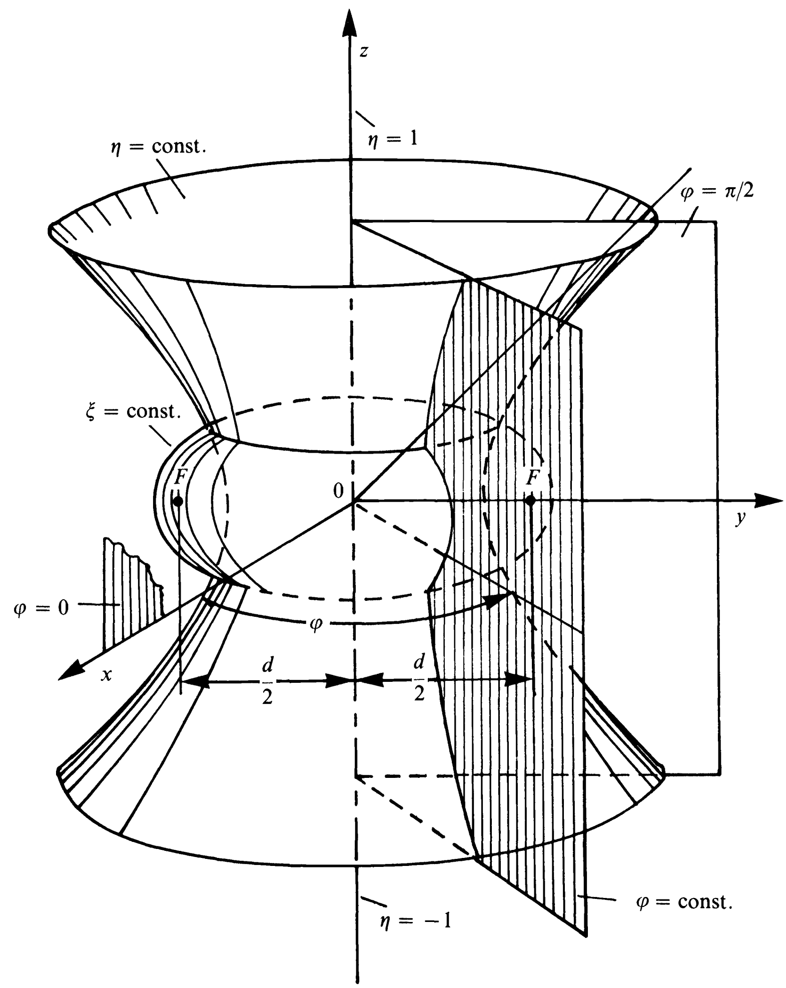

FIG. 2. *The oblate spheroidal coordinate system*

the recursion relation and the boundary condition

$$(d_{n+2}/d_n) \xrightarrow[n\to\infty]{} 0. \tag{6}$$

The normalization of $d_n(h|ml)$ is determined by imposing restrictions on the behavior of $S_{ml}^{(1)}(h, \eta)$. For example, Meixner [9] requires that

$$\int_{-1}^{1} [S_{ml}^{(1)}(h, \eta)]^2 \, d\eta = 2(l + m)!/[(2l + 1)(l - m)!], \tag{7}$$

while Flammer [3] requires that

$$\begin{aligned} &S_{ml}^{(1)}(h, 0) = P_l^m(0), \quad l - m \text{ even}, \\ &[dS_{ml}^{(1)}(h, \eta)/d\eta]_{\eta=0} = [dP_l^m(\eta)/d\eta]_{\eta=0}, \quad l - m \text{ odd}. \end{aligned} \tag{8}$$

Because of this arbitrariness, all subsequent equations will be written in a form that is independent of the normalization of the spheroidal angle wave functions, i.e., the equations will be valid for any $d_n(h|ml)$ which satisfy (5) and (6).

The spheroidal radial wave functions are also expanded in terms of the corresponding spherical functions:

$$R_{ml}^{(j)}(h, \xi) = \{[(\xi^2 - 1)/\xi^2]^{m/2}/M_{ml}\} \sum_{n=0,1}^{\infty}{}' i^{n+m-l} d_n(h|ml) \cdot [(n + 2m)!/n!]\Phi_{n+m}^{(j)}(h\xi), \quad j = 1, 2, 3, 4, \tag{9}$$

where

$$M_{ml} = \sum_{n=0,1}^{\infty}{}' d_n(h|ml)(n + 2m)!/n!, \tag{10}$$

and where the $\Phi_{n+m}^{(j)}(h\xi)$ are the spherical Bessel functions of the first and second kind and the Hankel functions of the first and second kind when $j = 1, 2, 3$ and 4, respectively. Since $R_{ml}^{(2)}(h, \xi) \to -\infty$ as $\xi \to 1$ in prolate geometry, the following discussion will be limited to $\xi > 1$ for prolate geometry when $j \neq 1$. Extensive tables of both prolate and oblate spheroidal radial wave functions and their first derivatives have recently been published [5], [6]. The FORTRAN computer programs used to generate these tables and a FORTRAN computer program used to calculate both prolate and oblate spheroidal angle wave functions and their first and second derivatives are described in [7], [16] and [8]. The availability of the tables and computer programs should greatly increase the usage of spheroidal wave functions.

2. A general addition theorem. Consider two rectangular coordinate systems A_1 and A_2 with completely arbitrary relative positions and orientations, as shown in Fig. 3. The corresponding spheroidal coordinates are given by (2) or (3). The relative positions of A_1 and A_2 are defined by the vector $\mathbf{r}_{12}$ extending from the origin O_2 to the origin O_1. The relative orientations are defined by the three Eulerian rotations with angles α, β and γ which when applied to A_1 will make this system parallel to A_2. An arbitrary point P has spheroidal coordinates $(\xi_1, \eta_1, \varphi_1)$ with respect to A_1 and $(\xi_2, \eta_2, \varphi_2)$ with respect to A_2.

The expansion of a spheroidal wave function in A_2 in terms of spheroidal wave functions in A_1 will be obtained using the following four-step procedure. First, a spheroidal wave function in A_2 will be expanded in terms of spherical wave functions in A_2. Second, the spherical wave functions in A_2 will be expanded in terms of spherical wave functions relative to a rectangular coordinate frame A_3 with origin O_1 and with coordinate axes parallel to those of A_2. Third, the spherical wave functions in A_3 will be expanded in terms of spherical wave functions in A_1. Finally, the spherical wave functions in A_1 will be expanded in terms of spheroidal wave functions in A_1.

The expansion of a spheroidal wave function in A_2 in terms of spherical wave functions in A_2 is given by

$$\begin{aligned} \psi_{ml}^{(j)}(h_2; \xi_2, \eta_2, \varphi_2) &= R_{ml}^{(j)}(h_2, \xi_2) S_{ml}^{(1)}(h_2, \eta_2) \exp(im\varphi_2) \\ &= \sum_{r=m,m+1}^{\infty}{}' i^{r-l} d_{r-m}(h_2|ml) \Phi_r^{(j)}(kr_2) \Omega_r^m(\theta_2, \varphi_2), \\ & \qquad j = 1, 2, 3, 4, \quad m \geqq 0, \end{aligned} \tag{11}$$

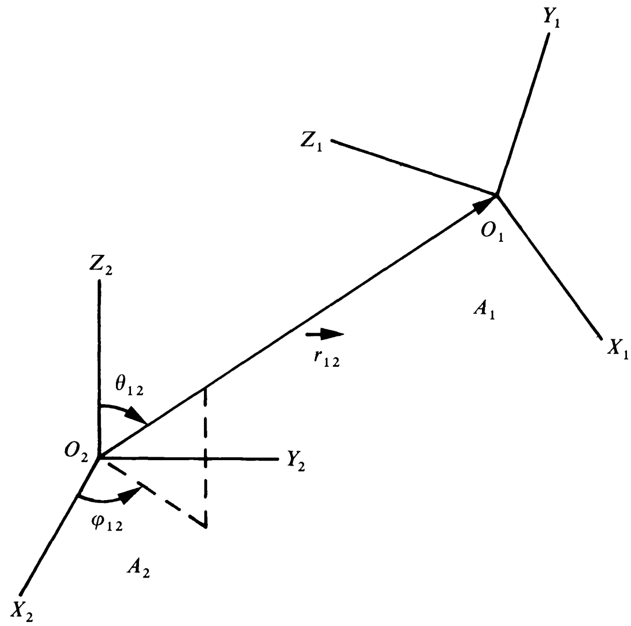

FIG. 3. *Two coordinate frames with arbitrary relative positions and orientations*

where the prime indicates that the sum includes terms for $r = m, m + 2, m + 4, \cdots$ if $l - m$ is even and $r = m + 1, m + 3, m + 5, \cdots$ if $l - m$ is odd. The angular dependence of the spherical wave functions is given by the unnormalized spherical harmonics $\Omega_r^m(\theta_2, \varphi_2) = P_r^m(\cos\theta_2)\exp(im\varphi_2)$. Here $(r_2, \theta_2, \varphi_2)$ are the spherical coordinates of the point P with respect to A_2. The spheroidal wave function and its corresponding expansion coefficients $d_{r-m}(h_2|ml)$ may be either prolate or oblate. This formula can be obtained as a special case of the addition theorem given by Meixner [9]. Flammer [3] also derives this formula, except for the unfortunate omission of the factor i^{r-l}, for the single case $j = 1$, although his method can be used to obtain the analogous expressions for $j = 2, 3, 4$.

Spherical wave functions in A_2 can be expanded in terms of spherical wave functions in A_3 by use of the spherical addition theorem first derived by Friedman and Russek [4]. An alternative formulation of the expansion coefficients was obtained by New [10] using a procedure outlined by Sack [14]. This latter form will be used below because it requires considerably less numerical evaluation. Let $(r_3, \theta_3, \varphi_3)$ be the spherical coordinates of the point P with respect to A_3. Let $(r_{12}, \theta_{12}, \varphi_{12})$ be the spherical coordinates of the origin O_1 with respect to A_2. Then

$$\Phi_r^{(j)}(kr_2)\Omega_r^m(\theta_2, \varphi_2) = \sum_{s=0}^{\infty} \sum_{u=-s}^{s} \sum_{\substack{t=|r-s| \\ t \geqq |m-u|}}^{r+s}{}' a(s, t, r, u, m) \tag{12}$$

$$\cdot \Phi_s^{(1)}(kr_<)\Omega_s^u(\theta_<, \varphi_<)\Phi_t^{(j)}(kr_>)\Omega_t^{m-u}(\theta_>, \varphi_>),$$

$$j = 1, 2, 3, 4, \quad m \geqq 0,$$

where

$$a(s,t,r,u,m) = i^{s+t-r}\left\{\frac{(2s+1)(2t+1)(s-u)!(t-m+u)!(r+m)![(s+t+r)/2]!}{[(r+t-s)/2]![(r+s-t)/2]![(s+t-r)/2]!(s+t+r+1)!}\right\} \cdot \sum_{w=w\min}^{w\max} (-1)^w \binom{s+t-r}{(s+t-r)/2+w}\binom{t+r-s}{(t+r-s)/2+m-u+w} \cdot \binom{s+r-t}{(s+r-t)/2-u+w}, \tag{13}$$

with

$$\binom{a}{b} = a!/[b!(a-b)!], \tag{14}$$

$$2w\min = \max(r-s-t, s-r-t-2m+2u, t-s-r+2u), \tag{15}$$

$$2w\max = \min(s+t-r, r+t-s-2m+2u, r+s-t+2u), \tag{16}$$

and where

$$(r_<, \theta_<, \varphi_<) = \begin{cases}(r_3, \theta_3, \varphi_3), & \text{if } r_3 < r_{12},\\ (r_{12}, \theta_{12}, \varphi_{12}), & \text{if } r_3 > r_{12},\end{cases} \qquad (r_>, \theta_>, \varphi_>) = \begin{cases}(r_3, \theta_3, \varphi_3), & \text{if } r_3 > r_{12},\\ (r_{12}, \theta_{12}, \varphi_{12}), & \text{if } r_3 < r_{12}.\end{cases} \tag{17}$$

The prime on the summation over t indicates that t is incremented in steps of two. Terms involving negative values for u or $m - u$ can be evaluated with the use of the definition

$$P_n^{-m}(x) = (-1)^m P_n^m(x)(n-m)!/(n+m)!. \tag{18}$$

The relation between the coefficients $a(s, t, r, u, m)$ and both the Wigner 3-j symbols [18] and the Clebsch–Gordon coefficients [2] is discussed in the Appendix.

Spherical wave functions in A_3 can be expanded in terms of spherical wave functions in A_1 by use of a rotation formula. Since A_3 and A_2 are parallel, α, β and γ are the angles for the Eulerian rotations which transform A_1 into A_3. Let $(r_1, \theta_1, \varphi_1)$ be the spherical coordinates of the point P with respect to A_1. The radial dependence of the spherical wave functions is unaffected by a rotation, i.e., $r_1 = r_3$. The required transformation for the angular dependence is given by Rose [13]:

$$\Omega_v^p(\theta_3, \varphi_3) = N_v^p \sum_{q=-v}^{v} \Omega_v^q(\theta_1, \varphi_1) D_{qp}^v(\alpha, \beta, \gamma)/N_v^q, \tag{19}$$

where

$$N_v^m = \{(2v+1)(v-m)!/[4\pi(v+m)!]\}^{1/2} \tag{20}$$

and

$$D^{v}_{qp}(\alpha, \beta, \gamma) = \exp(-iq\alpha) d^{v}_{qp}(\beta) \exp(-ip\gamma). \tag{21}$$

The matrix elements $d^{v}_{qp}(\beta)$ are defined by

$$\begin{aligned} d^{v}_{qp}(\beta) = {} & \left[\frac{(v-p)!(v+q)!}{(v+p)!(v-q)!}\right]^{1/2} [\cos(\beta/2)]^{2v+p-q}[-\sin(\beta/2)]^{q-p} \\ & \cdot {}_2F_1[q-v, -p-v; q-p+1; -\tan^2(\beta/2)]/(q-p)!, \quad q > p, \end{aligned} \tag{22}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. Values for $q < p$ are obtained from the relation

$$d^{v}_{qp}(\beta) = (-1)^{q-p} d^{v}_{pq}(\beta). \tag{23}$$

The transformation is completed by expanding the spherical wave functions in A_1 in terms of spheroidal wave functions in A_1. This can be accomplished using the formula

$$\begin{aligned} \Phi^{(j)}_{v}(kr_1)\Omega^{q}_{v}(\theta_1, \varphi_1) = {} & [C(v, q)/(2v+1)] \sum_{n=|q|,|q|+1}^{\infty}{}' (i^{v-n}/B_{|q|n}) \\ & \cdot d_{v-|q|}(h_1 \| q| n) R^{(j)}_{|q|n}(h_1, \xi_1) S^{(1)}_{|q|n}(h_1, \eta_1) \exp(iq\varphi_1), \qquad j = 1, 2, 3, 4, \end{aligned} \tag{24}$$

where

$$C(v, q) = \begin{cases} (v+q)!/(v-q)!, & q \geqq 0, \quad (25) \\ (-1)^{q}, & q < 0, \quad (26) \end{cases}$$

$$B_{|q|n} = \sum_{l=0,1}^{\infty}{}' d^{2}_{l}(h_1 \| q| n)(l + 2|q|)!/\{l![2(l+|q|)+1]\}, \tag{27}$$

and where the sum includes terms for $n = |q|, |q| + 2, \cdots$ if $v - |q|$ is even and $n = |q| + 1, |q| + 3, \cdots$ if $v - |q|$ is odd. This formula is a special case of the addition theorem given by Meixner [9]. Flammer [3] also derives this formula for the case $j = 1$. Note that the interfocal length $d_1 = 2h_1/k$ for the spheroidal coordinate frame in A_1 is not necessarily equal to the interfocal length $d_2 = 2h_2/k$ for the spheroidal coordinate frame in A_2. The spheroidal wave functions appearing in (24) and their corresponding expansion coefficients $d_{v-|q|}(h_1 \| q| n)$ may be either prolate or oblate.

The general addition theorem for spheroidal functions is obtained by combining the four transformations given by (11), (12), (19) and (24).

$$\psi^{(j)}_{ml}(h_2; \xi_2, \eta_2, \varphi_2) = \sum_{r=m,m+1}^{\infty}{}' i^{r-l} d_{r-m}(h_2|ml) \sum_{s=0}^{\infty} \sum_{u=-s}^{s} \sum_{\substack{t=|r-s| \\ t \geqq |m-u|}}^{r+s}{}' a(s, t, r, u, m)$$

((28) cont. on following page)

$$
(28)\cdot\begin{cases}
[N_t^{m-u}/(2t+1)]\Phi_s^{(1)}(kr_{12})\Omega_s^u(\theta_{12},\varphi_{12})\sum\limits_{q=-t}^{t}[C(t,q)D^t_{q,m-u}(\alpha,\beta,\gamma)/N_t^q] \\
\sum\limits_{n=|q|,|q|+1}^{\infty}{}' [i^{t-n}d_{t-|q|}(h_1\|q|n)/B_{|q|n}]R^{(j)}_{|q|n}(h_1,\xi_1)S^{(1)}_{|q|n}(h_1,\eta_1)\exp(iq\varphi_1), \\
\qquad\qquad r_1 > r_{12}, \\
\text{or} \\
[N_s^u/(2s+1)]\Phi_t^{(j)}(kr_{12})\Omega_t^{m-u}(\theta_{12},\varphi_{12})\sum\limits_{q=-s}^{s}[C(s,q)D^s_{qu}(\alpha,\beta,\gamma)/N_s^q] \\
\sum\limits_{n=|q|,|q|+1}^{\infty}{}' [i^{s-n}d_{s-|q|}(h_1\|q|n)/B_{|q|n}]R^{(1)}_{|q|n}(h_1,\xi_1)S^{(1)}_{|q|n}(h_1,\eta_1)\exp(iq\varphi_1), \\
\qquad\qquad r_1 < r_{12},
\end{cases}
$$

$$j = 1, 2, 3, 4, \quad m \geqq 0,$$

where $r_1 = d_1(\xi_1^2 + \eta_1^2 - 1)^{1/2}/2$ or $d_1(\xi_1^2 - \eta_1^2 + 1)^{1/2}/2$ depending on whether the spheroidal coordinates $(\xi_1, \eta_1, \varphi_1)$ are prolate or oblate, respectively. Here $(\xi_1, \eta_1, \varphi_1)$ are the spheroidal coordinates of the point P with respect to A_1, $(\xi_2, \eta_2, \varphi_2)$ are the spheroidal coordinates of the point P with respect to A_2, and $(r_{12}, \theta_{12}, \varphi_{12})$ are the spherical coordinates of the origin O_1 with respect to A_2. Note that in order to use (28) it may be necessary to obtain the spheroidal coordinates $(\xi_1, \eta_1, \varphi_1)$ in A_1 corresponding to the spheroidal coordinates $(\xi_2, \eta_2, \varphi_2)$ in A_2. This can be accomplished by transforming $(\xi_2, \eta_2, \varphi_2)$ to rectangular coordinates in A_2, performing the required translation and rotation in rectangular coordinates to obtain the corresponding rectangular coordinates in A_1, and then transforming back to spheroidal coordinates to obtain $(\xi_1, \eta_1, \varphi_1)$.

Equation (28) is valid only for nonnegative values of m. However, expansions in spheroidal wave functions usually contain terms for negative values of m. Consider the spheroidal wave functions $R^{(j)}_{ml}(h, \xi)S^{(1)}_{ml}(h, \eta)\exp(im\varphi)$, $m < 0$. Since the spheroidal radial and angle wave functions for m negative and those for m positive are not linearly independent but are related through the definitions

$$S^{(1)}_{-ml}(h, \eta) = (-1)^m(l - m)!S^{(1)}_{ml}(h, \eta)/(l + m)!, \tag{29}$$

$$R^{(j)}_{-ml}(h, \xi) = R^{(j)}_{ml}(h, \xi), \tag{30}$$

spheroidal wave functions with m negative can be replaced with the equally suitable wave functions $R^{(j)}_{ml}(h, \xi)S^{(1)}_{ml}(h, \eta)\exp(-im\varphi)$, $m > 0$. An addition theorem for these functions can be obtained by taking the complex conjugate of (28). Note that $R^{(3)}_{ml}(h, \xi)$, $R^{(4)}_{ml}(h, \xi)$ and $\Phi_l^{(3)}(kr)$, $\Phi_l^{(4)}(kr)$ are complex conjugate pairs, and that the complex conjugate of $\Omega_l^m(\theta, \varphi)$ is $P_l^m(\cos\theta)\exp(-im\varphi)$.

When the two coordinate frames A_1 and A_2 are parallel, $\alpha = \beta = \gamma = 0$, and the rotation given by (19) becomes the identity transformation. In this case the addition theorem is simplified by setting

$$D^t_{q,m-u}(0, 0, 0) = \delta_{q,m-u} \tag{31}$$

and

$$D^s_{qu}(0, 0, 0) = \delta_{qu}, \tag{32}$$

where δ_{ab} is the Kronecker delta.

When the two coordinate frames A_1 and A_2 are parallel and share a common z-axis, the axis of symmetry for spheroids, such that $\theta_{12} = 0$ or π, the addition theorem reduces to the form,

$$\psi_{ml}^{(j)}(h_2;\xi_2,\eta_2,\varphi_2) = \sum_{r=m,m+1}^{\infty}{}' i^{r-l} d_{r-m}(h_2|ml) \sum_{s=0}^{\infty}$$

$$(33)\quad \cdot \begin{cases} \displaystyle\sum_{\substack{t=|r-s|\\ t\geqq m}}^{r+s}{}' \{a(s,t,r,0,m)(t+m)!/[(t-m)!(2t+1)]\}\Phi_s^{(1)}(kr_{12})P_s(\cos\theta_{12}) \\ \displaystyle\qquad \sum_{n=m,m+1}^{\infty}{}' i^{t-n}[d_{t-m}(h_1|mn)/B_{mn}]\psi_{mn}^{(j)}(h_1;\xi_1,\eta_1,\varphi_1), \quad r_1 > r_{12}, \\ \text{or} \\ \displaystyle\sum_{t=|r-s|}^{r+s}{}' \{a(s,t,r,m,m)(s+m)!/[(s-m)!(2s+1)]\}\Phi_t^{(j)}(kr_{12})P_t(\cos\theta_{12}) \\ \displaystyle\qquad \sum_{n=m,m+1}^{\infty}{}' i^{s-n}[d_{s-m}(h_1|mn)/B_{mn}]\psi_{mn}^{(1)}(h_1;\xi_1,\eta_1,\varphi_1), \quad r_1 < r_{12}, \end{cases}$$

$$j = 1,2,3,4, \quad m \geqq 0,$$

where

$$(34)\qquad P_v(\cos\theta_{12}) = 1, \qquad \text{if } \theta_{12} = 0,$$

$$(35)\qquad P_v(\cos\theta_{12}) = (-1)^v, \quad \text{if } \theta_{12} = \pi.$$

For the special case where $m = 0$, the coefficient $a(s,t,r,0,0)$ can be evaluated from a single term, as shown in the Appendix.

Appendix. The coefficients $a(s, t, r, u, m)$ defined in (13) can be related to the Wigner 3-j symbols [18] and the Clebsch–Gordon coefficients [2]. Equation (13) can be written as

$$(\text{A.1})\qquad a(s,t,r,u,m) = (-1)^{\lambda+m+\Lambda}\left[\frac{(2s+1)(2t+1)(s-u)!(t-m+u)!(r+m)!\Lambda!}{(2\Lambda+1)!\lambda!\lambda_1!\lambda_2!}\right] \cdot U\begin{pmatrix} r & s & t \\ m & -u & u-m \end{pmatrix},$$

where

$$(\text{A.2})\qquad U\begin{pmatrix} r & s & t \\ m & -u & u-m \end{pmatrix} = \sum_{w=w\min}^{w\max} (-1)^{-\Lambda+w-m}\binom{2\lambda}{\lambda+w}\binom{2\lambda_1}{\lambda_1+m-u+w}\binom{2\lambda_2}{\lambda_2-u+w},$$

and

$$(\text{A.3})\qquad \Lambda = (r+s+t)/2, \quad \lambda_1 = \Lambda - s, \quad \lambda = \Lambda - r, \quad \lambda_2 = \Lambda - t.$$

Sack [14] relates the coefficients $a(s,t,r,u,m)$ to the Wigner 3-j symbols by use of the expression

$$(A.4)\quad U\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \cdot \left[\frac{(j_1+j_2+j_3+1)!(j_1+j_2-j_3)!(j_1+j_3-j_2)!(j_2+j_3-j_1)!}{(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!(j_3-m_3)!(j_3+m_3)!}\right]^{1/2},$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3-j symbol.

The relation between the Wigner 3-j symbols and the Clebsch–Gordon coefficients is given by [18]

$$(A.5)\quad \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = [(-1)^{j_1-j_2-m_3}/(2j_3+1)^{1/2}](j_1j_2m_1m_2|j_1j_2j_3-m_3),$$

where $(j_1j_2m_1m_2|j_1j_2j_3-m_3)$ is the Clebsch–Gordon coefficient.

Combining (A.1), (A.4) and (A.5) and simplifying, one obtains

$$(A.6)\quad a(s,t,r,u,m) = (-1)^{s+u}(2s+1)\Lambda!(rsm-u|rstm-u)/(\lambda!\lambda_1!\lambda_2!) \cdot \left[\frac{(r+m)!(s-u)!(t-m+u)!(2t+1)(2\lambda)!(2\lambda_1)!(2\lambda_2)!}{(r-m)!(s+u)!(t-u+m)!(2\Lambda+1)!}\right]^{1/2}.$$

For the special case $u = m = 0$, the Clebsch–Gordon coefficient $(rs00|rst0)$ has been expressed as a single term by Racah [11],

$$(A.7)\quad (rs00|\,rst0) = (-1)^{t+\Lambda}(2t+1)^{1/2}\Lambda! \cdot [(2\lambda)!(2\lambda_1)!(2\lambda_2)!]^{1/2}/\{\lambda!\lambda_1!\lambda_2![(2\Lambda+1)!]^{1/2}\}.$$

Thus the corresponding coefficient $a(s,t,r,0,0)$ can also be expressed as a single term,

$$(A.8)\quad a(s,t,r,0,0) = (-1)^{\lambda}(2s+1)(2t+1)(2\lambda)!(2\lambda_1)!(2\lambda_2)!(\Lambda!)^2/[(2\Lambda+1)!(\lambda!\lambda_1!\lambda_2!)^2].$$

It is interesting to note that the Clebsch–Gordon coefficients and, consequently, the coefficients $a(s,t,r,u,m)$ are related to the generalized hypergeometric functions ${}_3F_2(a,b,c;d,e;1)$ by the expression [12]

$$(j_1j_2m_1m_2|j_1j_2j_3m_3) = (-1)^{j_2+m_2} \cdot \left[\frac{(j_3+j_1-j_2)!(j_1+j_2-j_3)!(j_3-m_3)!(j_1-m_1)!(2j_3+1)}{(j_3-j_1+j_2)!(j_1+j_2+j_3+1)!(j_3+m_3)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!}\right]^{1/2} \cdot \frac{(j_3+j_2+m_1)!}{(j_1-j_2-m)!}\,{}_3F_2(-j_3+j_1-j_2, j_1-m_1+1, -j_3-m_3; j_1-j_2-m_3+1, -j_3-j_2-m_1; 1).$$

Many of the properties of the coefficients $a(s,t,r,u,m)$ can thus be obtained using the well-known properties of the generalized hypergeometric functions [1], [15]. For example, (A.7) and therefore (A.8) can be obtained from (A.9) by the use of Dixon's theorem [15] to evaluate ${}_3F_2(-j_3+j_1-j_2, j_1+1, -j_3; j_1-j_2+1, -j_3-j_2; 1)$.

REFERENCES

[1] W. N. BAILEY, *Generalized Hypergeometric Series*, Cambridge University Press, London, 1935.

[2] E. U. CONDON AND G. H. SHORTLEY, *The Theory of Atomic Spectra*, Cambridge University Press, London, 1959.

[3] C. FLAMMER, *Spheroidal Wave Functions*, Stanford University Press, Stanford, Calif., 1957.

[4] B. FRIEDMAN AND J. RUSSEK, *Addition theorems for spherical waves*, Quart. Appl. Math., 12 (1954), pp. 13–23.

[5] S. HANISH, R. V. BAIER, A. L. VAN BUREN AND B. J. KING, *Tables of Radial Spheroidal Wave Functions, Vols.* 1–3, *Prolate*, $m = 0, 1, 2$, Rep. 7088–7090, U.S. Naval Res. Lab., Washington, D.C., 1970.

[6] ———, *Tables of Radial Spheroidal Wave Functions, Vols.* 4–6, *Oblate*, $m = 0, 1, 2$, Rep. 7091–7093, U.S. Naval Res. Lab., Washington, D.C., 1970.

[7] B. J. KING, R. V. BAIER AND S. HANISH, *A* FORTRAN *computer program for calculating the prolate spheroidal radial functions of the first and second kind and their first derivatives*, Rep. 7012, U.S. Naval Res. Lab., Washington, D.C., 1970.

[8] B. J. KING AND A. L. VAN BUREN, *A* FORTRAN *computer program for calculting the prolate and oblate angle functions of the first kind and their first and second derivatives*, Rep. 7161, U.S. Naval Res. Lab., Washington, D.C., 1970.

[9] J. MEIXNER AND F. W. SCHÄFKE, *Mathieusche Funktionen und Sphäroidfunctionen*, Springer-Verlag, Berlin, 1954.

[10] R. NEW, *Acoustic radiation from multiple surfaces*, Rep. 71–8, Catholic Univ. Inst. of Ocean Sciences, Washington, D.C., 1971.

[11] G. RACAH, *Theory of complex spectra. II*, Phys. Rev., 62 (1942), pp. 438–462.

[12] M. E. ROSE, *Multipole Fields*, John Wiley, New York, 1955.

[13] ———, *Elementary Theory of Angular Momentum*, John Wiley, New York, 1957.

[14] R. A. SACK, *Three dimensional addition theorem for arbitrary functions involving expansions in spherical harmonics*, J. Mathematical Phys., 5 (1964), pp. 252–259.

[15] L. J. SLATER, *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1966.

[16] A. L. VAN BUREN, R. V. BAIER AND S. HANISH, *A* FORTRAN *computer program for calculating the oblate spheroidal radial functions of the first and second kind and their first derivatives*, Rep. 6959, U.S. Naval Res. Lab., Washington, D.C., 1970.

[17] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1966.

[18] E. P. WIGNER, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York, 1959.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

THE BEHAVIOR AS $\varepsilon \to 0^+$ OF SOLUTIONS TO $\varepsilon\nabla^2 w = (\partial/\partial y)w$ ON THE RECTANGLE $0 \leqq x \leqq l, |y| \leqq 1$*

L. PAMELA COOK AND G. S. S. LUDFORD†

Abstract. The title problem is first examined in the limit of the semi-infinite strip $l = \infty$, for boundary data $w(x, -1) = f(x)$, $w(x, 1) = g(x)$, $w(0, y) = h(y)$. Here f, g, h are infinitely differentiable except at the corners where one-sided derivatives of all orders exist. Previous work on the infinite strip covers cases where $h = 0$ so that (by superposition) the present discussion may be narrowed to cases where $f = g = 0$; for these the solution is asymptotically zero for $x \geqq x_0 > 0$. Near $x = 0$ four regions are distinguished: the parabolic boundary layer $y \leqq y_1 < 1$, excluding $\varepsilon^{-1/2}x \leqq X_0$, $1 + y \leqq y_{-1}$, which is determined by the singular region $\varepsilon^{-1}(1 + y) \leqq y_{*\infty}$; and the two parts of the hyperbolic boundary layer $\varepsilon^{-1}(1 - y) \leqq Y_\infty$, namely $\varepsilon^{-1/2}x \geqq X_1 > 0$ and the transition zone $\varepsilon^{-1}x \leqq x_{*\infty}$, both of which are determined by the parabolic layer. By means of Fourier sine transforms the method of matched asymptotic expansions is proved valid to all orders in ε in each of the regions, which can be extended to overlap. Other assumptions about h are also considered. Finally the corresponding results for the rectangle are shown to follow from the superposition of two semi-infinite strip problems.

1. Introduction. We propose to examine the asymptotic properties as $\varepsilon \to 0^+$ of the solution to the equation

$$\varepsilon(\partial^2/\partial x^2 + \partial^2/\partial y^2)W - (\partial/\partial y)W = 0 \tag{1a}$$

on the semi-infinite strip $|y| \leqq 1, x \geqq 0$, under the boundary conditions

$$W(x, 1) = g(x), \qquad W(x, -1) = f(x), \qquad W(0, y) = h(y). \tag{1b}$$

Our goal is to prove that the method of matched asymptotic expansions does give the correct approximation to W to all orders in ε.

The method of attack is similar to and an extension of that used in our previous paper (Cook and Ludford [2]). In § 10 we shall show how to extend these results to cover the asymptotics of the equation (1a) in a rectangular region, the latter being of greater physical interest than either the infinite or semi-infinite strip.

Before outlining the method of proof, we simplify the problem in the following manner. The solution W, under the boundary conditions (1b), is the sum of the solution w satisfying the boundary conditions

$$w(x, \pm 1) = 0, \tag{1c}$$

$$w(0, y) = h(y), \tag{1d}$$

and the solution v satisfying the boundary conditions

$$v(x, 1) = g(x), \qquad v(x, -1) = f(x), \qquad v(0, y) = 0.$$

Now consider $u(x, y)$ which solves (1a) on the infinite strip $-\infty < x < \infty$, with the boundary conditions

$$u(x, 1) = G(x), \qquad u(x, -1) = F(x),$$

---

* Received by the editors June 29, 1971.

† Center for Applied Mathematics, Cornell University, Ithaca, New York 14850. This work was supported in part by the U.S. Army Research Office–Durham and in part by the National Science Foundation under Grants GP-8711 and 28483.

where G, F are the odd extensions of g, f respectively. Clearly $u(x, y)$ is an odd function of x, so that $u(0, y) = 0$ and $v(x, y) = u(x, y)$ for $x \geqq 0$.

The proof that the method of matched asymptotic expansions is valid for u is found in Cook and Ludford [2]. The results depend on the differentiability properties of G and F. For example, if right and left derivatives of G, F exist at zero to order k_0, and if $G^{(k)}$, $F^{(k)}$ are integrable for $k \leqq k_0 + 1$, G and F being infinitely differentiable except at zero, then the results of the method of matched asymptotic expansions are valid to order ε^m where $m \leqq 2[(k_0 + 1)/4]$.

Since v is covered by the u of our previous paper, we may concentrate on w here. As in [2] the proof depends on having an explicit representation of the exact solution in terms of the Green's function, the latter consisting of the fundamental solution and its images in the (extended) boundaries. It can be seen immediately that, of the infinity of such terms, all but the first four can be ignored because they are a.e.s. (asymptotically exponentially small) throughout the strip. However, manipulating the remaining terms is difficult, and instead we consider their Fourier sine transforms. The latter are easily managed by expanding in Taylor series in ε. The basic difficulty is to prove that term-by-term inversion of the expansions in the transform plane does produce asymptotic expansions whose terms are those obtained by the method of matched asymptotic expansions.

2. The method of matched asymptotic expansions. As in our previous paper [2] we shall assume the reader is familiar with this method; see, for example,

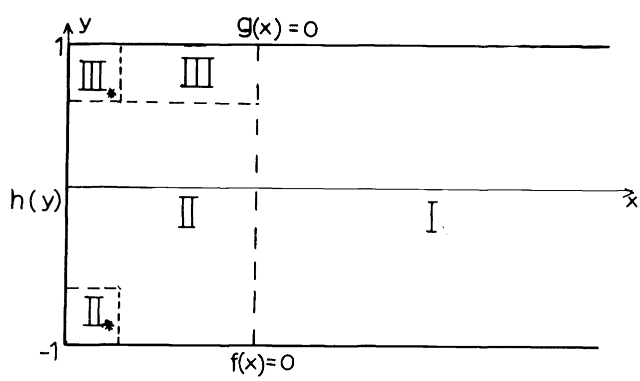

FIG. 1

Chapter 4 of Cole's book [1] where its application to problems such as ours is considered in some detail. In particular Cole discusses the locations of the boundary layers and their orders of magnitude, as well as the physical situations in which they occur (cf. also the Introduction of [2]). In this section we are solely concerned with collecting the results obtained by the method in a form that is suitable for our later proofs, without reproducing Cole's arguments for each step.

Figure 1 shows schematically the various regions of validity of the expansions referred to in the present section.

Assuming an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_k^{\mathrm{I}}(x, y) \tag{2}$$

of the solution to the boundary value problem (1a), (1c), (1d), we obtain the recurrence relation

$$(\partial/\partial y)w_k^{\mathrm{I}} = (\partial^2/\partial x^2 + \partial^2/\partial y^2)w_{k-1}^{\mathrm{I}} \tag{3}$$

for the coefficient functions, by direct substitution. We must therefore take

$$w_k^{\mathrm{I}}(x, y) = 0$$

in order to satisfy the boundary conditions (1c). It is clear that the expansion cannot be uniformly valid since it does not satisfy the boundary condition at $x = 0$.

Therefore, we consider the substitution

$$X = \varepsilon^{-1/2}x \tag{4}$$

in order to make $\varepsilon\partial^2/\partial x^2 = \partial^2/\partial X^2$ comparable to $\partial/\partial y$. Then with an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_{k-1}^{\mathrm{II}}(X, y), \tag{5}$$

we obtain the recurrence relation

$$(\partial^2/\partial X^2 - \partial/\partial y)w_k^{\mathrm{II}}(X, y) = -(\partial^2/\partial y^2)w_{k-1}^{\mathrm{II}}(X, y), \tag{6a}$$

for the coefficient functions. The appropriate boundary conditions for this inhomogeneous parabolic equation are

$$w_k^{\mathrm{II}}(X, -1) = 0, \qquad w_k^{\mathrm{II}}(0, y) = \begin{cases} h(y) & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases} \tag{6b}$$

Such an expansion cannot be valid near $X = 0$, $y = -1$, as is easily seen for the case $h = 1$: The functions

$$\begin{aligned} w_0^{\mathrm{II}} &= \operatorname{erfc}[X(y + 1)^{-1/2}/2], \\ w_1^{\mathrm{II}} &= -X^3/[8\pi^{1/2}(y + 1)^{5/2}] \exp[-X^2(y + 1)^{-1}/4] \end{aligned} \tag{7}$$

satisfy all conditions, and w_1^{II} becomes unbounded in a neighborhood of $X = 0$, $y = -1$. In fact, it is not even uniquely determined since, if singularities are admitted at $X = 0$, $y = -1$, certain solutions of the homogeneous diffusion equation may be added.

Such difficulties could have been anticipated since we are attempting to represent an elliptic singularity by means of solutions of parabolic equations. To consider the singular region we introduce the stretched coordinates

$$X_* = \varepsilon^{-1/2}X, \qquad y_* = \varepsilon^{-1}(y + 1) \tag{8}$$

in order to make all derivatives in (1a) of comparable order. With an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_k^{\mathrm{II}*}(X_*, y_*), \tag{9}$$

we then see that the coefficient functions must satisfy the full elliptic equation

$$(\partial^2/\partial X_*^2 + \partial^2/\partial y_*^2 - \partial/\partial y_*)w_k^{\mathrm{II}*} = 0 \tag{10a}$$

and the boundary conditions

$$w_k^{\mathrm{II}*}(X_*, 0) = 0, \qquad w_k^{\mathrm{II}*}(0, y_*) = h^{(k)}(-1)y_*^k/k!. \tag{10b}$$

The $w_k^{\mathrm{II}*}$ are unique if in addition we require that they do not grow exponentially as $y_* \to \infty$, a condition which is necessary in order to match with (5). This matching then uniquely determines the coefficients of the expansion (5). More precisely, X_*^2/y_* $(=X^2/(y+1))$ is fixed as $\varepsilon \to 0$ (though $y_* \to \infty$) and we find that only full powers of ε are involved. Consequently, no half powers of ε are required in region II, as was anticipated in writing the expansion (5). In particular, we find that the homogeneous solution $3X/[4\pi^{1/2}(y+1)^{7/2}]\exp[-X^2(y+1)^{-1}/4]$ must be added to the w_1^{II} in (7).

Finally we consider the boundary layer at $y = 1$ which is needed to correct the II-expansion for the boundary condition at $y = 1$. With X and the stretched variable

$$Y = \varepsilon^{-1}(1 - y) \tag{11}$$

(so that $\varepsilon\partial^2/\partial y^2 = \varepsilon^{-1}\partial^2/\partial Y^2$ is comparable to $\partial/\partial y = \varepsilon^{-1}\partial/\partial Y$), we assume an expansion

$$w \sim \sum_{k=0}^{\infty} \varepsilon^k w_k^{\mathrm{III}}(X, Y). \tag{12}$$

The recurrence relation for the coefficient functions is then

$$(\partial^2/\partial Y^2 + \partial/\partial Y)w_k^{\mathrm{III}} = -(\partial^2/\partial X^2)w_{k-1}^{\mathrm{III}} \tag{13a}$$

and the boundary condition

$$w_k^{\mathrm{III}}(X, 0) = 0. \tag{13b}$$

At each stage an integration constant is obtained, and is uniquely determined by matching with the expansion (5). It is now clear that the boundary layer must occur at $y = 1$, and not at $y = -1$, in order to obtain exponentially decreasing functions for the matching.

Once again this expansion cannot be uniformly valid in the boundary layer, as can be seen from $w_0^{\mathrm{III}} = C_0[1 - \exp(-Y)]$ where $C_0(X)$ is obtained by matching; in particular $C_0(0) = h(1)$. This violates the boundary condition on the y-axis for a distance ε down from $y = 1$.

Finally, to consider the top corner region we introduce

$$x_* = \varepsilon^{-1/2}X \tag{14}$$

so as to make $\varepsilon\partial^2/\partial X^2 = \partial^2/\partial x_*^2$ comparable to $\partial^2/\partial Y^2 + \partial/\partial Y$. Setting

$$w \sim \sum_{k=0}^{\infty} \varepsilon^{k/2} w_k^{\mathrm{III}*}(x_*, Y), \tag{15}$$

the recurrence relation for the coefficient functions is

$$(\partial^2/\partial x_*^2 + \partial^2/\partial Y^2 + \partial/\partial Y)w_k^{\mathrm{III}*} = 0, \tag{16a}$$

and the boundary conditions are

$$w_k^{\mathrm{III}*}(x_*, 0) = 0, \qquad w_k^{\mathrm{III}*}(0, Y) = \begin{cases} h^{(k)}(1)(-Y)^k/k! & \text{for } k \text{ even,} \\ 0 & \text{for } k \text{ odd.} \end{cases} \tag{16b}$$

Such Dirichlet problems have unique solutions under the additional requirement of exponential decay as $(x_*^2 + Y^2)^{1/2} \to \infty$ with $x_* \neq 0$. Without such a requirement there are solutions with algebraic growth, but these are precisely what are needed to match the expansion in II. We may think of the solution in II running through the region III_*, which reacts to the violation of its boundary conditions with a correction that dies out exponentially away from $Y = 0$. Note that half powers of ε are induced in the III_*-expansion, since the II-expansion (5) will involve powers of $\varepsilon^{1/2}$ after the substitution of (11) and (14). Such terms did not appear in II_*.

These then are the results obtained by the method of matched asymptotic expansions. We shall now show them to be valid approximations to order m, where m depends on the differentiability of h. However, to begin with we assume that

$$h \text{ is infinitely differentiable on } (-1, 1), \tag{17a}$$

$$h^{(k)}(-1 + 0) \quad \text{and} \quad h^{(k)}(+1 - 0) \quad \text{exist for all } k, \tag{17b}$$

for which the method as given above can be carried on indefinitely. The regions of validity for the expansions are (see Fig. 2)

I: $x_0 \leqq x, \quad -1 \leqq y \leqq 1$;

II: $0 \leqq X, \quad -1 \leqq y \leqq y_1 < 1$ excluding $0 \leqq X \leqq X_0$, $0 \leqq y + 1 \leqq y_{-1}$;

II_*: $0 \leqq X_*, \quad 0 \leqq y_* \leqq y_{*\infty}$;

III: $X_1 \leqq X, \quad 0 \leqq Y \leqq Y_\infty$;

III_*: $0 \leqq x_* \leqq x_{*\infty}, \quad 0 \leqq Y \leqq Y_\infty$.

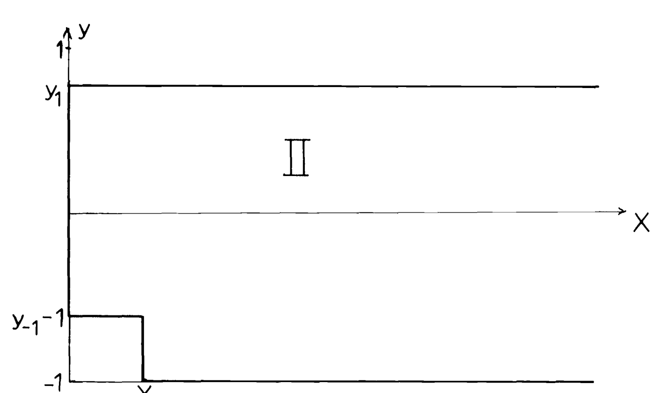

FIG. 2a

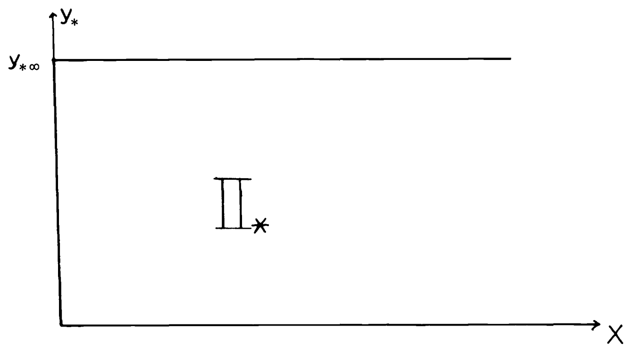

FIG. 2b

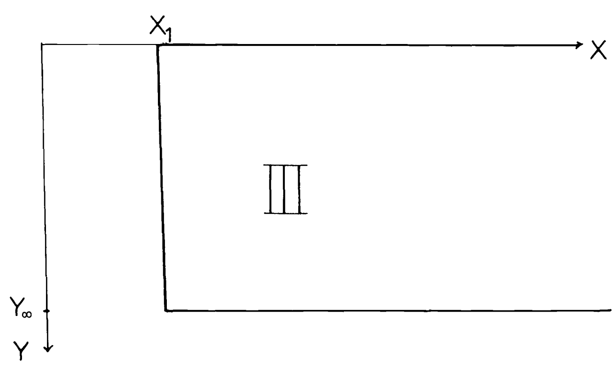

FIG. 2c

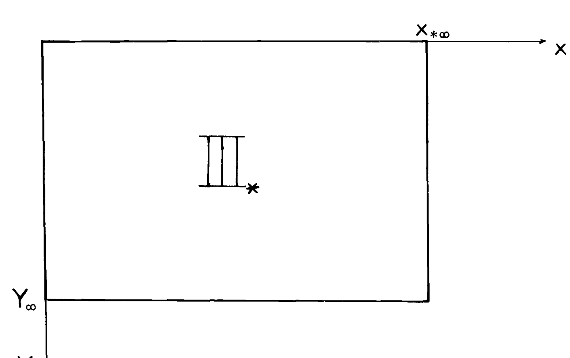

FIG. 2d

Here x_0, y_1, X_0, y_{-1}, $y_{*\infty}$, X_1, Y_∞, $x_{*\infty}$ are first assumed to be fixed positive numbers, but it is later shown that the regions of validity can be extended to

$$x_0 = O(\varepsilon^{1/2-\delta}),\quad y_1 = O(\varepsilon^{1-\delta}),\quad X_0 = O(\varepsilon^{1/4-\delta}),\quad y_{-1} = O(\varepsilon^{1/2-\delta}),$$
$$y_{*\infty} = O(\varepsilon^{-1+\delta}),\quad Y_\infty = O(\varepsilon^{-1+\delta}),\quad X_1 = O(\varepsilon^{1/2-\delta}),\quad x_{*\infty} = O(\varepsilon^{-1/2+\delta}).$$

Here $\delta > 0$ is arbitrarily small.

The regions II_* and III_* at the bottom and top corners both arise from discontinuities in the boundary data, but otherwise they are quite different: the expansion in II is determined by that in II_* whereas the opposite is true of II and III_*. We may say that the parabolic layer on $x = 0$ is completely determined by its singular origin $x = 0$, $y = -1$, and in turn determines the top singularity $x = 0, y = 1$. That the structures of II_* and III_* are the same as in [2] can be seen from dividing the solution into three parts. The first is an infinitely smooth solution satisfying the data on $x = 0$, but not on $y = \pm 1$ where it is in general nonzero. The second part nullifies the first on $y = -1$ and is zero elsewhere on the boundary, while the third does the same for $y = +1$. The first part does not require the regions II_* and III_* (nor for that matter III). As was shown in the Introduction, the other two are covered by our previous paper [2], albeit extended to data depending on ε. They therefore involve regions II_* and III_*, respectively, of the type found there.

We have been unable to exhibit these three parts explicitly, but at least the asymptotic existence of the third is clear from our analysis. As indicated above, the III_*-expansion has two components: the II-expansion written in the x_*, Y-variables, corresponding to the sum of the first two parts; and a correction for the boundary conditions on $y = 1$, corresponding to the third part.

3. The exact solution. Taking the Fourier sine transform

$$\overline{(\cdot)} = \int_0^\infty (\cdot)\sin \xi x\, dx$$

of the differential equation (1a), we find

$$(-\varepsilon\xi^2 + \varepsilon\partial^2/\partial y^2 - \partial/\partial y)\bar{w} = -\varepsilon\xi h(y),$$

where the boundary condition (1d) has been incorporated. The boundary conditions (1c) then give

$$\bar{w}(\xi, y) = -\varepsilon\xi \int_{-1}^{+1} \bar{\mathscr{G}}(y, y'; \xi, \varepsilon)h(y')\, dy',$$

where the Green's function

$$\bar{\mathscr{G}} = 2\exp[(y-y')/2\varepsilon]/\{(1+4\varepsilon^2\xi^2)^{1/2}\sinh[(1+4\varepsilon^2\xi^2)^{1/2}/2\varepsilon]\}$$

$$\cdot\begin{cases}\sinh[(1+4\varepsilon^2\xi^2)^{1/2}(y+1)/2\varepsilon]\sinh[(1+4\varepsilon^2\xi^2)^{1/2}(y'-1)/2\varepsilon] & \text{for } y < y',\\ \sinh[(1+4\varepsilon^2\xi^2)^{1/2}(y'+1)/2\varepsilon]\sinh[(1+4\varepsilon^2\xi^2)^{1/2}(y-1)/2\varepsilon] & \text{for } y' < y\end{cases} \tag{18}$$

is actually the transform of that for the original problem. The exact solution of

(1a) under the boundary conditions (1c), (1d) is then

$$w(x, y) = \frac{2}{\pi}\int_0^\infty \bar{w}(\xi, y)\sin \xi x\, dx.$$

We are not concerned with a.e.s. contributions to the solution, so that terms which are uniformly a.e.s. for ξ on the real axis may be neglected. Thus, since $\exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}/\varepsilon]$ is uniformly a.e.s., we may write

$$\begin{aligned} \bar{\mathscr{G}} \sim (1 + 4\varepsilon^2\xi^2)^{-1/2} \exp[(y - y')/2\varepsilon]\{&\exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' + 2)/2\varepsilon] \\ &- \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}|y - y'|/2\varepsilon] + \exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' - 2)/2\varepsilon] \\ &- \exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(|y - y'| - 4)/2\varepsilon]\}, \end{aligned} \tag{18'a}$$

as will be needed in regions III and III_*. But $\bar{\mathscr{G}}$ may be further simplified for y away from 1, so that

$$\begin{aligned} \bar{\mathscr{G}} \sim (1 + 4\varepsilon^2\xi^2)^{-1/2} \exp[(y - y')/2\varepsilon]\{&\exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' + 2)/2\varepsilon] \\ &- \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}|y - y'|/2\varepsilon]\} \end{aligned} \tag{18'b}$$

will be used in regions II and II_*.

We could also write the exact solution in terms of the fundamental solution of equation (1a) and its images in $y = \pm 1$ together with their images in $x = 0$. The same result is obtained by expanding the denominator of $\bar{\mathscr{G}}$ in (18), to obtain the terms

$$\begin{aligned} (1 + 4\varepsilon^2\xi^2)^{-1/2} \exp[(y - y')/2\varepsilon]\{&\exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' - 2 - 4r)/2\varepsilon] \\ &- \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(|y - y'| + 4r)/2\varepsilon] \\ &+ \exp[-(1 + 4\varepsilon^2\xi^2)^{1/2}(y + y' + 2 - 4r)/2\varepsilon] \\ &- \exp[(1 + 4\varepsilon^2\xi^2)^{1/2}(|y - y'| - 4 - 4r)/2\varepsilon]\} \end{aligned}$$

with $r = 0, 1, 2, \cdots$. The inverse sine transforms of these are the Bessel functions obtained by the imaging process above. To uniformly a.e.s. terms then the solution could also be written

$$\begin{aligned} -\frac{1}{\pi}\int_{-1}^{1} & h(y') \exp[(y - y')/2\varepsilon](\partial/\partial x) \\ &\cdot\{K_0[(x^2 + (y - y')^2)^{1/2}/2\varepsilon] - K_0[(x^2 + (y + y' + 2)^2)^{1/2}/2\varepsilon] \\ &- K_0[(x^2 + (y + y' - 2)^2)^{1/2}/2\varepsilon] + K_0[(x^2 + (y' - y + 4)^2/2\varepsilon]\}\, dy'. \end{aligned}$$

4. The core region I. Consider this last representation of the exact solution in terms of Bessel functions. In the core region we have

$$w(x, y) = 0$$

to a.e.s. terms since $x \geqq x_0 > 0$.

Extension of the core region inward is limited by the behavior of

$$\exp\{[y - y' - (x^2 + (y - y')^2)^{1/2}]/2\varepsilon\}$$

which arises from the first Bessel function when its argument is large. With

$$x_0 = \varepsilon^\mu$$

the argument in the exponential is negative and at least $O(\varepsilon^{2\mu-1})$ for $|y - y'| \leqq 2$ and $x \geqq x_0$. Hence $w(x, y)$ remains a.e.s. when

$$\mu < 1/2. \tag{19}$$

The core-region expansion is valid for $-1 \leqq y \leqq 1$, i.e., even into the boundary layer region for x restricted as above. This is to be expected since the zero expansion does in fact agree with the given boundary condition at $y = 1$. In other words, data at $x = 0$ has no asymptotic influence away from $x = 0$.

5. The side layer II. As suggested by the limitation (19) we introduce the stretched variable (4) in order to describe the solution near $x = 0$. In terms of the appropriate transform variable $\eta = \varepsilon^{1/2}\xi$,

$$w(X, y; \varepsilon) = \frac{2}{\pi}\int_0^\infty \sin \eta X \tilde{w}(\eta, y; \varepsilon)\, \partial\eta, \tag{20}$$

where

$$\begin{aligned}\tilde{w} \sim \eta(1 + 4\varepsilon\eta^2)^{-1/2}\int_{-1}^{1} & h(y')\exp[(y - y')/2\varepsilon] \\ & \cdot\{\exp[-(1 + 4\varepsilon\eta^2)^{1/2}(y + y' + 2)/2\varepsilon] \\ & \quad - \exp[-(1 + 4\varepsilon\eta^2)^{1/2}|y - y'|/2\varepsilon]\}\, dy'.\end{aligned} \tag{20'}$$

On expanding in a Taylor series in ε we obtain

$$\tilde{w}(\eta, y; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^k \tilde{w}_k^{\mathrm{II}}(\eta, y) + \varepsilon^m \tilde{R}_m(\eta, y; \varepsilon); \tag{21}$$

it is not necessary to write down $\tilde{w}_k^{\mathrm{II}}$ and $\tilde{R}_m$ explicitly. We now show that under inversion: (i) the coefficient functions $w_k^{\mathrm{II}}(X, y)$ satisfy the recurrence relation (6a) together with the boundary conditions (6b); and (ii) $R_m(X, y; \varepsilon)$ is bounded independently of ε in region II. Proof of matching with the expansion in II_* will however be postponed until the next section. From now on, we shall also use $\tilde{w}(\eta, y; \varepsilon)$ to denote its asymptotic approximation (20′).

(i) It can be checked that $(-\eta^2 + \varepsilon\partial^2/\partial y^2 - \partial/\partial y)\tilde{w}(\eta, y; \varepsilon) = -\eta h(y)$. Substituting the expansion (21) and equating coefficients of corresponding powers of ε yields

$$(\eta^2 + \partial/\partial y)\tilde{w}_k^{\mathrm{II}} = \begin{cases} \eta h(y) & \text{for } k = 0, \\ \partial^2/\partial y^2 \tilde{w}_{k-1}^{\mathrm{II}} & \text{for } k > 0. \end{cases}$$

Since $\tilde{w}(\eta, -1) = 0$, we also have

$$\tilde{w}_k^{\mathrm{II}}(\eta, -1) = 0 \quad \text{for all } k.$$

Hence, provided the $\tilde{w}_k^{\mathrm{II}}(\eta, y)$ are invertible, a fact that will be proved when $\tilde{R}_m$ is discussed, the inverses $w_k^{\mathrm{II}}(X, y)$ do satisfy the recurrence relation and boundary conditions as desired.

(ii) Except for a constant factor, $\tilde{R}_m(\eta, y; \varepsilon)$ is the mth derivative of $\tilde{w}(\eta, y; \varepsilon)$ with respect to ε, evaluated at εt, $0 < t < 1$. It is not immediately clear that the inverse of such a derivative exists from the form (20′), since expansion of $(1 + 4\varepsilon\eta^2)^{-1/2}$ alone generates powers of η^2 multiplying terms apparently bounded as $\eta \to \infty$. In order to see that the inverse does exist we integrate (20′) by parts m times. Noticing that terms of the form $(2\varepsilon)^k[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-k} \exp\{[(y + 1) \cdot (1 - (1 + 4\varepsilon\eta^2)^{1/2}) - 2(1 + (1 + 4\varepsilon\eta^2)^{1/2})]/2\varepsilon\}$ can be ignored since they are uniformly a.e.s. and those of the form

$$(2\varepsilon)^k(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k} \exp[(y - 1)(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon] \tag{22}$$

can be ignored since they are uniformly a.e.s. for y away from 1, we obtain

$$\begin{aligned}
\tilde{w} \sim \eta(1 + 4\varepsilon\eta^2)^{-1/2} & \\
\cdot\Bigg[\sum_{k=0}^{m-1} & \{(2\varepsilon)^{k+1}(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k-1} - (-1 - (1 + 4\varepsilon\eta^2)^{1/2})^{k+1}(2\eta^2)^{-k-1}\} \\
& \cdot \{h^{(k)}(-1) \exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})] - h^{(k)}(y)\} \\
- (-1 - (1 + 4\varepsilon\eta^2)^{1/2})^m & (2\eta^2)^{-m} \int_{-1}^{y} h^{(m)}(y') \\
& \cdot \exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\, dy' - (2\varepsilon)^m(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-m} \\
& \cdot \Bigg\{\int_{y}^{1} h^{(m)}(y') \exp[(y - y')(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon]\, dy' - \int_{-1}^{1} h^{(m)}(y') \\
& \cdot \exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2}) - (y' + 1)(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon]\, dy' \Bigg\}\Bigg].
\end{aligned}$$

Note immediately that the last two terms are invertible and $O(\varepsilon^m)$ in the X, y-plane, and hence can be dropped. This can be seen by integrating by parts once more to obtain

$$\eta(1 + 4\varepsilon\eta^2)^{-1/2}(2\varepsilon)^{m+1}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-m-1}$$

times terms which are bounded independently of η, y and ε. The result then follows on setting $\eta_* = \varepsilon^{1/2}\eta$ in the inversion integral.

Thus we need only consider the expansion of the remaining terms which, since $2\varepsilon[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-1} = [-1 + (1 + 4\varepsilon\eta^2)^{1/2}](2\eta^2)^{-1}$ can be written as

$$\begin{aligned}
\eta^{-1} \sum_{k=0}^{m-1} (-2\eta^2)^{-k} \sum_{t=0}^{[k/2]} \binom{k + 1}{2t + 1} & (1 + 4\varepsilon\eta^2)^t \\
& \cdot \{h^{(k)}(-1) \exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})] - h^{(k)}(y)\} \\
- \eta(1 + 4\varepsilon\eta^2)^{-1/2}[-1 - (1 + 4\varepsilon\eta^2)^{1/2}]^m & (2\eta^2)^{-m} \int_{-1}^{y} h^{(m)}(y') \\
& \cdot \exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\, dy'.
\end{aligned} \tag{23}$$

Differentiating m times with respect to ε gives terms of the form

$$\begin{aligned}
& \eta^{-1}(\eta^2)^{-j+m+l}(1 + 4\varepsilon\eta^2)^{t - m/2 - \alpha_1/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-2l-\alpha_2}(y + 1)^l h^{(j)}(-1) \\
& \quad \cdot \exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]
\end{aligned} \tag{24a}$$

and

$$\eta(\eta^2)^l(1 + 4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{\alpha_1 - l}\int_{-1}^{y} h^{(m)}(y')(y - y')^l \cdot \exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\, dy',$$

where

$$0 \leqq l + \alpha_1 + \alpha_2 \leqq m, \qquad 0 \leqq l, \alpha_1, \alpha_2 \leqq m, \qquad 0 \leqq j \leqq m - 1.$$

Integrating by parts $l + 1$ times on the last term in order to absorb the powers η^2 we can replace it with terms

(24b) $$\eta(\eta^2)^{l-\beta_1}(1 + 4\varepsilon\eta^2)^{-(1+m+\alpha_1)/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{-l+\alpha_1+\beta_1} \cdot (y + 1)^{l-\beta_1+\beta_2+1}h^{(m+\beta_2)}(-1)\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})],$$

(24c) $$\eta^{-1}(1 + 4\varepsilon\eta^2)^{-(1+m+\alpha_1)/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{1+\alpha_1} \cdot \{h^{(m)}(y) - h^{(m)}(-1)\exp[-2\eta^2(y + 1)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\},$$

(24d) $$\eta^{-1}(1 + 4\varepsilon\eta^2)^{-(1+m+\alpha_1)/2}[1 + (1 + 4\varepsilon\eta^2)^{1/2}]^{1+\alpha_1}\int_{-1}^{y} h^{(m+\beta_3+1)}(y')(y - y')^{\beta_3} \cdot \exp[-2\eta^2(y - y')/(1 + (1 + 4\varepsilon\eta^2)^{1/2})]\, dy',$$

where $0 \leqq \beta_2 \leqq \beta_1 - 1$, $0 \leqq \beta_1 \leqq l + 1$, $0 \leqq \beta_3 \leqq l$. The bounding of R_m is now reduced to the bounding of the inverses of (24).

The integrand in the inversion integral of (24d) can be rounded by $c/(1 + |\eta|^3)$, where c is independent of η, y and ε. (The bound for η large is obtained by integrating by parts once more and bounding the resulting terms.) Thus, (24d) is invertible and the result is bounded in ε. The terms in (24c) are invertible as they stand and the inverses are bounded as $X \to 0$. (The limit must be used to define the inverse functions at $X = 0$ since sine inversions automatically give zero there —this point is discussed further when we come to the coefficient functions.)

In dealing with (24a), (24b) we note that the same difficulty occurs as in the case of the infinite strip [2], leading to the exclusion of a region near $X = 0$, $y = -1$. The inversion integrals are convergent and $O(1)$ in ε for y away from -1, since we then obtain help from the exponentials. Although each integral is divergent for $y = -1$, we can obtain convergence for X bounded away from zero as follows. Rewrite $\sin \eta X$ as $(e^{i\eta X} - e^{-i\eta X})/2i$ and, in the resulting two integrals, bend the ends of the integration line upwards into the complex η-plane in the first and downwards in the second. They are then convergent for all y.

Thus we see that $R_m(X, y; \varepsilon)$ is $O(1)$ in ε in region II.

The terms $w_k^{\mathrm{II}}(X, y)$ may be treated similarly. Using the expansion (23) with $m = k + 1$ and noting that $\tilde{w}_k^{\mathrm{II}}(\eta, y)$ is the kth derivative with respect to ε evaluated at $\varepsilon = 0$, we find that $\tilde{w}_k^{\mathrm{II}}(\eta, y)$ is composed of terms

(25a) $$\eta^{-1}(\eta^2)^{-j+k+l}(y + 1)^l \exp[-\eta^2(y + 1)]h^{(j)}(-1),$$

(25b) $$\eta^{-1}(\eta^2)^{l-\beta_1}(y + 1)^{l-\beta_1+\beta_2+1}\exp[-\eta^2(y + 1)]h^{(k+1+\beta_2)}(-1),$$

(25c) $$\eta^{-1}\int_{-1}^{y}(y - y')^{\beta_3}\exp[-\eta^2(y - y')]h^{(k+\beta_3+1)}(y')\, dy',$$

where $0 \leqq l \leqq k$, $0 \leqq \beta_1 \leqq l$, $0 \leqq \beta_2 \leqq \beta_1 - 1$, $0 \leqq \beta_3 \leqq l$, and for $k = 0$, the additional term

$$\eta^{-1}h(y). \tag{25d}$$

The inversion of these terms follows easily (when we deform the contour again for the first two) except for $\eta^{-1}h(y)$. Then the inversion integral is not uniformly convergent, so we must first take $X \neq 0$ and let $X \to 0$. Thus the sine transform forces a zero value at $X = 0$, whereas we want the limiting value as $X \to 0$, which is in general not zero.

Extension of region II upwards is limited to $y_1 = \varepsilon^{1-\delta}$ because we have omitted terms of the form (22) from the expansion and ignored the last two terms in (18′a) as being uniformly a.e.s. Extension outwards is unlimited since no such limitation was required for the above bounding.

Extension of region II into the corner is limited by the inversion of (24a), (24b) which for points near $X_0 = \varepsilon^{\kappa}$, $y_{-1} = \varepsilon^{\lambda}$ with $\kappa, \lambda > 0$ involves the exponential of $-2\eta^2\varepsilon^{\lambda}/[1 + (1 + 4\varepsilon\eta^2)^{1/2}] \pm i\eta\varepsilon^{\kappa}$ in the integrands. After the deformation of the integration line both terms have negative real parts, one of which may be prevented from vanishing in the limit $\varepsilon \to 0$ by the transformation $\eta = \varepsilon^{-\kappa}\tau$ when $\lambda \geqq 2\kappa$ or $\eta = \varepsilon^{-\lambda/2}\tau$ when $\lambda \leqq 2\kappa$. The terms (24a), (24b) are then of order $\varepsilon^{-4\kappa m}$ or $\varepsilon^{-2\lambda m}$ at worst. Thus λ can be arbitrarily large so long as

$$\kappa < 1/4, \tag{26a}$$

and κ can be arbitrarily large so long as

$$\lambda < 1/2. \tag{26b}$$

6. The Singular Region II$_*$. The limitations (26) are misleading: we must in fact introduce the stretched variables (8) in order to describe the solution near $X = 0$, $y = -1$. Using a hat to denote the appropriate Fourier transform (with variable $\eta_* = \varepsilon^{1/2}\eta$) we obtain

$$\begin{aligned} \hat{w} \sim \eta_*(1 + 4\eta_*^2)^{-1/2} \int_0^{2/\varepsilon} & h(\varepsilon y'_* - 1) \\ & \cdot \{\exp[(1 - (1 + 4\eta_*^2)^{1/2})y_*/2 - (1 + (1 + 4\varepsilon\eta^2)^{1/2})y'_*/2] \\ & - \exp[(y_* - y'_*)/2 - |y_* - y'_*|(1 + 4\varepsilon\eta_*^2)^{1/2}/2]\}\, dy'_*. \end{aligned} \tag{27}$$

Expanding h in a Taylor series in ε we find

$$\hat{w}(\eta_*, y_*; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^k \hat{w}_k^{\mathrm{II}*}(\eta_*, y_*) + \varepsilon^m \hat{R}_m^*(\eta_*, y_*; \varepsilon), \tag{28}$$

where

$$\begin{aligned} \hat{w}_k^{\mathrm{II}*} = (\eta_*(1 + 4\eta_*^2)^{-1/2}h^{(k)}(-1)/k!) \int_0^{\infty} & y'^k_* \\ & \cdot \{\exp[(1 - (1 + 4\eta_*^2)^{1/2})y_*/2 - (1 + (1 + 4\eta_*^2)^{1/2})y'_*/2] \\ & - \exp[(y_* - y'_*)/2 - |y_* - y'_*|(1 + 4\eta_*^2)^{1/2}/2]\}\, dy'_*, \end{aligned} \tag{28'a}$$

$$\hat{R}_m^* = (\eta_*(1 + 4\eta_*^2)^{-1/2}/m!)\int_0^{2/\varepsilon} y_*'^m h^{(m)}(\varepsilon t y_*' - 1)$$

$$\tag{28'b} \cdot\{\exp[(1 - (1 + 4\eta_*^2)^{1/2})y_*/2 - (1 + (1 + 4\eta_*^2)^{1/2})y_*'/2]$$

$$- \exp[(y_* - y_*')/2 - |y_* - y_*'|(1 + 4\eta_*^2)^{1/2}/2]\}\, dy_*'.$$

The integrals in $\hat{w}_k^{\mathrm{II}*}$ have been extended to infinity, the added pieces being uniformly a.e.s. for y_* bounded. This renders the coefficient functions independent of ε.

It will now be shown that: (i) the $w_k^{\mathrm{II}*}$ do satisfy the proper recurrence relation and boundary conditions for region II_*; and (ii) R_m^* is bounded independently of ε there. We shall in addition complete the treatment of region II by showing that its expansion matches the one here.

(i) Direct substitution shows that

$$(\eta_*^2 + \partial/\partial y_* - \partial^2/\partial y_*^2)\hat{w} = \eta_* h(\varepsilon y_* - 1),$$

where $\hat{w}$ denotes its own asymptotic expansion (27). Expanding about $\varepsilon = 0$ for $h(\varepsilon y_* - 1)$ and equating coefficients of corresponding powers of ε yields the relations

$$(\eta_*^2 + \partial/\partial y_* - \partial^2/\partial y_*^2)\hat{w}_k^{\mathrm{II}*} = \eta_* h^{(k)}(-1)y_*^k/k!.$$

Also it is clear that $\hat{w}(\eta_*, 0; \varepsilon) = 0$. So in the original plane we have

$$(\partial^2/\partial X_*^2 + \partial^2/\partial y_*^2 - \partial/\partial y_*)w_k^{\mathrm{II}*} = 0,$$

$$w_k^{\mathrm{II}*}(X_*, 0) = 0, \qquad w_k^{\mathrm{II}*}(0, y_*) = h^{(k)}(-1)y_*^k/k!,$$

as desired, provided $\hat{w}_k^{\mathrm{II}*}$ is invertible. The latter is covered by our treatment of R_m^* below.

To investigate the matching of the II- and II_*-expansions we note that for $y \neq -1$ the integral (20') can be written in the form

$$I(\varepsilon/(y + 1), X/(y + 1)^{1/2}, h(-1 + \varepsilon y_*'))$$

$$= \sum_{k=0}^{m-1} (\varepsilon/(y + 1))^k I_k(X/(y + 1)^{1/2}, h(-1 + \varepsilon y_*')) + O(\varepsilon^m),$$

where I_k is a linear operator on functions of y_*' which depends only on $X/(y + 1)^{1/2}$ and the order symbol refers to fixed X, y. This result can be obtained by setting $\eta = \tau/(y + 1)^{1/2}$, $y' = -1 + \varepsilon y_*'$, letting the y_*'-integration range to infinity instead of $2/\varepsilon$ (thereby introducing a.e.s. error), and expanding on $\varepsilon/(y + 1)$ as in § 5 (i.e., by integration by parts). Beyond $2/\varepsilon$ the function $h(-1 + \varepsilon y_*')$ is defined as the polynomial $\sum_{p=1}^{N} h^{(p)}(1)(y_*' - 2/\varepsilon)^p/p!$, where N is sufficiently large to ensure whatever continuity of derivatives at $y_*' = 2/\varepsilon$ is required in the following. It is easily checked that, in operating on functions $O(1)$ in ε, I_k produces $O(1)$ functions. Consequently Taylor-series remainders can be ignored in writing the m-term II-expansion

$$\sum_{k=0}^{m-1} (\varepsilon/(y + 1))^k I_k(X/(y + 1)^{1/2}, \sum_{j=0}^{m-1-k} \varepsilon^j h^{(j)}(-1)y_*'^j/j!)$$

so that its n-term II_*-expansion is

$$(29)\qquad \sum_{k=0}^{m-1} y_*^{-k} \sum_{j=0}^{\min(n-1,m-k-1)} \varepsilon^j h^{(j)}(-1) I_k(X_*/y_*^{1/2}, y_*'^j)/j!.$$

On the other hand, when η_* is replaced by $\tau/y_*^{1/2}$ the n-term II_*-expansion given by (28) is seen to be

$$I\Big(1/y_*, X_*/y_*^{1/2}, \sum_{j=0}^{n-1} \varepsilon^j h^{(j)}(-1) y_*'^j/j!\Big)$$

and its m-term II-expansion is

$$(30)\qquad \sum_{k=0}^{m-1} (\varepsilon/(y+1))^k \sum_{j=0}^{\min(n-1,m-k-1)} \varepsilon^j h^{(j)}(-1) I_k(X/(y+1)^{1/2}, y_*'^j)/j!.$$

Clearly the expressions (29) and (30) are identical under the transformation $X = \varepsilon^{1/2} X_*$, $y + 1 = \varepsilon y_*$ so that matching is established.

(ii) Integrating by parts the terms in $\hat{R}_m^*$ which have $y_*'(1 + (1 + 4\eta_*^2)^{1/2})$ in the exponential, and noticing that $(2/\varepsilon)^j \exp[-(1 + (1 + 4\eta_*^2)^{1/2})/\varepsilon]$ is uniformly a.e.s., we obtain

$$\begin{aligned} m!\hat{R}_m^* \sim\ & -2\eta_*(1 + 4\eta_*^2)^{-1/2}(1 + (1 + 4\eta_*^2)^{1/2})^{-1} \\ &\cdot \Big\{ y_*^m h^{(m)}(\varepsilon t y_* - 1) + \int_0^{2/\varepsilon} [m y_*'^{m-1} h^{(m)}(\varepsilon t y_*' - 1) + y_*'^m \varepsilon t h^{(m+1)}(\varepsilon t y_*' - 1)] \\ &\qquad \cdot \exp[y_*(1 - (1 + 4\eta_*^2)^{1/2})/2 - y_*'(1 + (1 + 4\eta_*^2)^{1/2})/2]\, dy_*' \\ &\qquad + \int_{y_*}^{2/\varepsilon} [m y_*'^{m-1} h^{(m)}(\varepsilon t y_*' - 1) + y_*'^m \varepsilon t h^{(m+1)}(\varepsilon t y_*' - 1)] \\ &\qquad \cdot \exp[(y_* - y_*')(1 + (1 + 4\eta_*^2)^{1/2})/2]\, dy_*' \Big\} \\ & + \eta_*(1 + 4\eta_*^2)^{-1/2} \int_0^{y_*} y_*'^m h^{(m)}(\varepsilon t y_*' - 1) \\ &\qquad \cdot \exp[(y_* - y_*')(1 - (1 + 4\eta_*^2)^{1/2})/2]\, dy_*'. \end{aligned}$$

All but the first and last terms are clearly invertible and their inverses are $O(1)$ in ε since, after integrating by parts once more, they behave like η_*^{-2} for η_* large, independently of ε. The first term can be rewritten as $h^{(m)}(\varepsilon t y_* - 1) y_*^n$ times $(1 - (1 + 4\eta_*^2)^{1/2})/(2\eta_*(1 + 4\eta_*^2)^{1/2}) = (2\eta_*(1 + 4\eta_*^2)^{1/2})^{-1} - (2\eta_*)^{-1}$ and hence is invertible with inverse $O(1)$. Note that the value for $X_* = 0$ must again be interpreted as the limit for $X_* \to 0$. The last term inverts to

$$\begin{aligned} X_* \int_0^{y_*} & y_*'^m h^{(m)}(\varepsilon t y_*' - 1) \exp[(y_* - y_*')/2] K_1[(X_*^2 + (y_* - y_*')^2)^{1/2}/2] \\ & \cdot (X_*^2 + (y_* - y_*')^2)^{-1/2}\, dy_*', \end{aligned}$$

and thus is $O(1)$ in ε.

The invertibility of $\hat{w}_k^{\mathrm{II}*}$ is also covered by the above analysis, since the same terms, with m replaced by k and ε set zero, are involved.

Extension of region II_* outward is limited only by

$$y_{*\infty} = \varepsilon^{-1+\delta},$$

which ensures that the original asymptotic expression (27) is still valid and that the inverse of (28′b) is small compared to ε^{-m}. Nowhere was it necessary to bound X_*.

7. The boundary layer III. The expansion found in region II is not asymptotic to w near $y = 1$ since it does not include boundary layer terms. Such terms arise from the parts of the exact solution which were omitted for being uniformly a.e.s. away from $y = 1$ (see (18′) and (22)). Written in the stretched variable $Y = (1 - y)/\varepsilon$, these parts are given by the inverse of

$$\begin{aligned}\tilde{\tau} = {} & \eta(1 + 4\varepsilon\eta^2)^{-1/2} \exp[-Y(1 + 4\varepsilon\eta^2)^{1/2}/2] \\ & \cdot\left\{\int_{-1}^{1} h(y') \exp\,[(1 - y')(1 - (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon]\,dy'\right. \\ & - \int_{-1}^{1} h(y') \exp[\{(1 - y')(1 + (1 + 4\varepsilon\eta^2)^{1/2} - 4(1 + 4\varepsilon\eta^2)^{1/2}\}/2\varepsilon]\,dy' \\ & \left. + \sum_{k=0}^{m} [2\varepsilon(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k-1} h^{(k)}(1)\right\}.\end{aligned} \tag{31}$$

Thus to obtain the expansion in the boundary layer we must consider the contribution from the boundary layer correction τ as well as that from the asymptotic form used previously, now applied in the boundary layer.

Expansion of these two in a Taylor series in ε gives

$$\tilde{w}(\eta, Y; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^k \tilde{w}_k^{\mathrm{III}}(\eta, Y) + \varepsilon^m \tilde{\mathscr{R}}_m(\eta, Y; \varepsilon),$$

where it is not necessary to write out $\tilde{w}_k^{\mathrm{III}}$ and $\tilde{\mathscr{R}}_m$ explicitly. The task is now to demonstrate that (i) the $w_k^{\mathrm{III}}(X, Y)$ satisfy the recurrence relation (13a) together with the boundary conditions and matching mentioned there; and (ii) $\mathscr{R}_m(X, Y; \varepsilon)$ is $O(1)$ in ε in region III.

(i) By direct substitution we obtain

$$(\partial^2/\partial Y^2 + \partial/\partial Y - \varepsilon\eta^2)\tilde{w} = -\eta h(1 - \varepsilon Y), \tag{32}$$

where again $\tilde{w}$ stands for its asymptotic form; furthermore $\tilde{w}(\eta, 0; \varepsilon) = 0$. It follows that

$$(\partial^2/\partial Y^2 + \partial/\partial Y)\tilde{w}_k^{\mathrm{III}} = -\eta^2 \tilde{w}_{k-1}^{\mathrm{III}} - (-Y)^k \eta h^{(k-1)}(1)/k!,$$

$$\tilde{w}_k^{\mathrm{III}}(\eta, 0; \varepsilon) = 0,$$

so that the transforms of the recurrence relation and boundary conditions are satisfied. If the $\tilde{w}_k^{\mathrm{III}}$ are invertible, as will be proved along with $\tilde{\mathscr{R}}_m$, only the matching remains.

Notice that $\tilde{\tau}(\eta, Y; \varepsilon)$ is a solution of the homogeneous differential equation (32) with $\tilde{\tau}(\eta, 0; \varepsilon)$ the negative value of (20′) at $y = 1$; it is therefore purely the correction for the boundary condition at $y = 1$. Since $\tilde{\tau}(\eta, (1 - y)/\varepsilon; \varepsilon)$ is a.e.s. for fixed $y \neq 1$, we need only prove that $\tilde{w} - \tilde{\tau}$ matches with the $\tilde{w}$ of region II, that is, we need only show that the $\tilde{w}$ used in region II satisfies the matching principle in the variables y, Y. But this follows directly from Fraenkel's Theorem 1 [3]: the $\tilde{\mathscr{R}}_m(\eta, y)$ in (21) remains $O(1)$ for all y in $-1 < y \leqq 1$ and the $\tilde{w}_k^{\mathrm{II}}(\eta, y)$, since they have the forms (25), clearly satisfy assumption 2 of Fraenkel's theorem. The expansion in region III is now seen to match that in region II since inversion preserves matching.

(ii) To consider the existence of the coefficient functions $w_k^{\mathrm{III}}(X, Y)$ and the bounding of $\mathscr{R}_m(X, Y; \varepsilon)$, recall that the corresponding expansion is the superposition of that for $\tau(X, Y; \varepsilon)$ and the expansion in region II rewritten in terms of the boundary layer variable Y.

In treating the terms from region II, i.e. omitting the boundary layer terms, the estimate of the remainder remains valid for y near 1. In other words the expansion (21) holds uniformly up to the top boundary. Thus for its contribution to the expansion in region III we need only substitute the boundary layer variable Y and expand to order ε^m. Since the $\tilde{w}_s^{\mathrm{II}}(\eta, y)$ have the forms (25), the remainder after Taylor series expansion in ε has terms of the form

(33a) $$\eta^{-1}(\eta^2)^{-j+s+l+p}Y^{m-s}(2 - \varepsilon t Y)^{l-m+p}\exp(-2\eta^2 + \varepsilon t\eta^2 Y)h^{(j)}(-1),$$

(33b) $$\eta(\eta^2)^{l-s+p}Y^{m-s}(2 - \varepsilon t Y)^{l-\beta_1+1+\beta_2-m+p}\exp(-2\eta^2 + \varepsilon t\eta^2 Y)h^{(s+1+\beta_2)}(-1),$$

(33c) $$\eta^{-1}Y^m h^{(m)}(1 - \varepsilon t Y),$$

$$\eta^{-1}\eta^{2l_1}Y^{m-s}(2 - \varepsilon Y)^{\beta_3-\gamma_2}\int_0^1 (1 - y')^{\beta_3+l_1}y'^{\gamma_1} \cdot \exp[-\eta^2(1 - y')(2 - \varepsilon t Y)]h^{(s+\beta_3+\gamma_1)}(y'(2 - \varepsilon t Y) - 1)\,dy',$$

where $0 \leqq p \leqq m - s$, $l_1 + \gamma_1 - 1 + \gamma_2 = m - s$ and the other parameters satisfy the previous conditions still. The last term was obtained by replacing y' by $(y + 1)y' - 1$ as the integration variable in (25c).

Inversion in the first two terms is valid and $O(1)$ in ε for Y bounded because of the exponential convergence. The third term is also invertible and its inverse, defined for $X = 0$ again by the limit, is $O(1)$ for Y bounded. In dealing with the last term we must integrate by parts l_1 times, to obtain terms of the form

(33d) $$\eta^{-1}(\eta^2)^{\alpha_1}(2 - \varepsilon Y)^{\beta_3-\gamma_2-l+\alpha_1}Y^{m-s}\exp[-\eta^2(2 - \varepsilon t Y)]h^{(s+\beta_3+\gamma_1+\alpha_2-1)}(-1),$$

(33e) $$\eta^{-1}(2 - \varepsilon Y)^{\beta_3-\gamma_2-l_1}Y^{m-s}\int_0^1 (1 - y')^{\beta_3+r_2+r_3}y'^{\gamma_1-r_2} \cdot \exp[-\eta^2(2 - \varepsilon t Y)(1 - y')]h^{(s+l_1+\gamma_1+r_3)}(y'(2 - \varepsilon t Y) - 1)\,dy',$$

where $\alpha_1 \leqq l_1$ and $0 \leqq r_2 + r_3 \leqq l_1$. Again the terms are invertible and $O(1)$ in ε for Y bounded. We conclude that the contribution to the remainder is uniformly $O(\varepsilon^m)$ in region III.

Treating now the contribution to the expansion from $\tau(\eta, Y; \varepsilon)$ we first integrate (31) by parts m times:

$$
\begin{aligned}
\tilde{\tau} = \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]\Bigg\{ & \sum_{k=0}^{m-1} \eta^{-1}(-2\eta^2)^{-k} \sum_{t=0}^{[k/2]} (1+4\varepsilon\eta^2)^t \binom{k+1}{2t+1} \\
& \cdot [h^{(k)}(1) - h^{(k)}(-1)\exp[-4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2})]] \\
& + \eta(1+4\varepsilon\eta^2)^{-1/2} \int_{-1}^{1} [(-1)^{m-1}(1+(1+4\varepsilon\eta^2)^{1/2})^m(2\eta^2)^{-m}h^{(m)}(y') \\
& \cdot \exp[-2\eta^2(1-y')/(1+(1+4\varepsilon\eta^2)^{1/2})] + (2\varepsilon)^m(1+(1+4\varepsilon\eta^2)^{1/2})^{-m}h^{(m)}(y') \\
& \cdot \exp[\{(1-y')(1+(1+4\varepsilon\eta^2)^{1/2}) - 4(1+4\varepsilon\eta^2)^{1/2}\}/2\varepsilon]]\, dy' \Bigg\}.
\end{aligned} \tag{34}
$$

On integrating by parts once more and letting $\eta_* = \varepsilon^{1/2}\eta$ in the inversion integral, it is clear that the last term is invertible and $O(\varepsilon^m)$. Consequently we need only consider the expansion of the first three groups of terms. By expansion in Taylor series, their contribution to $\tilde{\mathscr{R}}_m$ is their mth derivative with respect to ε evaluated at $t\varepsilon$, which produces terms of the form

$$\eta^{2(m-k)} \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]h^{(k)}(1)/[\eta(1+4\varepsilon\eta^2)^{m-t-\alpha_1/2}], \tag{35a}$$

$$
\begin{aligned}
&\eta^{2(m-k+l)} \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2 - 4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2})] \\
&\quad \cdot h^{(k)}(-1)/[\eta(1+4\varepsilon\eta^2)^{m-t-\alpha_1/2}(1+(1+4\varepsilon\eta^2)^{1/2})^{2l+\alpha_2}],
\end{aligned} \tag{35b}
$$

$$
\begin{aligned}
&\eta^{2(l-\beta_1)}\eta \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2 - 4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2}) \\
&\quad \cdot h^{(m+\beta_2)}(-1)/[(1+4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}(1+(1+4\varepsilon\eta^2)^{1/2})^{l-\alpha_1-\beta_1}],
\end{aligned} \tag{35c}
$$

$$
\begin{aligned}
&\{\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]h^{(m)}(1) \\
&\quad - \exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2 - 4\eta^2/(1+(1+4\varepsilon\eta^2)^{1/2})]h^{(m)}(-1)\} \\
&\quad \cdot (1+(1+4\varepsilon\eta^2)^{1/2})^{\alpha_1+1}/[\eta(1+4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}],
\end{aligned} \tag{35d}
$$

$$
\begin{aligned}
&\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2}/2] \int_{-1}^{1} h^{(m+\beta_3+1)}(y')(1-y')^{\beta_3} \\
&\quad \cdot \exp[-2\eta^2(1-y')/(1-(1+4\varepsilon\eta^2)^{1/2})]\, dy' \\
&\quad \cdot (1+(1+4\varepsilon\eta^2)^{1/2})^{\alpha_1+1}/[\eta(1+4\varepsilon\eta^2)^{(1+m+\alpha_1)/2}],
\end{aligned} \tag{35e}
$$

where

$$0 \leqq \alpha_1 \leqq m, \quad 0 \leqq \alpha_1 + l \leqq m, \quad 0 \leqq \beta_1 \leqq l-1, \quad 0 \leqq \beta_3 \leqq l, \quad 0 \leqq \beta_2 \leqq \beta_1 - 1.$$

Notice that (as in region II) it was necessary to integrate by parts, after having differentiated, to remove the powers of η^2 which emerged.

The terms (35d), (35e) can be bounded under inversion in precisely the same manner as (24c), (24d) in region II. The extra factor $\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]$ only improves convergence of the inversion integral. The terms (35b), (35c) are invertible and their inverses are bounded in ε because they provide exponential convergence.

The first term (35a) is clearly invertible for $\varepsilon \neq 0$, $Y \neq 0$ but we require its inverse for all ε, Y. To this end bound X away from zero, so that the inversion line can be bent upwards/downwards respectively for the $\exp(\pm i\eta X)$, of which $\sin \eta X$

is composed, to provide convergent integrals $O(1)$ in ε uniformly for $Y \geqq 0$. (There is no question of reaching $X = 0$ by bounding Y away from zero, as there was in region II.)

Thus we are assured that $\mathscr{R}_m$ is $O(1)$ in region III, as desired.

The sth coefficient function involves the same terms (33), (35) with m replaced by s and ε set equal to zero. With the exception of terms corresponding to (35a), which are now positive powers of η, the existence of the coefficient function is therefore covered by our discussion above. The exceptional terms are also covered if the inversion line is deformed before expanding in Taylor series.

Extension of region III downwards to $Y_\infty = \varepsilon^{-\lambda}$ is restricted to

$$\lambda < 1$$

because we have implicitly assumed $2 - \varepsilon t Y$ is positive. Extension outwards in X is unrestricted since it was nowhere necessary to bound it (e.g. terms of the form $\eta^{2p+1} \exp(-2\eta^2)$ invert into $X^p \exp[-X^2/2\sqrt{2}]$ which are bounded as $X \to \infty$).

Extension into the corner is limited by the integrals resulting from deformation of the inversion line. Letting $\eta X = t$ shows that the terms behave at worst like X^{-2m}, so that for $X = \varepsilon^\kappa$ we must have

$$\kappa < 1/2.$$

8. The transition zone III$_*$. Motivated by the last restriction, we introduce the stretched variable $x_* = \varepsilon^{-1/2} X$ in order to describe the boundary layer near $X = 0$. The structure of the III$_*$-expansion is similar to that of the III-expansion in that it is composed of two parts, namely the II-expansion, expanded in the III$_*$-variables, and the boundary layer correction. We shall show that

$$w(x_*, Y; \varepsilon) \sim \sum_{k=0}^{m-1} \varepsilon^{k/2} w_k^{\mathrm{III}*}(x_*, Y) + \varepsilon^{m/2} S_m(x_*, Y; \varepsilon), \tag{36}$$

where: (i) the $w_k^{\mathrm{III}*}$ satisfy the recurrence relation (16a), the boundary conditions (16b), and the appropriate matching conditions, and (ii) S_m is $O(1)$ in ε uniformly in region III$_*$.

(i) Once it is known that the asymptotic expansion (36) holds in region III$_*$, the fact that w (as formed from (18′a)) satisfies (1a) and the boundary conditions (1c), (1d) to within a.e.s. terms is sufficient to ensure that the $w_k^{\mathrm{III}*}$ satisfy (16a), (16b). The validity of (36) is established by the boundedness of S_m and the existence of the $w_k^{\mathrm{III}*}$ as proved in (ii) below.

As noted above, the III$_*$-expansion is a superposition of the II-expansion, expanded in the III$_*$-variables, and the boundary layer correction terms. The latter terms, as the name suggests, are a.e.s. out of the boundary layer; and so the matching of the III$_*$-expansion with the II-expansion is assured if the II-expansion matches with itself expanded in the III$_*$-variables. In the next section it is proved that the w_k^{II} have asymptotic expansions in the III$_*$-variables which are polynomials in $\varepsilon^{1/2} x_*$ and εY. Therefore we are assured of matching by Fraenkel's theorem.

(ii) To order $[(m+1)/2]$ the III-expansion of the II-expansion is

$$\omega = \sum_{k=0}^{[(m-1)/2]} \varepsilon^k \frac{2}{\pi} \int_0^\infty \tilde{w}_k^{\mathrm{II}}(\eta, Y) \sin \eta X \, d\eta + \varepsilon^{[(m+1)/2]} \mathscr{S}_{[(m+1)/2]}, \tag{37}$$

where we have already shown that $\mathscr{S}_{[(m+1)/2]}$ is bounded in ε not only in III but also in III_*. To obtain the contribution of (37) to the expansion (36) we shall deform the inversion line near $\eta = 0$ in the complex η-plane so as to avoid later convergence difficulties at that point. Now, in the boundary layer $\tilde{\omega}_k^{\text{II}}$ is composed of terms of the form (33), with $\varepsilon = 0$ and m replaced by k, and hence is an odd function of η. Thus the integrals in (37), with $X = \varepsilon^{-1/2}x_*$, can be written

(38)
$$\int_0^\infty \tilde{\omega}_k^{\text{II}}(\eta, Y) \sin(\eta\varepsilon^{1/2}x_*)\, d\eta = (2i)^{-1} \oint_{-\infty}^{\infty} \tilde{\omega}_k^{\text{II}}(\eta, Y) \exp(i\eta\varepsilon^{1/2}x_*)\, d\eta - (\pi/2) r_k(Y),$$

where $r_k(Y)$ is the residue of $\tilde{\omega}_k^{\text{II}}$ at its simple pole $\eta = 0$ and hence is a polynomial in Y. Instead of changing to the corresponding transform variable $\eta_* = \varepsilon^{1/2}\eta$, which would lead us to troublesome terms $\exp(-2\eta_*^2/\varepsilon)$, we would like to expand $\exp(i\eta\varepsilon^{1/2}x_*)$. But this contributes powers of η to the integrand which apparently destroy the convergence as $\eta \to \infty$. Therefore we first rewrite the integral terms of $\tilde{\omega}_k^{\text{II}}$ by integrating by parts $[(m - 2k)/2] + 1$ times with respect to y', to obtain

(39a) $\eta^{-1}\eta^{-2\gamma_2} Y^{\gamma_1} h^{(\gamma_3)}(1)$,

(39b) $\eta^{-1}\eta^{2\gamma_4} Y^{\gamma_5} h^{(\gamma_6)}(1) \exp(-2\eta^2)$,

(39c) $\eta^{-1}\eta^{-2\{[(m-2k)/2]+1\}} Y^{\gamma_7} \int_0^1 (1 - y')^{\gamma_8} y'^{\gamma_9} h^{(\gamma_{10})}(2y' - 1) \exp[-2\eta^2(1 - y')]\, dy'$,

where

$$0 \leqq \gamma_2 \leqq \left[\frac{m - 2k}{2}\right] + 1, \qquad 0 \leqq \gamma_4.$$

The set of terms (39a), (39b) can be integrated explicitly in (38) to give powers of εx_*^2 and Y; so that they, as well as the residue terms, provide $O(1)$ contributions to the coefficient functions and remainder in III_*. Having taken care of these terms, we can expand the exponential in the integrand (38) for the remaining terms without losing convergence. If $\tilde{\omega}_k$ denotes their contribution to $\tilde{\omega}_k^{\text{II}}$, we have

$$(2i)^{-1} \oint_{-\infty}^{\infty} \tilde{\omega}_k \exp(i\varepsilon^{1/2}\eta x_*)\, d\eta = \sum_{j=0}^{m-1-2k} (i)^{j-1} \varepsilon^{j/2} x_*^j (2j!)^{-1} \oint_{-\infty}^{\infty} \tilde{\omega}_k \eta^j\, d\eta$$
$$+ (i)^{m-2k-1} (\varepsilon^{1/2}x_*)^{m-2k} (2(m - 2k)!)^{-1} \oint_{-\infty}^{\infty} \tilde{\omega}_k \eta^{m/2-k} \exp(i\eta\varepsilon^{1/2}x_*)\, d\eta,$$

where the Taylor series has been taken to a remainder providing $O(\varepsilon^{m/2})$ in (38). Two things must be proved about this last expansion, namely that the integrals in the sum exist and that the remainder integral is $O(1)$ uniformly in III_*. But these facts are clear since for the former the integrands behave exponentially as $n \to \infty$, while for the latter they converge at least as well as η^{-2}. It was to obtain this last convergence property that integration by parts was performed on the integral terms of $\tilde{\omega}_k^{\text{II}}$.

The boundary layer correction terms (34) can be written as

$$\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]$$

(40) $$\cdot\left\{\tilde{\varpi}(\eta,\varepsilon)+\eta(1+4\varepsilon\eta^2)^{-1/2}\sum_{k=0}^{m-1}[2\varepsilon/(1+(1+4\varepsilon\eta^2)^{1/2})]^{k+1}h^{(k)}(1)\right\},$$

where

$$\tilde{\varpi}=\eta(1+4\varepsilon\eta^2)^{-1/2}\int_{-1}^{1}h(y')\{\exp[(1-y')(1-(1+4\varepsilon\eta^2)^{1/2})/2\varepsilon] - \exp[\{(1-y')(1+(1+4\varepsilon\eta^2)^{1/2})-4(1+4\varepsilon\eta^2)^{1/2}\}/2\varepsilon]\}\,dy$$

is the approximation (20′) from which the II-expansion was obtained, evaluated at $y=1$. So, as proved in the preceding paragraph,

$$\varpi(x_*,\varepsilon)=\sum_{k=0}^{m-1}\varepsilon^{k/2}\varpi_k^{\mathrm{II}}(x_*)+\varepsilon^{m/2}\mathscr{S}_{m/2}(x_*,\varepsilon)$$

holds in III_*. But we must now incorporate the exponential factor in (40), and the fact that ϖ is an odd function of η enables us to do this by convolution. Changing to the variables x_* and $\eta_*=\varepsilon^{1/2}\eta$, we find

$$\begin{aligned}
&\frac{2}{\pi}\int_0^\infty \exp[-Y(1+(1+4\eta_*^2)^{1/2})/2]\varepsilon^{-1/2}\tilde{\varpi}(\varepsilon^{-1/2}\eta_*,\varepsilon)\sin\eta_*x_*\,d\eta_*\\
&=\frac{1}{\pi i}\int_{-\infty}^{\infty}\exp[-Y(1+(1+4\eta^2)^{1/2})/2]\varepsilon^{-1/2}\tilde{\varpi}(\varepsilon^{-1/2}\eta_*,\varepsilon)\exp(i\eta_*x_*)\,d\eta_*\\
&=\frac{1}{2\pi}\exp(-Y/2)\int_{-\infty}^{\infty}\varpi(x_*',\varepsilon)YK_1[((x_*-x_*')^2+Y^2)^{1/2}/2]\\
&\qquad\cdot((x_*-x_*')^2+Y^2)^{-1/2}\,dx_*'\\
&=\sum_{k=0}^{m-1}\varepsilon^{k/2}Y\exp(-Y/2)\int_{-\infty}^{\infty}\varpi_k^{\mathrm{II}}(x_*')K_1[((x_*-x_*')^2+Y^2)^{1/2}/2]\\
&\qquad\cdot[2\pi((x_*-x_*')^2+Y^2)^{1/2}]^{-1}\,dx_*'\\
&\quad+\varepsilon^{m/2}Y\exp(-Y/2)\int_{-\infty}^{\infty}\mathscr{S}_{m/2}(x_*',\varepsilon)K_1[((x_*-x_*')^2+Y^2)^{1/2}/2]\\
&\qquad\cdot[2\pi((x_*-x_*')^2+Y^2)^{1/2}]^{-1}\,dx_*'.
\end{aligned}$$

Since ϖ_k, S_m behave as polynomials in x_* and

$$\int_0^\infty x^{2\mu+1}K_1[(x^2+Y^2)^{1/2}/2](x^2+Y^2)^{-1/2}\,dx=2^{2\mu+1}\Gamma(\mu+1)Y^\mu K_{-\mu}(Y/2),$$

these integrals are bounded independently of ε throughout III_*.

Finally we must deal with the contribution to the expansion (36) from the series in (40). But, under change of transform variable to η_*, these terms become

$$\exp[-Y(1+(1+4\varepsilon\eta^2)^{1/2})/2]\eta_*(1+4\eta_*^2)^{-1/2}$$
$$\cdot\sum_{k=0}^{m-1}2(2\varepsilon)^kh^{(k)}(1)/(1+(1+4\varepsilon\eta^2)^{1/2})^{k+1},$$

each of which is clearly invertible. Each therefore contributes to one of the coefficient functions of (36) or the remainder.

This completes the proof that there is indeed a valid asymptotic expansion throughout region III_*.

Extension of region III_* to

$$Y_\infty = \varepsilon^{-\lambda}, \qquad x_{*\infty} = \varepsilon^{-\mu}$$

is limited to

$$\lambda < 1, \qquad \mu < 1/2.$$

The restriction on λ is a carry-over from the bounding of the remainder in region III. The limitation on μ arises from the terms

$$\varepsilon^k(\varepsilon x_*^2)^{\gamma_4}$$

which come from (39b). We obtain the worst behavior for $k = 0$, when they behave like $(\varepsilon^{1-2\mu})^{\gamma_4}$—hence the restriction on μ.

9. Other assumptions about $h(y)$. It is of interest to see the effect of varying the conditions (17) on $h(y)$. First we strengthen them with

$$h^{(k)}(-1+0) = 0 \quad \text{or} \quad h^{(k)}(1-0) = 0 \quad \text{for } k \leqq k_0,$$

and ask at what stage in the approximation of w it is necessary to introduce the corner region II_* or III_*. Secondly we weaken them to

$$h \text{ is infinitely differentiable on } (-1, 1) \text{ except at } y = a,$$

$$h^{(k)}(a \pm 0),\ h^{(k)}(-1+0),\ h^{(k)}(1-0) \text{ exist for all } k,$$

when it is necessary to introduce a new region about $y = a$.

If $h^{(k)}(-1+0) = 0$ for $k \leqq k_0$, it is not necessary to introduce region II_* until $m = [k_0/2] + 1$. Thus, the exclusion of the lower corner from II arises in bounding the terms (24a), (24b) and these now vanish for $m \leqq [k_0/2]$. All remaining bounds extend into the corner. Similarly if $h^{(k)}(1-0) = 0$ for $k \leqq k_0$, it is not necessary to introduce the region III_* until $m = k_0 + 2$, since the terms (35a) in III are zero for $m \leqq k_0 + 1$ and again all bounds extend into the corner.

To facilitate the discussion when h and its derivatives are allowed a discontinuity at $y = a$, we define a new function

$$h_a(y) = \begin{cases} h(y) & \text{for } y < a, \\ \sum_{k=0}^{M} (y-a)^k h^{(k)}(a-0)/k! & \text{for } y \geqq a, \end{cases}$$

where M depends on m. The original boundary value problem (1a), (1c), (1d) is now written as the superposition of two semi-infinite strip problems with the respective boundary data

$$w_1(0, y) = h_a(y), \qquad w_1(x, \pm 1) = 0,$$

$$w_2(0, y) = h(y) - h_a(y), \qquad w_2(x, \pm 1) = 0.$$

Since the function h_a has continuous derivatives to order M, the proof that the method of matched asymptotic expansions is valid for w_1 to order m is contained in the preceding portion of this paper, when M is taken sufficiently large for all derivatives involved to be continuous.

For w_2 the results for region II follow as before except that we must also deal with terms

$$\text{(41a)}\qquad -\eta(1 + 4\varepsilon\eta^2)^{-1/2} \sum_{k=0}^{m-1} (-1 - (1 + 4\varepsilon\eta^2)^{1/2})^{k+1}(2\eta^2)^{-k-1} \cdot \{h^{(k)}(a + 0) - h^{(k)}(a - 0)\} \exp[-2\eta^2(y - a)/(1 + (1 + 4\varepsilon\eta^2)^{1/2})],$$

for $y > a$, and

$$\text{(41b)}\qquad -\eta(1 + 4\varepsilon\eta^2)^{-1/2} \sum_{k=0}^{m-1} (2\varepsilon)^{k+1}(1 + (1 + 4\varepsilon\eta^2)^{1/2})^{-k-1} \cdot \{h^{(k)}(a + 0) - h^{(k)}(a - 0)\} \exp[(y - a)(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2\varepsilon],$$

for $y \leqq a$, which arise on integrating (20′) by parts due to the discontinuity at $y = a$.

The terms (41a) present no problem for $y \geqq a + \delta$ ($\delta > 0$) since even after m differentiations with respect to ε the inversion integral retains exponential convergence. As for region II_* (see § 5), if we rewrite $\sin \eta X$ as exponentials and deform the integration contour, we obtain convergence for all $y \geqq a$ when $X \geqq X_0 > 0$.

The terms (41b) are uniformly a.e.s. if $y \leqq a - \delta$, and analysis as in the previous paragraph shows they have similar properties for $y \leqq a$, $X \geqq X_0 > 0$.

It is therefore necessary to introduce a "corner" region near $X = 0$, $y = a$, similar to the II_*-region. In terms of the stretched variables $y_a = \varepsilon^{-1}(y - a)$, $x_* = \varepsilon^{-1/2}X$ and the corresponding transform variable $\eta_* = \varepsilon^{1/2}\eta$, we have

$$\hat{w} \sim \eta_*(1 + 4\eta_*^2)^{-1/2} \int_0^{(1-a)/\varepsilon} \{h(\varepsilon y_a' + a) - h_a(\varepsilon y_a' + a)\} \cdot \exp[\{(y_a - y_a') - |y_a - y_a'|(1 + 4\eta_*^2)^{1/2}\}/2]\, dy_a',$$

to uniformly a.e.s. terms.

Now expand $h - h_a$ in its Taylor series about a to m terms and note that, as in region II_*, the upper limit of integration can be extended to ∞ for these terms (but not the remainder), thereby introducing only uniformly a.e.s. terms and making the integral independent of ε. Bounding follows precisely as in region II_*.

The regions III and III_* present no problem since the additional terms due to the discontinuity at $y = a$ are invertible and the corresponding remainders are $O(\varepsilon^m)$. For instance in the formula (34) we must add

$$\eta(1 + 4\varepsilon\eta^2)^{-1/2} \exp[-Y(1 + (1 + 4\varepsilon\eta^2)^{1/2})/2 + (a - 1)2\eta^2/(1 + (1 + 4\varepsilon\eta^2)^{1/2})] \cdot [(-1 - (1 + 4\varepsilon\eta^2)^{1/2})/2\eta^2]^{k+1}\{h^{(k)}(a + 0) - h^{(k)}(a - 0)\},$$

which always provides exponentially convergent inversion integrals.

10. The rectangle. We are now in a position to show that the method of matched asymptotic expansions is valid for (1a) on the rectangle $0 \leqq x \leqq l$, $|y| \leqq 1$ with the boundary data

$$
\begin{aligned}
w(x, -1) &= f(x), \qquad w(0, y) = h(y), \\
w(x, 1) &= g(x), \qquad w(l, y) = \mathscr{s}(y).
\end{aligned} \tag{42}
$$

The proof consists in writing w as the superposition of three functions, the first two of which are solutions on semi-infinite strips and the third a solution on the rectangle with data which is a.e.s.

In order to formulate these problems we define new functions f_1, f_2, g_1, g_2, where

$$
f_1 = \begin{cases} fH & \text{for } 0 \leqq x \leqq l, \\ 0 & \text{for } x > l, \end{cases} \qquad f_2 = \begin{cases} f - f_1 & \text{for } 0 \leqq x \leqq l, \\ 0 & \text{for } x < 0, \end{cases}
$$

and g_1, g_2 are similar. H is an infinitely differentiable function of x such that

$$
H = \begin{cases} 1 & \text{for } -\infty < x \leqq l/4, \\ 0 & \text{for } 3l/4 \leqq x < \infty, \end{cases}
$$

and is introduced purely as an artifice. That is, if we now let w_1, w_2 be the solutions satisfying the respective boundary conditions

$$
\begin{aligned}
&w_1(0, y) = h(y), \\
&w_1(x, -1) = f_1(x), \qquad w_1(x, 1) = g_1(x) \quad \text{for } 0 \leqq x < \infty, \\
&w_2(l, y) = \mathscr{s}(y), \\
&w_2(x, -1) = f_2(x), \qquad w_2(x, 1) = g_2(x) \quad \text{for } -\infty < x \leqq l,
\end{aligned}
$$

then the boundary data for w_1 is zero in a neighborhood of $x = l$ and that for w_2 is zero in a neighborhood of $x = 0$. It follows that $w_1(l, y)$ and $w_2(0, y)$ are a.e.s., a result on which the proof hinges. (The infinite differentiability of H ensures that no spurious layers are introduced by w_1 and w_2.)

On the rectangle we may write $w(x, y) = w_1(x, y) + w_2(x, y) - w_3(x, y)$, where w_3 is the solution with the boundary values

$$
\begin{aligned}
w_3(x, -1) &= 0, \qquad w_3(0, y) = w_2(0, y), \\
w_3(x, 1) &= 0, \qquad w_3(l, y) = w_1(l, y).
\end{aligned} \tag{43}
$$

Note that these boundary values depend on, and are a.e.s. in, ε; and that w_3 is the correction needed to annihilate the boundary values of w_1 and w_2 at the sides $x = l$ and $x = 0$, respectively (which were omitted from their definitions as solutions of semi-infinite strip problems). Note also that the data (43) is continuous on the boundary, including the corners. It now follows from the maximum principle that $|w_3|$ is bounded by its maximum on the boundary and hence is a.e.s. throughout the rectangle. Its contribution to w may therefore be ignored.

The proof that the method of matched asymptotic expansions is valid for w_1 and w_2 separately is the preceding portion of this paper. Note that the entire problem is linear so that the approximation to w obtained by the method of matched asymptotic expansions in any region is the sum of the approximations obtained there for w_1 and w_2. This shows that the method is valid for w.

No mention has been made in this section about the differentiability properties of the boundary data. The order to which the approximations can be carried out will depend on these in the ways described in § 9 and our earlier paper [2].

REFERENCES

[1] J. COLE, *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, Mass., 1968.

[2] L. PAMELA COOK AND G. S. S. LUDFORD, *The behavior as* $\varepsilon \to 0^+$ *of solutions to* $\varepsilon\nabla^2 w = (\partial/\partial y)w$ *in* $|y| \leqq 1$ *for discontinuous boundary data*, this Journal, 2 (1971), pp. 567–594.

[3] L. E. FRAENKEL, *On the method of matched asymptotic expansions. Part I: A matching principle*, Proc. Cambridge Philos. Soc., 65 (1969), pp. 209–231.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

BOUNDARY PROBLEMS OF STURMIAN TYPE ON AN INFINITE INTERVAL*

WILLIAM T. REID†

Abstract. Two types of Sturmian boundary problems on $(-\infty, \infty)$, with boundary conditions specifying that the proper functions are of integrable square, are considered with the aid of the principal solutions of the involved differential equation at ∞ and $-\infty$, and the reduction of the given problem to an associated Sturmian problem on a finite interval. For problems of the first type it follows from classical Sturmian theory that the totality of proper values may be ordered as a simple sequence, with the proper function corresponding to the jth proper value possessing exactly $j-1$ zeros on $(-\infty, \infty)$. Problems of the second type involve "turning points," and in this instance Sturmian comparison theorems are used to establish the existence of a sequence of sets of proper values such that the proper functions corresponding to parameter values in the jth set possess exactly $j-1$ zeros on $(-\infty, \infty)$.

1. Introduction. This paper is concerned with two types of boundary problems of Sturmian type for a real linear homogeneous ordinary differential equation on the real line, in which the boundary condition prescribes that the proper functions are of integrable square on $(-\infty, \infty)$. Problems of the first type involve differential equations whose coefficient functions possess monotoneity properties of the sort appearing in the classical Sturmian theory, and the results obtained for this problem are generalizations of those obtained many years ago by Milne [7]. Problems of the second type involve "turning points," and the coefficients of the differential equation are monotonic of opposite character on complementary subintervals of $(-\infty, \infty)$. The present consideration of problems of this latter sort was stimulated by the paper [1] of Harris and Sibuya, who utilized complex variable methods to establish the existence and asymptotic form of the proper values for a problem of this kind.

The central feature of the treatment of the present paper is the reduction of each of the considered problems to an associated problem on a finite interval, through the use of the principal solutions of the involved differential equation at the end points ∞ and $-\infty$. For problems of the first type the associated finite interval problem satisfies the hypotheses of the classical Sturmian theory, so that the totality of real proper values may be ordered as a sequence $\{\lambda_j\}$, with the proper function corresponding to λ_j possessing exactly $j-1$ zeros on $(-\infty, \infty)$. For problems of the second type the associated finite interval problem does not possess all the monotoneity properties so requisite for the Sturmian theory. However, with the aid of Sturmian comparison theorems one may establish the existence of a sequence Λ_j of sets of proper values, such that if $\lambda \in \Lambda_j$, then λ is a proper value for which corresponding proper functions possess exactly $j-1$ zeros on $(-\infty, \infty)$. At the moment there remains unanswered the determination of specific conditions on the coefficient functions of the differential equation which will insure that each set Λ_j reduces to a single value.

* Received by the editors August 31, 1971, and in revised form March 6, 1972.

† Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73069. This work was supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant AFOSR-71-2069.

The real number line $(-\infty, \infty)$ will be denoted by $\mathbf{R}$, and $\mathbf{R}^+$ will designate the nonnegative ray $\{t|t \in \mathbf{R}, t \geqq 0\}$. If $[a, b] \subset \mathbf{R}$ and a function $f:[a, b] \to \mathbf{R}$ is a.c. (absolutely continuous), then $f'(t)$ denotes the derivative of $f(t)$ at values where the derivative exists and zero elsewhere. For a given compact interval $[a, b] \subset \mathbf{R}$ the symbols $\mathfrak{C}[a, b]$, $\mathfrak{L}[a, b]$, $\mathfrak{L}^2[a, b]$, $\mathfrak{L}^\infty[a, b]$ and $\mathfrak{A}[a, b]$ are used to denote the class of functions $f:[a, b] \to \mathbf{R}$ which are respectively continuous, (Lebesgue) integrable, (Lebesgue) measurable and f^2 integrable, measurable and essentially bounded, and a.c. on $[a, b]$. If functions f and g are equal a.e. (almost everywhere), on a common domain of definition, we write simply $f = g$. If $f:[a, b] \to \mathbf{R}$ is essentially bounded, then $\operatorname{ess\,sup}_{t \in [a,b]} f(t)$ denotes the essential supremum of f on $[a, b]$, that is, the smallest nonnegative number M such that the set $\{t|t \in [a, b], |f(t)| > M\}$ has measure zero. A function $f:\mathbf{R} \to \mathbf{R}$ is said to be locally integrable, of class $\mathfrak{L}^2$, of class $\mathfrak{L}^\infty$, or a.c., if $f \in \mathfrak{L}[a, b]$, $f \in \mathfrak{L}^2[a, b]$, $f \in \mathfrak{L}^\infty[a, b]$, or $f \in \mathfrak{A}[a, b]$, for arbitrary compact subintervals $[a, b]$ of $\mathbf{R}$.

2. A problem of classical Sturmian type on an infinite interval. In this section we shall consider a boundary problem

$$[r(t, \lambda)u'(t)]' - p(t, \lambda)u(t) = 0, \tag{2.1}$$

$$\int_{-\infty}^{\infty} u^2(t)\, dt < \infty, \tag{2.2}$$

involving the real parameter λ, under the following hypotheses, where

$$\mathbf{R} = (-\infty, \infty) \quad \text{and} \quad \Delta = \{\lambda|\Lambda_1 < \lambda < \Lambda_2\}.$$

(i) On $\mathbf{R} \times \Delta$ the functions r and p are real-valued, and $r > 0$.

(ii) For $\lambda_0 \in \Delta$, the functions $r(t, \lambda_0)$, $1/r(t, \lambda_0)$, and $p(t, \lambda_0)$ are locally of class $\mathfrak{L}^\infty$; moreover, for t a.e. on $\mathbf{R}$ these functions are continuous in λ at λ_0.

(iii) For $t \in \mathbf{R}$ the functions $r(t, \lambda)$, $p(t, \lambda)$ are monotone nonincreasing functions of λ on Δ, and such that for $[a, b]$, an arbitrary nondegenerate

($\mathfrak{H}$) compact subinterval of $\mathbf{R}$, the functional

$$J_0[\eta; \lambda|a, b] = \int_a^b \{r(t, \lambda)\eta'^2(t) + p(t, \lambda)\eta^2(t)\}\, dt$$

is a strictly monotone decreasing function of λ for arbitrary $\eta(t) \not\equiv 0$ belonging to the class $\mathscr{D}[a, b]$ defined by

$$\mathscr{D}[a, b] = \{\eta|\eta \in \mathfrak{A}[a, b], \eta' \in \mathfrak{L}^2[a, b]\}.$$

(iv) For t a.e. on $\mathbf{R}$,

$$\lim_{\lambda \to \Lambda_1} p(t, \lambda) = +\infty.$$

(v) There exists a compact subinterval I_0 of $\mathbf{R}$ such that

$$\lim_{\lambda \to \Lambda_2} p(t, \lambda) = -\infty \quad \text{for } t \text{ a.e. on } I_0.$$

(vi) For $\lambda \in \Delta$, $\liminf_{t \to \pm\infty} p(t, \lambda) > 0$.

Under hypotheses ($\mathfrak{H}$ (i), (ii)) a function $u(t)$ is said to be a *solution* of (2.1) if there exists an associated $v(t)$ such that $(u(t); v(t))$ is a solution in the Carathéodory sense of the differential system

$$u'(t) = [1/r(t, \lambda)]v(t), \qquad v'(t) = p(t, \lambda)u(t); \tag{2.1'}$$

that is, $u(t)$ and $v(t)$ are locally a.c. on **R** and (2.1′) holds a.e. on **R**.

It is to be noted that the above conditions ($\mathfrak{H}$ (i)–(vi)) hold for $\Delta = \mathbf{R}$ under the following hypothesis:

$r(t, \lambda) \equiv 1$, while on $\mathbf{R} \times \mathbf{R}$ the real-valued function $p(t, \lambda)$ is continuous, has a negative partial derivative with respect to λ, and

$$(\mathfrak{H}^0) \qquad \lim_{\lambda \to -\infty} p(t, \lambda) = +\infty, \qquad \lim_{\lambda \to \infty} p(t, \lambda) = -\infty,$$
$$\lim_{t \to \pm\infty} p(t, \lambda) = +\infty.$$

The conditions of ($\mathfrak{H}^0$) are those imposed by Milne [7]; in particular, they are satisfied if $r(t, \lambda) \equiv 1$ and $p(t, \lambda) = p_0(t) - \lambda$, where $p_0(t)$ is a real-valued continuous function on **R** such that $\lim_{t \to \pm\infty} p_0(t) = +\infty$.

LEMMA 2.1. *Suppose that hypotheses* ($\mathfrak{H}$ (i)–(v)) *are satisfied*, $[a, b]$ *is a compact subinterval containing the interval* I_0 *of* ($\mathfrak{H}$ (v)), *while* $\beta_a(\lambda)$, $\beta_b(\lambda)$ *are functions of* λ *on* Δ *which are real-valued, continuous and nonincreasing on* Δ. *Then:*

(i) *for* $\eta \in \mathscr{D}[a, b]$, *and* $\eta(t) \not\equiv 0$ *on* $[a, b]$, *the functional*

$$J[\eta; \lambda | a, b] = \beta_a(\lambda)\eta^2(a) + \beta_b(\lambda)\eta^2(b) + J_0[\eta; \lambda | a, b] \tag{2.3}$$

is a continuous strictly monotone decreasing function of λ *on* Δ;

(ii) *for* $\lambda \in \Delta$, *the real proper values of the Sturmian boundary problem*

$$\begin{gathered} [r(t, \lambda)u'(t)] + [\mu - p(t, \lambda)]u(t) = 0, \\ \beta_a(\lambda)u(a) - r(a, \lambda)u'(a) = 0, \\ \beta_b(\lambda)u(b) + r(b, \lambda)u'(b) = 0, \end{gathered} \tag{2.4}$$

involving the characteristic parameter μ, *form a sequence*

$$\mu_1(\lambda) < \mu_2(\lambda) < \cdots \tag{2.5}$$

such that:

(a) *a proper function* $u = u_j(t; \lambda)$ *of* (2.4) *corresponding to the proper value* $\mu = \mu_j(\lambda)$ *has exactly* $j - 1$ *zeros on* (a, b);

(b) $\lim_{j \to \infty} \mu_j(\lambda) = +\infty$ *for* $\lambda \in \Delta$;

(c) *for* $j = 1, 2, \cdots$, *the proper value* $\mu_j(\lambda)$ *is a strictly monotone decreasing continuous function of* λ *on* Δ;

(d) *there exists a* $\hat{\lambda} \in \Delta$ *such that*

$$\mu_1(\lambda) > 0 \quad \textit{if } \lambda \in (\Lambda_1, \hat{\lambda}).$$

In view of hypothesis ($\mathfrak{H}$ (ii)), for $\eta \in \mathscr{D}[a, b]$ the functional $J_0[\eta; \lambda | a, b]$ exists for all $\lambda \in \Delta$, and conclusion (i) is a ready consequence of hypothesis ($\mathfrak{H}$ (iii)). For each $\lambda \in \Delta$ the existence of a sequence of proper values (2.5) of (2.4) satisfying conclusions (ii (a), (b)) is a consequence of classical Sturm–Liouville theory. In case the coefficient functions $r(t, \lambda)$, $p(t, \lambda)$ satisfy hypotheses ($\mathfrak{H}$ (i), (ii)), and are also continuous in (t, λ) on $\mathbf{R} \times \Delta$, this result is given in Ince [5, Chap. X, § 10.71], Hartman [3, Chap. XI, Th. 4.1], or Reid [10, Chap. V, Th. 7.4]. Also, under ($\mathfrak{H}$ (iii)) and these additional hypotheses the strict monotone decreasing character of the proper values $\mu_j(\lambda)$ follows from a well-known comparison theorem for Sturm–Liouville systems (see, for example, Reid [10, Chap. VI, Th. 4.1]), and the continuity of each proper value $\mu_j(\lambda)$ as a function of λ is also a ready consequence of the extremizing property of these proper values (see, for example, Reid [10, Chap. VI, Prob. 5.8]). The proof for each of the above results may be modified to yield the stated conclusions (ii (a), (b), (c)) when hypotheses ($\mathfrak{H}$ (i), (ii), (iii)) hold, and the functions r, p are not required to be continuous on $\mathbf{R} \times \Delta$.

Now in view of hypothesis ($\mathfrak{H}$ (iv)) we have that

$$\lim_{\lambda \to \Lambda_1} J[\eta; \lambda | a, b] = +\infty \quad \text{for } \eta \in \mathscr{D}[a, b] \quad \text{and} \quad \eta(t) \not\equiv 0 \quad \text{on } [a, b]. \tag{2.6}$$

We shall proceed to show that there exists a value $\hat{\lambda} \in \Delta$ such that

$$J[\eta; \lambda | a, b] > 0 \quad \text{for } \lambda \in (\Lambda, \hat{\lambda}) \quad \text{and} \quad \eta \in \mathscr{D}[a, b], \quad \eta(t) \not\equiv 0 \quad \text{on } [a, b]. \tag{2.7}$$

Indeed, if there exists no value $\hat{\lambda}$ such that (2.7) holds, then there exists a sequence $\{\hat{\lambda}_k, \eta_k\}$, $k = 1, 2, \cdots$, such that

$$\begin{aligned} &\hat{\lambda}_{k+1} \in (\Lambda_1, \hat{\lambda}_k) \subset \Delta, \quad k = 1, 2, \cdots, \qquad \lim_{k \to \infty} \hat{\lambda}_k = \Lambda_1; \\ &\eta_k(t) \not\equiv 0 \text{ on } [a, b], \quad \eta_k \in \mathscr{D}[a, b], \qquad J[\eta_k, \hat{\lambda}_k | a, b] \leqq 0. \end{aligned} \tag{2.8}$$

Without loss of generality we may suppose that the functions η_k are normed so that

$$\|\eta_k\|^2 \equiv \eta_k^2(a) + \eta_k^2(b) + \int_a^b \eta_k^2(t)\, dt = 1, \qquad k = 1, 2, \cdots. \tag{2.9}$$

As

$$J[\eta_k; \hat{\lambda}_1 | a, b] \leqq J[\eta_k; \hat{\lambda}_k | a, b] \leqq 0, \qquad k = 1, 2, \cdots,$$

if

$$\kappa_0(\hat{\lambda}_1) = \max \{|\beta_a(\hat{\lambda}_1)|, |\beta_b(\hat{\lambda}_1)|, \operatorname*{ess\,sup}_{t \in [a,b]} |p(t, \hat{\lambda}_1)|\},$$

and

$$\kappa(\hat{\lambda}_1) = \kappa_0(\hat{\lambda}_1) \operatorname*{ess\,sup}_{t \in [a,b]} r^{-1}(t, \hat{\lambda}_1),$$

it follows that

$$\int_a^b [\eta_k'(t)]^2\, dt \leqq \kappa(\hat{\lambda}_1) \|\eta_k\|^2 = \kappa(\hat{\lambda}_1), \qquad k = 1, 2, \cdots. \tag{2.10}$$

Relations (2.9), (2.10) imply that the sequence of functions $\{\eta_k(t)\}$ is uniformly bounded and equicontinuous on $[a, b]$. Moreover, (2.10) states that the sequence $\{\eta_k'(t)\}$ belongs to a bounded ball in $\mathfrak{L}^2[a, b]$, and hence there is a subsequence of $\{\eta_k'(t)\}$ which converges weakly in $\mathfrak{L}^2[a, b]$ to a function ϕ. Also by the Arzelà–Ascoli theorem there is a subsequence of the first determined subsequence which converges uniformly to a continuous function $\eta_\infty(t)$. Combining these results we obtain a subsequence of $\{\eta_k(t)\}$, which will still be denoted by $\{\eta_k(t)\}$, that converges uniformly on $[a, b]$ to a limit function $\eta_\infty(t)$, and the sequence $\{\eta_k'(t)\}$ converges weakly in $\mathfrak{L}^2[a, b]$ to $\phi(t) = \eta_\infty'(t)$. Moreover, in view of (2.9) we have $\|\eta_\infty\| = 1$, so that $\eta_\infty(t) \not\equiv 0$ on $[a, b]$. Then $\eta_\infty \in \mathscr{D}[a, b]$, and since $r(t, \lambda_h) > 0$ for $h = 1, 2, \cdots$, we have the lower semicontinuity result

$$J[\eta_\infty; \hat{\lambda}_h | a, b] \leqq \liminf_{k \to \infty} J[\eta_k; \hat{\lambda}_h | a, b]$$
$$\leqq \liminf_{k \to \infty} J[\eta_k; \hat{\lambda}_k | a, b] \leqq 0,$$

which is contradictory to (2.6). Now if $u(t)$ is a proper function of (2.4) corresponding to a proper value $\mu = \mu(\lambda)$, then an integration by parts yields the relation

$$J[u; \lambda | a, b] = \mu(\lambda) \int_a^b u^2(t)\, dt.$$

Consequently, if $\hat{\lambda}$ is a value satisfying (2.7) we have that $\mu_1(\lambda) > 0$ for $\lambda \in (\Lambda_1, \hat{\lambda})$.

The principal result to be established in this section is that of the following theorem. For the particular case of $r(t, \lambda)$ independent of λ and $p(t, \lambda)$ of the form $-\lambda + q(t)$, results of the nature of this theorem have a long history, dating from the basic work of H. Weyl [12], [13]; in this connection the reader is also referred to the paper of Wolfson [14], and to references given by Hartman in connection with §§ 4 and 6 of Chapter XI of [3].

THEOREM 2.1. *Under the above hypotheses* ($\mathfrak{H}$ (i)–(vi)) *the real proper values of the system* (2.1), (2.2) *form an infinite sequence*

$$\lambda_1 < \lambda_2 < \cdots$$

such that $\lambda_j \to \Lambda_2$ *as* $j \to \infty$; *moreover, for* $j = 1, 2, \cdots$ *the corresponding proper function* $u_j(t) = u(t, \lambda_j)$ *of this system has exactly* $j - 1$ *zeros on* $\mathbf{R}$.

In view of hypotheses ($\mathfrak{H}$ (i), (ii), (iii), (v)) it follows from oscillation results of the Sturmian theory (see, for example, Ince [5, Chap. X, § 10.6], Hartman [3, Chap. XI, § 3], or Reid [10, Chap. V, § 5]) for an arbitrary positive integer m there exists a value l_m such that if $u = u(t, \lambda)$ is a solution of (2.1) for $\lambda > l_m$, then $u(t, \lambda)$ has at least m zeros on I_0, where I_0 is the compact subinterval of ($\mathfrak{H}$ (v)). Moreover, in view of hypothesis ($\mathfrak{H}$ (vi)), for $\lambda \in \Delta$ there exists a $\tau = \tau(\lambda) > 0$ and a $\kappa = \kappa(\lambda) > 0$ such that

$$p(t, \lambda) \geqq \kappa(\lambda) \quad \text{for } t \in (-\infty, -\tau(\lambda)] \cup [\tau(\lambda), \infty). \tag{2.11}$$

In particular, (2.1) is nonoscillatory on each of the intervals $[\tau(\lambda), \infty)$ and $(-\infty, -\tau(\lambda)]$, and consequently this equation has principal solutions $u_\infty(t, \lambda)$ and $u_{-\infty}(t, \lambda)$ at ∞ and $-\infty$, respectively.

The principal solution $u_\infty(t, \lambda)$, $\{u_{-\infty}(t, \lambda)\}$, is characterized by the property that if for the given value of λ the function $u(t)$ is a solution of (2.1) which is not a multiple of $u_\infty(t, \lambda)$, $\{u_{-\infty}(t, \lambda)\}$, then

$$\lim_{t \to \infty} \frac{u_\infty(t, \lambda)}{u(t)} = 0, \qquad \lim_{t \to -\infty} \frac{u_{-\infty}(t, \lambda)}{u(t)} = 0.$$

The corresponding functions $w = w_\infty(t, \lambda) \equiv r(t, \lambda)u_\infty'(t, \lambda)/u_\infty(t, \lambda)$, $\{w = w_{-\infty}(t, \lambda) \equiv r(t, \lambda)u_{-\infty}'(t, \lambda)/u_{-\infty}(t, \lambda)\}$ are solutions of the associated Riccati differential equation

$$w'(t) + \frac{w^2(t)}{r(t, \lambda)} - p(t, \lambda) = 0, \tag{2.12}$$

and are called the *distinguished solutions* of (2.12) at the respective end-values ∞ and $-\infty$. For the case of real scalar linear homogeneous differential equations of the second order the concept of a principal solution was introduced independently by Leighton and Morse [6] and Hartman and Wintner [4], using different definitive properties for such solutions. The extension to matrix systems is due to Hartman [2] (see also [3, Chap. XI]) and Reid [8] (see also [10, Chap. VII]). Moreover, if $\tau \in [\tau(\lambda), \infty)$ and $s \in (\tau, \infty)$, there is a unique solution $u = u_s(t, \tau, \lambda)$, $v = v_s(t, \tau, \lambda)$ of (2.1′) satisfying the boundary conditions

$$u_s(\tau, \tau, \lambda) = 1, \qquad u_s(s, \tau, \lambda) = 0, \tag{2.13}$$

and $v_s(\tau, \tau, \lambda)$ is a monotone nondecreasing function of s on (τ, ∞) which is bounded above. If $v_\infty(\tau, \lambda) = \lim_{s\to\infty} v_s(\tau, \tau, \lambda)$, then a principal solution $u = u_\infty(t, \tau, \lambda)$ of (2.1) at ∞ is determined by the initial conditions

$$u_\infty(\tau, \tau, \lambda) = 1, \qquad v_\infty(\tau, \tau, \lambda) = v_\infty(\tau, \lambda). \tag{2.14}$$

Also the distinguished solution $w_\infty(t, \lambda)$ of (2.12) at ∞ is given by

$$w_\infty(t, \lambda) = v_\infty(t, \tau, \lambda)/u_\infty(t, \tau, \lambda).$$

In particular,

$$w_\infty(\tau, \lambda) = v_\infty(\tau, \tau, \lambda) = v_\infty(\tau, \lambda) \quad \text{for } \tau \in [\tau(\lambda), \infty). \tag{2.14′}$$

As $r(t, \lambda)$ and $p(t, \lambda)$ are both positive on $[\tau(\lambda), \infty)$ it follows that for $\tau \in [\tau(\lambda), \infty)$ the integral

$$\int_\tau^\infty \{r(t, \lambda)[u'_\infty(t, \tau, \lambda)]^2 + p(t, \lambda)[u_\infty(t, \tau, \lambda)]^2\}dt \tag{2.15}$$

converges and has the value $-v_\infty(\tau, \lambda) = -w_\infty(\tau, \lambda)$; for this latter result see, in particular, Reid [8, Th. 8.1]. Since $p(t, \lambda) \geqq \kappa(\lambda) > 0$ for $t \in [\tau(\lambda), \infty)$, it follows from (2.15) that $u_\infty(\cdot, \lambda) \in \mathfrak{L}^2[\tau(\lambda), \infty)$. Now if for given $\lambda \in \Delta$ and $\tau \in [\tau(\lambda), \infty)$ we have that $u = u_0(t)$, $v = v_0(t)$ is the solution of the equation (2.1) which satisfies the initial conditions

$$u_0(\tau) = 1, \qquad v_0(\tau) = 1,$$

it follows that $u_0(t) \geqq 1$ for $t \in [\tau, \infty)$ and the integral $\int_\tau^\infty u_0^2(t)\, dt$ is divergent. As every solution $u(t)$ of (2.1) is a linear combination of $u_\infty(t, \lambda)$ and $u_0(t)$, it then follows that a real-valued solution $u(t)$ of (2.1) is such that the integral $\int^\infty u^2(t)\, dt$ is convergent if and only if $u(t)$ is a multiple of $u_\infty(t, \lambda)$.

Similarly, if $\tau \in (-\infty, -\tau(\lambda)]$ and $s \in (-\infty, \tau)$, then there is a unique solution $u = u_s(t, \tau, \lambda)$, $v = v_s(t, \tau, \lambda)$ satisfying the boundary conditions (2.13) and $v_s(\tau, \tau, \lambda)$ is a monotone nondecreasing function which is bounded below on $(-\infty, \tau)$. If $v_{-\infty}(\tau, \lambda) = \lim_{s\to-\infty} v_s(\tau, \tau, \lambda)$, then a principal solution $u_{-\infty}(t, \tau, \lambda)$ of (2.1) at $-\infty$ is determined by the boundary conditions

$$u_{-\infty}(\tau, \tau, \lambda) = 1, \qquad v_{-\infty}(\tau, \tau, \lambda) = v_{-\infty}(\tau, \lambda). \tag{2.14$_0$}$$

Also, the distinguished solution $w_{-\infty}(t, \lambda)$ of (2.12) at $-\infty$ is given by $w_{-\infty}(t, \lambda) = v_{-\infty}(t, \tau, \lambda)/u_{-\infty}(t, \tau, \lambda)$, and

$$(2.14'_0) \qquad w_{-\infty}(\tau, \lambda) = v_{-\infty}(\tau, \tau, \lambda) = v_{-\infty}(\tau, \lambda) \quad \text{for } \tau \in (-\infty, -\tau(\lambda)].$$

Corresponding to the above results for the endpoint ∞, we have that a real-valued solution $u(t)$ of (2.1) for a value λ is such that the integral $\int_{-\infty} u^2(t)\, dt$ is convergent if and only if $u(t)$ is a multiple of $u_{-\infty}(t, \lambda)$.

Now under hypotheses ($\mathfrak{H}$ (i), (ii), (iii)) it follows that if $\Delta_0 = \{\lambda | \lambda' \leqq \lambda \leqq \lambda''\}$ is a compact subinterval of Δ, then $0 \leqq r(t, \lambda) \leqq r(t, \lambda')$ and $|p(t, \lambda)| \leqq |p(t, \lambda')| + |p(t, \lambda'')|$ for $(t, \lambda) \in \mathbf{R} \times \Delta_0$. Consequently, if $[a, b]$ is a compact subinterval of $\mathbf{R}$, and $\tilde{\lambda}_j \in \Delta_0$, $j = 0, 1, 2, \cdots$, with $\lim_{j \to \infty} \tilde{\lambda}_j = \tilde{\lambda}_0$, then with the aid of the Lebesgue dominated convergence theorem it follows that the sequences $\{r(t, \tilde{\lambda}_j)\}$ and $\{p(t, \tilde{\lambda}_j)\}$ converge strongly in $\mathfrak{L}[a, b]$ to the respective limit functions $r(t, \tilde{\lambda}_0)$ and $p(t, \tilde{\lambda}_0)$; that is,

$$\lim_{j \to \infty} \int_a^b |r(t, \tilde{\lambda}_j) - r(t, \tilde{\lambda}_0)|\, dt = 0, \qquad \lim_{j \to \infty} \int_a^b |p(t, \tilde{\lambda}_j) - p(t, \tilde{\lambda}_0)|\, dt = 0.$$

Consequently (see, for example, Reid [9, Th. 3.1]), if $(t_j, u_j^0, v_j^0) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, with the sequences $\{t_j\}$, $\{u_j^0\}$, $\{v_j^0\}$ converging to t_0, u_0^0, v_0^0, respectively, then for the solutions $u = u_j(t)$, $v = v_j(t)$ of (2.1′) for $\lambda = \tilde{\lambda}_j$ satisfying the initial conditions

$$u_j(t_j) = u_j^0, \qquad v_j(t_j) = v_j^0,$$

we have that the sequences $\{u_j(t)\}$, $\{v_j(t)\}$ converge on $\mathbf{R}$ to $u_0(t)$, $v_0(t)$, respectively, and the convergence is uniform on arbitrary compact subintervals of $\mathbf{R}$. Also, since $p(t, \lambda) \geqq 0$ for $t \in [\tau(\lambda_0), \infty) \cup (-\infty, -\tau(\lambda_0)]$ and $\lambda \in (\Lambda_1, \lambda_0]$, for the principal solutions $u_\infty(t, \lambda)$, $v_\infty(t, \lambda)$ and $u_{-\infty}(t, \lambda)$, $v_{-\infty}(t, \lambda)$ of (2.1) satisfying

$$u_\infty(\tau(\lambda_0), \lambda) = 1, \quad u_{-\infty}(-\tau(\lambda_0), \lambda) = 1 \quad \text{for } \lambda \in (\Lambda_1, \lambda_0],$$

we have that $v_\infty(\tau(\lambda_0), \lambda) = w_\infty(\tau(\lambda_0), \lambda)$ and $v_{-\infty}(-\tau(\lambda_0), \lambda) = w_{-\infty}(-\tau(\lambda_0), \lambda)$ are continuous functions of λ on $(\Lambda_1, \lambda_0]$. For a proof of this result stated precisely in these terms, the reader is referred to Reid [11, Th. 3.2, Cor.]. For the case of $r(t, \lambda) \equiv 1$ and $p(t, \lambda)$ continuous in (t, λ) on $\mathbf{R} \times \Delta$, this result is a consequence of Hartman [3, Chap. XI, Cor. 6.6].

As a first step in the proof of Theorem 2.1, for a given $\lambda_0 \in \Delta$ satisfying $\lambda_0 > l_m$ let $\tau_0 = \tau(\lambda_0)$ and $\kappa_0 = \kappa(\lambda_0)$ be determined as in (2.11). In view of the monotoneity property of ($\mathfrak{H}$ (iii)) we have

$$(2.11') \qquad p(t, \lambda) \geqq p(t, \lambda_0) \geqq \kappa(\lambda_0) \quad \text{for } \lambda \in (\Lambda_1, \lambda_0], \quad t \in (-\infty, -\tau_0] \cup [\tau_0, \infty).$$

For $\lambda \in (\Lambda_1, \lambda_0]$ let $u = u_s(t, \lambda) = u_s(t, \tau_0, \lambda)$, $v = v_s(t, \lambda) = v_s(t, \tau_0, \lambda)$ be the solution of (2.1′) determined by the corresponding boundary conditions (2.13). For $\Lambda_1 < \lambda_1 < \lambda_2 \leqq \lambda_0$ we then have

$$\begin{aligned} -v_s(\tau_0, \lambda_1) &= -u_s(\tau_0, \lambda_1) v_s(\tau_0, \lambda_1) \\ &= \int_{\tau_0}^s \{r(t, \lambda_1)[u_s'(t, \lambda_1)]^2 + p(t, \lambda_1)[u_s(t, \lambda_1)]^2\}\, dt \\ &\geqq \int_{\tau_0}^s \{r(t, \lambda_2)[u_s'(t, \lambda_1)]^2 + p(t, \lambda_2)[u_s(t, \lambda_1)]^2\}\, dt \end{aligned}$$

(cont.)

$$\geqq \int_{\tau_0}^{s} \{r(t,\lambda_2)[u_s'(t,\lambda_2)]^2 + p(t,\lambda_2)[u_s(t,\lambda_2)]^2\}\,dt,$$

where the last inequality is a consequence of the fact that in view of the disconjugacy of (2.1) on $[\tau_0, s]$ for $\lambda = \lambda_2$ the solution $u_s(t,\lambda_2)$ minimizes $J_0[\eta,\lambda_2|\tau_0,s]$ in the class of arcs $\eta \in \mathscr{D}[\tau_0,s]$ satisfying $\eta(\tau_0) = u_s(\tau_0,\lambda_2)$, $\eta(s) = u_s(s,\lambda_2)$ (see, for example, Reid [10, Chap. V, Cor. 2 of Th. 3.1]). Moreover, we have

$$\int_{\tau_0}^{s} \{r(t,\lambda_2)[u_s'(t,\lambda_2)]^2 + p(t,\lambda_2)[u_s(t,\lambda_2)]^2\}\,dt = -u_s(\tau_0,\lambda_2)v_s(\tau_0,\lambda_2)$$

$$= -v_s(\tau_0,\lambda_2),$$

and consequently $-v_s(\tau_0,\lambda_1) \geqq -v_s(\tau_0,\lambda_2)$ for $\Lambda_1 < \lambda_1 < \lambda_2 \leqq \lambda_0$. Finally, since $v_s(\tau_0,\lambda_\alpha) \to v_\infty(\tau_0,\lambda_\alpha) = w_\infty(\tau_0,\lambda_\alpha)$ as $s \to \infty$, we have

$$-w_\infty(\tau_0,\lambda_1) \geqq -w_\infty(\tau_0,\lambda_2) \quad \text{for } \Lambda_1 < \lambda_1 < \lambda_2 \leqq \lambda_0.$$

In a similar fashion, we may establish that

$$w_{-\infty}(-\tau_0,\lambda_1) \geqq w_{-\infty}(-\tau_0,\lambda_2) \quad \text{for } \Lambda_1 < \lambda_1 < \lambda_2 \leqq \lambda_0.$$

Now, for $\eta \in \mathscr{D}[-\tau_0,\tau_0]$ and $\lambda \in \Delta$, define

$$J[\eta;\lambda] = w_{-\infty}(-\tau_0,\lambda)\eta^2(-\tau_0) - w_\infty(\tau_0,\lambda)\eta^2(\tau_0) + J_0[\eta;\lambda| -\tau_0,\tau_0] \quad \text{for } \lambda \in (\Lambda_1,\lambda_0]; \tag{2.15'}$$

$$J[\eta;\lambda] = w_{-\infty}(-\tau_0,\lambda_0)\eta^2(-\tau_0) - w_\infty(\tau_0,\lambda_0)\eta^2(\tau_0) + J_0[\eta;\lambda| -\tau_0,\tau_0] \quad \text{for } \lambda \in (\lambda_0,\Lambda_2). \tag{2.15''}$$

The functional $J[\eta;\lambda]$ specified by (2.15′), (2.15″) is clearly of the form (2.3) with $a = -\tau_0$, $b = \tau_0$, and

$$\begin{aligned} &\beta_a(\lambda) = w_{-\infty}(-\tau_0,\lambda), \quad \beta_b(\lambda) = -w_\infty(\tau_0,\lambda) \quad \text{for } \lambda \in (\Lambda_1,\lambda_0],\\ &\beta_a(\lambda) = \beta_a(\lambda_0), \quad \beta_b(\lambda) = \beta_b(\lambda_0) \quad \text{for } \lambda \in (\lambda_0,\Lambda_2). \end{aligned} \tag{2.16}$$

Consequently, in view of Lemma 2.1 the corresponding Sturmian boundary problem (2.4) has an infinite sequence of proper values (2.5) satisfying the conclusions (a)–(d) of that lemma. Moreover, for $\lambda \in (\Lambda_1,\lambda_0]$ the boundary conditions of this problem (2.4) specify that the proper solutions are multiples of $u_\infty(t,\lambda)$ and $u_{-\infty}(t,\lambda)$ on the respective intervals $[\tau_0,\infty)$, $(-\infty,-\tau_0]$, so that for such values of λ the boundary condition (2.2) is equivalent to the two-point boundary conditions of this associated problem (2.4).

As λ_0 has been chosen so that any solution of (2.1) for $\lambda = \lambda_0$ has at least m zeros on I_0, and $I_0 \subset (-\tau_0,\tau_0)$, we have that $\mu_m(\lambda_0) < 0$. Since each of the proper values $\mu_j(\lambda)$ is a strictly decreasing continuous function of λ on Δ, and $\mu_1(\lambda) > 0$ for $\lambda \in (\Lambda_1,\hat{\lambda})$, it then follows that for $j = 1,\cdots,m$ there is a unique value $\lambda_j \in \Delta$ such that $\mu_j(\lambda_j) = 0$. Moreover, in view of the inequalities (2.5), we have that

$$\Lambda_1 < \lambda_1 < \lambda_2 < \cdots < \lambda_m \leqq \lambda_0,$$

and for $j = 1,2,\cdots,m$ the proper solution $u_j(t) = u_j(t;\lambda_j)$ of the associated Sturmian boundary problem (2.4) is a solution of (2.1), (2.2) which has exactly $j-1$ zeros on $\mathbf{R}$.

In view of the above result, for a given $\lambda_0 \in \Delta$ there are only a finite number of proper values λ of (2.1), (2.2) on the interval $(-\Lambda_1, \lambda_0]$, and consequently for the sequence of proper values $\{\lambda_j\}$ determined above we have that $\lambda_j \to \Lambda_2$ as $j \to \infty$.

It is to be emphasized, however, that there is no condition on the order of growth of the λ_j specified by the hypotheses of Theorem 2.1. Indeed, let $b(\lambda)$ be any continuous, positive monotone increasing function of λ on $[0, \infty)$ such that $b(\lambda) \to \infty$ as $\lambda \to \infty$, and for $\Delta = (0, \infty)$ define r, p or $\mathbf{R} \times \Delta$ by:

$$(2.17) \qquad r(t, \lambda) \equiv 1, \quad p(t, \lambda) = \begin{cases} -\lambda^2 & \text{for } t \in [-b(\lambda), b(\lambda)], \\ 1 & \text{for } t \in (-\infty, -b(\lambda)) \cup (b(\lambda), \infty). \end{cases}$$

In this case, for $\lambda \in (0, \infty)$ one may choose $\tau(\lambda) = b(\lambda)$. Moreover, one may verify readily that $u_\infty(t, \lambda) = \exp\{-t + b(\lambda)\}$, $u_{-\infty}(t, \lambda) = \exp\{t + b(\lambda)\}$, and the corresponding distinguished solutions of (2.12) are $w_\infty(t, \lambda) \equiv -1$, $w_{-\infty}(t, \lambda) \equiv 1$. Hence $u(t)$ is a proper solution of the corresponding system (2.1), (2.2) if and only if

$$(2.18) \qquad \begin{aligned} u''(t) + \lambda^2 u(t) &= 0 \quad \text{for } t \in [-b(\lambda), b(\lambda)], \\ u(-b(\lambda)) - u'(-b(\lambda)) &= 0, \\ u(b(\lambda)) + u'(b(\lambda)) &= 0. \end{aligned}$$

By elementary computation it may be verified that $\lambda \in \Delta$ is a proper value of (2.18) if and only if it is a root of the equation

$$(2.19) \qquad \operatorname{ctn}[2\lambda b(\lambda)] = \frac{1}{2}\left\{\lambda - \frac{1}{\lambda}\right\}.$$

An associated proper function of (2.18) is then a nonzero multiple of

$$(2.20) \qquad \lambda^{-1} \sin(\lambda[t + b(\lambda)]) + \cos(\lambda[t + b(\lambda)]),$$

and a corresponding proper solution of (2.1), (2.2) is a nonzero multiple of the function $u(t, \lambda)$ defined as

$$\begin{aligned} u(t, \lambda) &= \exp\{t + b(\lambda)\} \quad (t \in (-\infty, -b(\lambda))) \\ &= \lambda^{-1} \sin(\lambda[t + b(\lambda)]) + \cos(\lambda[t + b(\lambda)]) \quad (t \in [-b(\lambda), b(\lambda)]) \\ &= \{\lambda^{-1} \sin(2\lambda b(\lambda)) + \cos(2\lambda b(\lambda))\} \exp\{-t + b(\lambda)\} \quad (t \in (b(\lambda), \infty)). \end{aligned}$$

In particular, if $\{\lambda_j\}$ is any monotone increasing sequence of positive numbers which tends to ∞ as $j \to \infty$ there exists a continuous positive monotone increasing function of λ on $[0, \infty)$ such that $b(\lambda) \to \infty$ as $\lambda \to \infty$ and $\lambda = \lambda_j$ is the jth root of the equation (2.19).

In the above example the function $p(t, \lambda)$ defined in (2.17) is discontinuous along the curves $t = b(\lambda)$ and $t = -b(\lambda)$ in (t, λ)-space. However, by suitably modifying $p(t, \lambda)$ "near" these curves one may obtain a boundary problem with continuous $p(t, \lambda)$ that exhibits the same type of phenomenon as illustrated by the presented example.

3. A Sturmian problem on an infinite interval involving "turning points." Let $g(t, \lambda)$ be a real-valued function defined for $(t, \lambda) \in \mathbf{R} \times \mathbf{R}^+$ which satisfies the following conditions:

(a) For $\lambda_0 \in \mathbf{R}^+$, $g(t, \lambda_0)$ is locally of class $\mathfrak{L}^\infty$ on $\mathbf{R}$; moreover, for t.a.e. on $\mathbf{R}$ this function is continuous in λ at λ_0.

(b) $g(t, 0) = 0$.

(c) For $t \in (-1, 1)$, g is monotone nonincreasing in λ on R^+, and the functional

$$\int_{-1}^{1} \{\eta'^2(t) + g(t, \lambda)\eta^2(t)\}\, dt$$

($\mathfrak{H}'$) is a strictly monotone decreasing function of λ for arbitrary $\eta(t) \not\equiv 0$ belonging to the class $\mathscr{D}[-1, 1]$.

(d) For $t \in (-\infty, -1) \cup (1, \infty)$, g is monotone nondecreasing in λ on $\mathbf{R}^+$.

(e) There exists a subinterval $[a, b] \subset (-1, 1)$ such that $\lim_{\lambda \to \infty} g(t, \lambda) = -\infty$ for t a.e. on $[a, b]$.

(f) For $\lambda \in (0, \infty)$ there exists a $\tau(\lambda) \geqq 1$ and a $\kappa(\lambda) > 0$ such that $g(t, \lambda) \geqq \kappa(\lambda)$ for $t \in (-\infty, -\tau(\lambda)) \cup (\tau(\lambda), \infty)$.

We shall be concerned with the boundary problem

$$u''(t) - g(t, \lambda)u(t) = 0, \tag{3.1}$$

$$\int_{-\infty}^{\infty} u^2(t)\, dt < \infty, \tag{3.2}$$

for which we shall establish the following theorem.

THEOREM 3.1. *Under the hypotheses* ($\mathfrak{H}'$ (a)–(f)) *the real proper values of the system* (3.1), (3.2) *comprise a sequence of disjoint subsets* Λ_j, $j = 1, 2, \cdots$, *of* $\mathbf{R}^+$ *such that if* $\lambda \in \Lambda_j$ *then there is a corresponding proper solution* $u = u(t, \lambda)$ *of* (3.1), (3.2) *which has exactly* $j - 1$ *zeros on* $\mathbf{R}$, *and these zeros all occur on the open interval* $(-1, 1)$. *Moreover,* $\{\Lambda_j\} \to \infty$ *in the sense that for each* $\hat{\lambda} \in \mathbf{R}^+$ *there exists an integer* $k = k(\hat{\lambda})$ *such that* $\Lambda_j \subset (\hat{\lambda}, \infty)$ *for* $j \geqq k$.

In view of hypotheses ($\mathfrak{H}'$ (a), (b), (d)), for arbitrary $\lambda \in \mathbf{R}^+$ the differential equation (3.1) is disconjugate on each of the intervals $[1, \infty)$ and $(-\infty, -1]$, and consequently for each such λ at these respective endpoints there exist principal solutions $u_\infty(t, \lambda)$, $u_{-\infty}(t, \lambda)$ which are different from zero on the respective intervals $[1, \infty)$, $(-\infty, -1]$, and corresponding distinguished solutions

$$w_\infty(t, \lambda) = u'_\infty(t, \lambda)/u_\infty(t, \lambda), \quad w_{-\infty}(t, \lambda) = u'_{-\infty}(t, \lambda)/u_{-\infty}(t, \lambda)$$

of the associated Riccati differential equation. Moreover, by argument similar to that employed in the preceding section, the functions $u_\infty(t, \lambda)$, $u'_\infty(t, \lambda)$, $u_{-\infty}(t, \lambda)$, $u'_{-\infty}(t, \lambda)$ are continuous in (t, λ) on $\mathbf{R} \times \mathbf{R}^+$, and therefore $w_\infty(t, \lambda)$ and $w_{-\infty}(t, \lambda)$ are continuous in (t, λ) on $[1, \infty) \times \mathbf{R}^+$ and $(-\infty, -1] \times \mathbf{R}^+$, respectively. Also, in view of hypothesis ($\mathfrak{H}'$ (f)), a solution $u(t)$ of (3.1) is such that condition (3.2) is satisfied if and only if $u(t)$ is a multiple of $u_\infty(t, \lambda)$ on $[1, \infty)$, and a multiple of $u_{-\infty}(t, \lambda)$ on $(-\infty, -1]$. That is, $u(t)$ is a proper function of (3.1), (3.2) for a value λ if and only if $u(t)$ is a proper function corresponding to a proper value λ of the finite interval boundary problem

$$\begin{gathered} u''(t) - g(t, \lambda)u(t) = 0, \\ w_{-\infty}(-1, \lambda)u(-1) - u'(-1) = 0, \\ -w_\infty(1, \lambda)u(1) + u'(1) = 0. \end{gathered} \tag{3.3}$$

The quadratic functional associated with this finite interval boundary problem is

(3.4) $$J[\eta|\lambda] = w_{-\infty}(-1,\lambda)\eta^2(-1) - w_{\infty}(1,\lambda)\eta^2(1) + \int_{-1}^{1} \{\eta'^2(t) + g(t,\lambda)\eta^2(t)\}\, dt.$$

Now if $u(t) = u_s(t,\lambda)$ is for $s > 1$ the solution of (3.1) satisfying the end-conditions $u_s(1,\lambda) = 1, u_s(s,\lambda) = 0$, then as in the previous section we have that $u_\infty(t,\lambda) = \lim_{s\to\infty} u_s(t,\lambda)$ is the principal solution at ∞ satisfying $u(1,\lambda) = 1$, and in view of conditions ($\mathfrak{H}'$ (b), (d)) we have that the principal solution of (3.1) at ∞ thus determined is such that $u_\infty(t,\lambda) > 0$ for $t \in [1,\infty)$ and $u'_\infty(1,\lambda) = w_\infty(1,\lambda)$ is nonpositive and monotone nonincreasing in λ on $\mathbf{R}^+$. Similarly, the principal solution of (3.1) at $-\infty$ satisfying $u_{-\infty}(-1,\lambda) = 1$ is such that $u_{-\infty}(t,\lambda) > 0$ for $t \in (-\infty, -1]$, and $u'_{-\infty}(-1,\lambda) = w_{-\infty}(-1,\lambda)$ is nonnegative and monotone nondecreasing in λ on $\mathbf{R}^+$. Consequently, the boundary term

(3.5) $$w_{-\infty}(-1,\lambda)\eta^2(-1) - w_\infty(1,\lambda)\eta^2(1)$$

of (3.4) is a monotone nondecreasing function of λ for arbitrary real-valued functions $\eta(t)$, while for $\eta \in \mathfrak{A}[-1,1]$ with $\eta' \in \mathfrak{L}^2[-1,1]$ the integral term

(3.6) $$\int_{-1}^{1} \{\eta'^2(t) + g(t,\lambda)\eta^2(t)\}\, dt$$

is a monotone nonincreasing function of λ, so that the functional (3.4) does not possess the monotone properties requisite to insure the classical Sturmian theory for the boundary problem (3.3). As λ increases on $\mathbf{R}^+$ the functional (3.6) tends to attract zeros of solutions of (3.1) into the interval $[-1,1]$, whereas the functional (3.5) tends to repulse zeros of solutions away from this interval.

As in the case of the problem treated in the preceding section, however, we shall discuss the existence of proper values of (3.3) by the consideration of the related system

(3.7) $$\begin{aligned} u''(t) - g(t,\lambda)u(t) + \sigma u(t) &= 0,\\ w_{-\infty}(-1,\lambda)u(-1) - u'(-1) &= 0,\\ -w_\infty(1,\lambda)u(1) + u'(1) &= 0, \end{aligned}$$

which is linear in the characteristic parameter σ. From standard Sturm–Liouville theory (see, for example, Reid [10, Chap. V, Th. 7.4]) it follows that for each $\lambda \in \mathbf{R}^+$ the proper values $\sigma = \sigma(\lambda)$ of (3.7) are real, may be ordered as a sequence

$$\sigma_1(\lambda) < \sigma_2(\lambda) < \cdots < \sigma_j(\lambda) < \cdots,$$

and a proper function $u = u_j(t,\lambda)$ of (3.7) corresponding to $\sigma = \sigma_j(\lambda)$ possesses exactly $j - 1$ zeros on $(-1, 1)$. Clearly $u(t)$ is a proper function of (3.3) corresponding to a proper value $\lambda = \lambda_0$ if and only if there exists a j such that $\sigma_j(\lambda_0) = 0$ and $u = u_j(t,\lambda)$ is a proper value of (3.7) corresponding to $\sigma = \sigma_j(\lambda_0)$.

As $g(t,0) \equiv 0$ by hypothesis ($\mathfrak{H}'$ (b)), for $\lambda = 0$ the equation (3.1) reduces to $u'' = 0$, so that $u_\infty(t,0) \equiv u_{-\infty}(t,0) \equiv 1$, $w_\infty(t,0) \equiv w_{-\infty}(t,0) \equiv 0$, and consequently $\sigma_1(0) = 0$. Also, in view of hypothesis ($\mathfrak{H}'$ (e)), from the extremizing properties of the proper values of (3.7) it follows that each $\sigma_j(\lambda)$ is negative for λ sufficiently large. Finally, as in the discussion of the boundary problem in § 2, the condition ($\mathfrak{H}'$ (a)) and the continuity of the functions $w_{-\infty}(-1,\lambda)$ and $w_\infty(1,\lambda)$ on $\mathbf{R}^+$ imply that each $\sigma_j(\lambda)$ is a continuous function on $\mathbf{R}^+$, and consequently the conclusion of Theorem 3.1 holds for Λ_j defined as the set of values λ such that $\sigma_j(\lambda) = 0$. As $\lim_{j\to\infty} \sigma_j(\lambda) = \infty$ for each $\lambda \in \mathbf{R}^+$, if $[0, \hat{\lambda}]$ is a compact subinterval of $\mathbf{R}^+$, and k is an integer such that $\sigma_k(\hat{\lambda}) > 0$, then $\sigma_j(\lambda) > 0$ for $\lambda \in [0, \hat{\lambda}]$ and $j = k, k+1, \cdots$, so that $\Lambda_j \subset (\hat{\lambda}, \infty)$ for $j \geqq k$.

It is to be noted that one can estimate the position of the sets Λ_j through the consideration of a second related boundary problem

$$\begin{aligned} u'' - g(t,\lambda)u(t) + \nu u(t) &= 0, \\ -w_{-\infty}(-1,\lambda)u(-1) - u'(-1) &= 0, \\ w_{\infty}(1,\lambda)u(1) + u'(1) &= 0, \end{aligned} \tag{3.8}$$

which is linear in the characteristic parameter ν. For (3.8) the related functional is

$$J_0[\eta|\lambda] = -w_{-\infty}(-1,\lambda)\eta^2(-1) + w_{\infty}(1,\lambda)\eta^2(1) + \int_{-1}^{1} \{\eta'^2(t) + g(t,\lambda)\eta^2(t)\}\, dt, \tag{3.9}$$

which is a monotone strictly decreasing function of λ for each nonidentically vanishing $\eta \in \mathscr{D}[-1,1]$. Consequently, the proper values $\nu = \nu(\lambda)$ of (3.8) are real, may be ordered as a sequence

$$\nu_1(\lambda) < \nu_2(\lambda) < \cdots < \nu_j(\lambda) < \cdots, \tag{3.10}$$

and a proper function of (3.8) corresponding to a proper value $\nu = \nu_j(\lambda)$ has exactly $j-1$ zeros on $(-1,1)$. Also, each proper value $\nu_j(\lambda)$ is a strictly monotone decreasing function of λ on $\mathbf{R}^+$ and $\nu_1(0) = 0$. In particular, the real proper values of the system

$$\begin{aligned} u''(t) - g(t,\lambda)u(t) &= 0, \\ -w_{-\infty}(-1,\lambda)u(-1) - u'(-1) &= 0, \\ w_{\infty}(1,\lambda)u(1) + u'(1) &= 0, \end{aligned} \tag{3.11}$$

may be ordered as a sequence

$$\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_j < \cdots, \tag{3.12}$$

where each $\hat{\lambda}_j$ has the definitive property of being the unique value $\hat{\lambda}$ on $\mathbf{R}^+$ such that $\nu_j(\hat{\lambda}) = 0$, and a proper solution of (3.11) corresponding to $\lambda = \hat{\lambda}_j$ has exactly $j-1$ zeros on $(-1,1)$.

Now, as noted above, for $\lambda \in \mathbf{R}^+$ we have $w_{\infty}(1,\lambda) \leqq 0$ and $w_{-\infty}(-1,\lambda) \geqq 0$, so that the functionals $J[\eta|\lambda]$ and $J_0[\eta|\lambda]$ defined by (3.4) and (3.9) satisfy the inequality

$$J[\eta|\lambda] \geqq J_0[\eta|\lambda] \quad \text{for } \eta \in \mathscr{D}[-1,1]. \tag{3.13}$$

Therefore, by classical comparison theorems (see, for example, Reid [10, Chap. VI, § 4]), we have

$$\nu_j(\lambda) \leqq \sigma_j(\lambda) \leqq \nu_{j+2}(\lambda), \qquad j = 1, 2, \cdots, \quad \lambda \in \mathbf{R}^+. \tag{3.14}$$

Consequently, the set $\Lambda_j = \{\lambda | \lambda \in \mathbf{R}^+, \sigma_j(\lambda) = 0\}$ must lie in the compact interval $[\hat{\lambda}_j, \hat{\lambda}_{j+2}]$.

Added in proof. Subsequent to the completion of the present paper and the associated one listed as [11] in the References, Philip Hartman has considered a related problem. His paper, *Boundary value problems for second order, ordinary differential equations involving a parameter*, has appeared in the Journal of Differential Equations, 12 (1972), pp. 194–212. In particular, for second order linear equations he has presented some results on principal solutions in more detail than I have given, as well as an extension of the principal existence theorem to some more general nonlinear problems.

REFERENCES

[1] W. Harris and Y. Sibuya, *Asymptotic distribution of eigenvalues*, Abstract 683-B24, Notices Amer. Math. Soc., 18 (1971); Proc. NRL-MRC Conference on Ordinary Differential Equations. (June 14–23, 1971, Washington, D.C.), Academic Press, New York, 1972, pp. 121–125.

[2] P. Hartman, *Self-adjoint, nonoscillatory systems of ordinary, second-order, linear differential equations*, Duke Math. J., 24 (1957), pp. 25–35.

[3] ———, *Ordinary Differential Equations*, John Wiley, New York, 1964.

[4] P. Hartman and A. Wintner, *On the assignment of asymptotic values for the solutions of linear differential equations of the second order*, Amer. J. Math., 77 (1955), pp. 475–483.

[5] E. L. Ince, *Ordinary Differential Equations*, Longmans, Green and Co., London, 1927.

[6] W. Leighton and M. Morse, *Singular quadratic functionals*, Trans. Amer. Math. Soc., 40 (1936), pp. 252–286.

[7] W. E. Milne, *The behavior of a boundary problem as the interval becomes infinite*, Ibid., 30 (1928), pp. 797–802.

[8] W. T. Reid, *Principal solutions of non-oscillatory self-adjoint linear differential systems*, Pacific J. Math., 8 (1958), pp. 147–169.

[9] ———, *Some limit theorems for ordinary differential systems*, J. Differential Equations, 3 (1967), pp. 423–439.

[10] ———, *Ordinary Differential Equations*, John Wiley, New York, 1971.

[11] ———, *A continuity property of principal solutions of linear Hamiltonian systems*, Scripta Math., to appear.

[12] H. Weyl, *Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen*, Göttingen Nachr., 1909, pp. 37–63; 1910, pp. 442–467.

[13] ———, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklung willkurlicher Funktionen*, Math. Ann., 68 (1910), pp. 220–269.

[14] K. O. Wolfson, *On the spectrum of a boundary value problem with two singular endpoints*, Amer. J. Math., 72 (1950), pp. 713–719.

SIAM J. MATH. ANAL.
Vol. 4, No. 1, February 1973

GENERALIZED CAUCHY AND POISSON INTEGRALS AND DISTRIBUTIONAL BOUNDARY VALUES*

RICHARD D. CARMICHAEL†

Abstract. Let C be an open connected cone, and let $O(C)$ denote its convex envelope. Cauchy and Poisson integrals of distributions in $\mathscr{D}'_{L^p}$, $1 < p \leqq 2$, corresponding to tubular radial domains $T^{O(C)} = \mathbb{R}^n + iO(C)$ are defined; and properties of these integrals are obtained. The boundary values of these integrals are obtained in the distributional sense on the distinguished boundary of $T^{O(C)}$. Functions which are analytic and have a specified growth condition in $T^{O(C)}$ are related to the Cauchy and Poisson integrals of their distributional boundary values. The results concerning these functions extend some well-known theorems concerning the Hardy $H^p(T^{O(C)})$-spaces to our distributional setting. Further, functions which are analytic in disconnected tubular cones are considered; and in particular conditions are obtained under which such a function has an analytic extension to the convex envelope of the tubular cone.

1. Introduction. The concept of distributional boundary values in $\mathscr{D}'_{L^p}$ has been introduced by Tillman [1]. He has characterized functions which are analytic in an octant $B_\delta = \{z \in \mathbb{C}^n : \delta_j(\operatorname{Im}(z_j)) > 0, \delta = (\delta_1, \cdots, \delta_n), \delta_j = \pm 1, j = 1, \cdots, n\}$ and which have distributional boundary values in the topology of $\mathscr{D}'_{L^p}$ using the "Indikatrix" of $U \in \mathscr{D}'_{L^p}$, $1 < p < \infty$, which is the analytic function

$$f(z) = (2\pi i)^{-n} \left\langle U_t, \prod_{j=1}^{n} \frac{1}{z_j - t_j} \right\rangle, \qquad \operatorname{Im}(z_j) \neq 0, \quad j = 1, \cdots, n.$$

Tillman proves that the "Indikatrix" satisfies the following properties:

$$(1) \qquad |f(z)| \leqq M \prod_{j=1}^{n} (\eta_j^{-1/q} + \eta_j^{-1/q-m}), \qquad \eta_j = |\operatorname{Im}(z_j)|, \quad \frac{1}{p} + \frac{1}{q} = 1;$$

$$(2) \qquad \left\{ f_\varepsilon(x) = \sum_{\delta} \prod_{j=1}^{n} \delta_j f_\delta(x + i\varepsilon\delta) \right\}_{\varepsilon > 0} \text{ is bounded in } \mathscr{D}'_{L^p};$$

$$(3) \qquad f_\varepsilon \to U \text{ in the topology of } \mathscr{D}'_{L^p} \text{ as } \varepsilon \to 0.$$

Furthermore, Tillmann obtains the converse that if $f(z)$ is analytic and (1) and (2) hold, then $f(z)$ is the Indikatrix of a distribution $U \in \mathscr{D}'_{L^p}$; thus (3) holds for this U. Luszcyki and Zieleźny [2] have obtained results similar to Tillmann but for only one dimension.

Beltrami and Wohlers [3], [4], [5], [6] have obtained results in which an element $U \in \mathscr{D}'_{L^2}$ is the boundary value of an analytic function from the upper or lower half-plane. The topology used is that of $\mathscr{S}'$; furthermore, they confine their attention to the space $\mathscr{D}'_{L^2}$ and work in only one dimension. By using the $\mathscr{S}'$ topology and thus considering $\mathscr{D}'_{L^2}$ as a subspace of $\mathscr{S}'$, Beltrami and Wohlers are able to associate the generalized Cauchy integral (Indikatrix) with analytic functions in the upper (lower) half-plane which satisfy

$$(4) \qquad |f(z)| \leqq C_\gamma(1 + |z|)^m, \qquad \operatorname{Im}(z) \geqq \gamma > 0,$$

a more general boundedness condition than that of Tillmann. Also available in

* Received by the editors November 15, 1971.

† Department of Mathematics, Wake Forest University, Winston-Salem, North Carolina 27109.

this setting is the use of the Fourier transform in $\mathscr{S}'$ with which Beltrami and Wohlers show that the generalized Cauchy integral of $U \in \mathscr{D}'_{L^2}$ equals the Fourier–Laplace transform of an element $V \in \mathscr{S}'$ having support in a half-line. A generalized Poisson integral for $U \in \mathscr{D}'_{L^2}$ has been defined by Beltrami and Wohlers [5, pp. 70–71], and this generalized integral has been related to analytic functions having growth as in (4).

In Carmichael [7] we considered $\mathscr{D}'_{L^p}$ as a subspace of $\mathscr{S}'$ and generalized several results of Beltrami and Wohlers concerning Cauchy and Poisson integrals to octants and to distributions $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Further results concerning distributional boundary values in $\mathscr{D}'_{L^p}$ considered as a subspace of $\mathscr{S}'$ were obtained in Carmichael [8], [9]; however, in [8], [9] we were concerned with Cauchy or Poisson integrals.

In the present paper we define and obtain properties of a generalized Cauchy integral (Indikatrix) and a generalized Poisson integral for distributions $U \in \mathscr{D}'_{L^p}$ corresponding to tube domains $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$, where C is an open connected cone, of which the half-plane in $\mathbb{C}^1$ and the octant in $\mathbb{C}^n$ are examples. These integrals (and their properties) have as special cases the Indikatrix of Tillmann and Beltrami, Wohlers and the Poisson integrals defined by Beltrami, Wohlers in 1 dimension and by Carmichael in n dimensions. In fact the present definitions of the Cauchy and Poisson integrals are considerably more general; for also included as special cases are integrals corresponding to such tube domains in $\mathbb{C}^n$ as the forward and backward light cones, which are important domains in quantum field theory. These results will be contained in § 3 of this paper along with some needed preliminary theorems.

In § 4 we shall prove that the generalized Cauchy and Poisson integrals of $U \in \mathscr{D}'_{L^p}$ attain boundary values in $\mathscr{D}'_{L^p}$. The boundary value theorem for the Poisson integral of $U \in \mathscr{D}'_{L^p}$ generalizes a well-known result concerning the boundary value attained by the classical Poisson integral corresponding to tube domains of L^p-functions.

We relate the Cauchy and Poisson integrals of $U \in \mathscr{D}'_{L^p}$ in tube domains to a space of analytic functions which will be denoted as G^b_C (see § 3) and whose elements satisfy a more general boundedness condition than that of Tillmann or Beltrami and Wohlers (recall (1) and (4)). Our theorems concerning the distributional boundary values of functions in G^b_C (and in a special case of G^b_C) and the relation of the functions to the Cauchy and Poisson integrals of the boundary values, which are elements of $\mathscr{D}'_{L^p}$, are of importance because they may be viewed as generalizations of known results concerning the Hardy H^p-spaces in tube domains. In our theorems the set of functions having boundary values and representable by Cauchy and Poisson integrals of the boundary values is enlarged from the H^p-spaces to the space G^b_C ($H^p \subset G^b_C$ for all $p, 1 \leqq p \leqq \infty$) and the set of admissible boundary values is enlarged from the L^p-functions to the $\mathscr{D}'_{L^p}$-distributions ($L^p \subset \mathscr{D}'_{L^p}$.) We accomplish this without altering the structure of the classical H^p setting; however, we must replace pointwise and norm convergence by weak convergence. Further, some of our theorems concerning boundary values of elements in the space G^b_C have as special cases results obtained by Tillmann, Carmichael, and Beltrami, Wohlers; but again the theorems presented here are considerably more general.

In § 5 we obtain theorems concerning functions analytic in tubular cones $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$, where the cone C is not necessarily connected. We are especially interested in obtaining conditions under which a function analytic in T^C has an analytic extension to $T^{O(C)}$, the convex envelope (hull) of T^C.

Topics which will be considered in future research will be briefly discussed in § 6. To obtain most of the results of this paper, we must restrict our attention to $\mathscr{D}'_{L^p}$, $1 < p \leqq 2$.

2. Notation and definitions. The n-dimensional notation to be used in this paper will be the same as that in Carmichael [8]. In particular we recall that by D^α, α being an n-tuple of nonnegative integers, we mean $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_j = (1/2\pi i)(\partial/\partial t_j)$ or $D_j = (1/2\pi i)(\partial/\partial z_j)$, $j = 1, \cdots, n$. We put D_t^α or D_z^α to distinguish between differentiating on the real variable $t = (t_1, \cdots, t_n)$ or the complex variable $z = (z_1, \cdots, z_n)$ whenever there is a possibility of confusion. The Fourier transform for L^1-functions ϕ is defined as in [8] and is denoted by $\hat{\phi}(x)$ or $\mathscr{F}[\phi(t); x]$. The inverse Fourier transform of $\phi(t)$ will be denoted $\mathscr{F}^{-1}[\phi(t); x]$.

For the definitions of the function spaces $\mathscr{E}$, $\mathscr{S}$, and $\mathscr{D}_{L^p}$ and the distribution spaces $\mathscr{S}'$ and $\mathscr{D}'_{L^p}$ we refer to Schwartz [10]. We recall that $\mathscr{D}'_{L^p}$, $1 < p \leqq \infty$, is the dual space of $\mathscr{D}_{L^q}$, $1/p + 1/q = 1$. We shall particularly be concerned with the notions of convergence in $\mathscr{D}_{L^q}$ and $\mathscr{S}$. A sequence of functions $\{\phi_\lambda\} \in \mathscr{D}_{L^q}$ converges to $\phi \in \mathscr{D}_{L^q}$ as $\lambda \to \lambda_0$ if

$$\lim_{\lambda \to \lambda_0} \|D^\alpha(\phi_\lambda(t) - \phi(t))\|_{L^q} = 0,$$

where α is an arbitrary n-tuple of nonnegative integers. A sequence of functions $\{\phi_\lambda\} \in \mathscr{S}$ converges to $\phi \in \mathscr{S}$ as $\lambda \to \lambda_0$ if

$$\lim_{\lambda \to \lambda_0} \sup_t |t^\beta D^\alpha(\phi_\lambda(t) - \phi(t))| = 0,$$

where α and β are arbitrary n-tuples of nonnegative integers.

For all terminology concerning distributions we refer to Schwartz [10]. We recall that the convolution of two distributions U and V is defined, when it exists, by

$$\langle U * V, \phi \rangle = \langle U_\xi, \langle V_\eta, \phi(\xi + \eta) \rangle \rangle,$$

where ϕ is an element of the appropriate function space. The Fourier transform is a continuous isomorphism of $\mathscr{S}$ onto $\mathscr{S}$ with the same being true of $\mathscr{S}'$ under the definition

$$\langle \hat{U}, \phi \rangle = \langle U, \hat{\phi} \rangle, \qquad U \in \mathscr{S}', \quad \phi \in \mathscr{S}.$$

The support of a distribution U will be denoted by supp (U) with the same notation for the support of a function.

The definitions of cone and compact subcone are the same as in [8]. If C is a cone, $O(C)$ will denote the convex envelope (hull) of C; and $T^C = \mathbb{R}^n + iC$ will be called a tubular radial domain if C is a connected cone. If C is not connected, then T^C will be called a tubular cone. The function

$$u_C(t) = \sup_{y \in \operatorname{pr} C} (-\langle t, y \rangle)$$

is the Indicatrix of C. (pr C denotes the projection of C, which is the intersection of C with the unit sphere.) $C^* = \{t : u_C(t) \leqq 0\} = \{t : \langle t, y\rangle \geqq 0,\ y \in C\}$ is the dual cone of C; and the number

$$\rho_C = \sup_{t \in C_*} \frac{u_{O(C)}(t)}{u_C(t)}, \qquad C_* = \mathbb{R}^n \backslash C^*,$$

characterizes the nonconvexity of the cone C.

Let C be an open connected cone. Let $f(z)$ be a function of $z \in T^C \subset \mathbb{C}^n$, and let U be a distribution. By $f(z) \to U$ in the topology (i.e., weak topology) of the distribution space as $y = \operatorname{Im}(z) \to 0$ (i.e., $y_j \to 0$, $j = 1, \cdots, n$), $y \in C$, we mean $\langle f(z), \phi(x)\rangle \to \langle U, \phi(x)\rangle$ as $y \to 0$, $y \in C$, where ϕ is an element of the appropriate function space. U is then called the boundary value of $f(z)$; and we note that it is defined on the distinguished boundary of T^C, $\{z = x + iy : x \in \mathbb{R}^n, y = (0, \cdots, 0)\}$, which is not necessarily the topological boundary of T^C.

Throughout this paper L^p denotes $L^p(\mathbb{R}^n)$, the equivalence class of Lebesgue measurable functions over $\mathbb{R}^n$ whose pth power is absolutely Lebesgue integrable.

3. Generalized Cauchy and Poisson integrals and preliminary theorems. Throughout this section C will denote an open connected cone. Consider

$$K(z - t) = \int_{C^*} \exp(2\pi i\langle z - t, \eta\rangle)\, d\eta,$$

where $C^* = \{\eta : u_C(\eta) \leqq 0\}$ is the dual cone of C and z is an arbitrary but fixed point in $T^{O(C)} = \mathbb{R}^n + iO(C)$. We call $K(z - t)$ the Cauchy kernel function and note that kernel functions similar to $K(z - t)$ were first studied by Bochner [11]. If $\overline{O(C)}$ contains an entire straight line, then by a result of Vladimirov [12, Lemma 1, p. 222] the cone C^* lies in some $(n - 1)$-dimensional plane. Thus C^* has measure zero, and $K(z - t) = 0$. To avoid this triviality when working with the generalized Cauchy and Poisson integrals, we assume throughout this paper that all open connected cones C are such that $\overline{O(C)}$ contains no entire straight line.

THEOREM 1. *$K(z - t) \in \mathscr{D}_{L^q}$, $1/p + 1/q = 1$, $1 < p \leqq 2$, as a function of t for fixed $z \in T^{O(C)}$.*

Proof. Let α be an arbitrary but fixed n-tuple of nonnegative integers. For $z = x + iy$ fixed in $T^{O(C)}$, it follows from a result of Vladimirov [12, Lemma 2, p. 223] that there exists a real number $\sigma > 0$ such that $\langle \eta, y\rangle \geqq \sigma|\eta|$ for all $\eta \in C^*$. From this and a well-known result concerning Lebesgue integrals (see Schwartz [13, Theorem 32, p. 39]) we have

$$\left|(-1)^{|\alpha|} \int_{C^*} \eta^\alpha e^{2\pi i\langle z - t, \eta\rangle} d\eta\right| \leqq \int_{C^*} |\eta^\alpha|\, e^{-2\pi\sigma|\eta|}\, d\eta \tag{5}$$
$$\leqq S_n \int_0^\infty r^{|\alpha| + \eta - 1}\, e^{-2\pi\sigma r}\, dr < \infty,$$

where S_n is the area of the unit sphere in $\mathbb{R}^n$. By (5),

$$D_t^\alpha K(z - t) = (-1)^{|\alpha|} \int_{C^*} \eta^\alpha e^{2\pi i\langle z - t, \eta\rangle}\, d\eta \tag{6}$$

exists, where the integral converges uniformly with respect to t. Thus $K(z - t) \in C^\infty$

since α is arbitrary. It remains to show that $D_t^\alpha K(z-t) \in L^q$, $1/p + 1/q = 1$, $1 < p \leqq 2$; arguing as in (5), we have that $(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle} \in L^1 \cap L^p$, $1 < p \leqq 2$, for fixed $z \in T^{O(C)}$, where $I_{C^*}(\eta)$ is the characteristic function of C^*. From (6), $D_t^\alpha K(z-t) = \mathscr{F}^{-1}[(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}; t]$; and the Fourier inverse transform can be interpreted as a limit in the mean. Thus by a well-known result from Fourier transform theory, we have

$$\|D_t^\alpha K(z-t)\|_{L^q} \leqq \|(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}\|_{L^p} < \infty;$$

and $D_t^\alpha K(z-t) \in L^q$ as desired. The proof is complete.

We note that if the tubular radial domain T^C is either the upper half-plane in $\mathbb{C}^1$ or the octant $B_{(1,\cdots,1)}$ in $\mathbb{C}^n$, then $K(z-t)$ becomes $(2\pi i)^{-1}[1/(t-z)]$ or $(2\pi i)^{-n}\prod_{j=1}^n [1/(t_j - z_j)]$, respectively. These are the classical Cauchy kernels.

Now let $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$; and let z be an arbitrary but fixed point in $T^{O(C)}$. Put

$$C(U; z) = \langle U_t, K(z-t)\rangle. \tag{7}$$

From Theorem 1 we see that $C(U; z)$ is well-defined; and if T^C is the upper (lower) half-plane in $\mathbb{C}^1$ or an octant in $\mathbb{C}^n$, then $C(U;z)$ is Tillmann's Indikatrix. We prove that $C(U; z)$ is an analytic function and satisfies a certain boundedness condition.

THEOREM 2. *Let $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Then $C(U; z)$ is analytic in $T^{O(C)}$; and for any compact subcone C' of $O(C)$,*

$$|C(U; z)| \leqq \frac{K(C')}{|y|^{n/p}} + \sum_{0<|\alpha|\leqq m} \frac{Q(C'; \alpha)}{|y|^{d_\alpha/p}}, \tag{8}$$

where $K(C')$ and $Q(C'; \alpha)$ are constants which depend on C' and C' and α, respectively, m is a fixed positive integer depending on U, and d_α is a positive integer depending on α.

Proof. By the characterization theorem of Schwartz [10, Theorem 25, p. 201], $U = \sum_{|\alpha|\leqq m} D^\alpha f_\alpha(t)$, $f_\alpha(t) \in L^p$, $1 < p \leqq 2$. Thus

$$\begin{aligned} C(U; z) &= \Big\langle \sum_{|\alpha|\leqq m} D^\alpha f_\alpha(t), K(z-t)\Big\rangle \\ &= \sum_{|\alpha|\leqq m} (-1)^{2|\alpha|} \int_{\mathbb{R}^n} f_\alpha(t) \int_{\mathbb{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z-t,\eta\rangle}\, d\eta\, dt. \end{aligned} \tag{9}$$

Now consider a fixed α, $|\alpha| \leqq m$. Let z_0 be an arbitrary but fixed point in $T^{O(C)}$, and suppose $N(z_0, \delta)$ is a neighborhood of z_0 of radius δ such that the closure of $N(z_0, \delta)$ is a proper subset of $T^{O(C)}$. Let z be an arbitrary point of $N(z_0, \delta)$; and suppose β is an arbitrary but fixed n-tuple of nonnegative integers. Using an argument as in (5) we have that $I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z,\eta\rangle} \in L^1 \cap L^p$, $1 < p \leqq 2$. Further,

$$\int_{\mathbb{R}^n} I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z-t,\eta\rangle}\, d\eta = \mathscr{F}^{-1}[I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z,\eta\rangle}; t],$$

and

$$\|\mathscr{F}^{-1}[I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z,\eta\rangle}; t]\|_{L^q} \leqq \|I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z,\eta\rangle}\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{10}$$

Using Hölders inequality, (10), and an estimate as in (5), we obtain

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^n} f_\alpha(t) \int_{\mathbb{R}^n} I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \, dt \right| \\
(11) \qquad &\leqq \|f_\alpha\|_{L^p} \|\mathscr{F}^{-1}[I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z,\eta\rangle}; t]\|_{L^q} \\
&\leqq \|f_\alpha\|_{L^p} \|I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z,\eta\rangle}\|_{L^p} \\
&\leqq \|f_\alpha\|_{L^p} \left(S_n \int_0^\infty r^{|p\alpha+p\beta|+n-1} e^{-2\pi p\sigma r} \, dr \right)^{1/p} < \infty.
\end{aligned}
$$

Thus for each fixed α, $|\alpha| \leqq m$, and any fixed β,

$$
\begin{aligned}
D_z^\beta &\int_{\mathbb{R}^n} f_\alpha(t) \int_{\mathbb{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \, dt \\
&= \int_{\mathbb{R}^n} f_\alpha(t) \int_{\mathbb{R}^n} I_{C^*}(\eta)\eta^{\alpha+\beta} e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \, dt;
\end{aligned}
$$

and by (11), the differentiated integral converges uniformly with respect to $z \in N(z_0, \delta)$. From this and the arbitrariness of the point $z_0 \in T^{O(C)}$ we have that each term in the sum on the right-hand side of (9) is analytic in $T^{O(C)}$. Thus $C(U; z)$ is analytic in $T^{O(C)}$ as desired.

It remains to prove the boundedness condition (8). Let z be an arbitrary but fixed point in $T^{C'}$, C' being an arbitrary compact subcone of $O(C)$. By the characterization theorem of Carmichael [9, Theorem 6], there exists a constant R such that if $1/p + 1/q = 1$,

$$
\begin{aligned}
|C(U; z)| &\leqq R \sum_{|\alpha| \leqq m} \left\| D_t^\alpha \int_{C^*} e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \right\|_{L^q} \\
(12) \qquad &\leqq R \sum_{|\alpha| \leqq m} \left\| \int_{\mathbb{R}^n} (-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \right\|_{L^q}.
\end{aligned}
$$

For any fixed α, $|\alpha| \leqq m$, we have as in (10) that

$$
(13) \qquad \left\| \int_{\mathbb{R}^n} (-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \right\|_{L^q} \leqq \|(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}\|_{L^p}.
$$

By the result of Vladimirov [12, Lemma 2, p. 223], there exists a real number $\sigma > 0$ depending on C' such that $\langle y, \eta\rangle \geqq \sigma|y||\eta|$, $\eta \in C^*$. From this and (13) we have

$$
\begin{aligned}
\left\| \int_{\mathbb{R}^n} (-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z-t,\eta\rangle} \, d\eta \right\|_{L^q}^p &\leqq \int_{C^*} |\eta^{p\alpha}| \, e^{-2\pi p\sigma|y||\eta|} \, d\eta \\
(14) \qquad &\leqq S_n \int_0^\infty r^{|p\alpha|+n-1} e^{-2\pi p\sigma r|y|} \, dr,
\end{aligned}
$$

where again S_n is the area of the unit sphere in $\mathbb{R}^n$. For the fixed α, $|\alpha| \leqq m$ and $\alpha \neq (0, \cdots, 0)$, we integrate by parts d_α times on the last integral in (14), where d_α is an arbitrary but fixed positive integer depending on α such that $|p\alpha| + n - 1$

$-\ d_\alpha \geqq 0$. We obtain

$$
\begin{aligned}
&S_n \int_0^\infty r^{|p\sigma|+n-1} e^{-2\pi p\sigma r|y|}\, dr \\
(15)\qquad &= S_n \frac{\prod_{j=0}^{d_\alpha - 1} (|p\alpha| + n - 1 - j)}{(2\pi p\sigma|y|)^{d_\alpha}} \int_0^\infty r^{|p\alpha| + n - 1 - d_\alpha} e^{-2\pi p\sigma r|y|}\, dr \\
&\leqq M(C')S_n \frac{\prod_{j=0}^{d_\alpha - 1} (|p\alpha| + n - 1 - j)}{(2\pi p\sigma|y|)^{d_\alpha}},
\end{aligned}
$$

where $M(C')$ is a constant depending on σ which depends on C'. If $\alpha = (0, \cdots, 0)$, then integrating by parts $n - 1$ times we obtain

$$
(16)\qquad
\begin{aligned}
S_n \int_0^\infty r^{n-1} e^{-2\pi p\sigma r|y|}\, dr &= S_n \frac{\prod_{j=0}^{n-2} (n - 1 - j)}{(2\pi p\sigma|y|)^{n-1}} \int_0^\infty e^{-2\pi p\sigma r|y|}\, dr \\
&= S_n \frac{\prod_{j=0}^{n-2} (n - 1 - j)}{(2\pi p\sigma|y|)^{n}}.
\end{aligned}
$$

Applying (14), (15) and (16) in (12) we have

$$
\begin{aligned}
|C(U; z)| \leqq R \Bigg\{ &\frac{(S_n \prod_{j=0}^{n-2} (n - 1 - j))^{1/p}}{(2\pi p\sigma|y|)^{n/p}_{d-1}} \\
&+ \sum_{0 < |\alpha| \leqq m} \frac{(S_n M(C') \prod_{j=0}^{d_\alpha - 1} (|p\alpha| + n - 1 - j))^{1/p}}{(2\pi p\sigma|y|)^{d_\alpha/p}} \Bigg\};
\end{aligned}
$$

and (8) follows from this estimate. The proof is complete.

We note the following more general setting for Theorems 1 and 2. Let b be a fixed nonnegative real number, and let C be an open connected cone. Put $S_b = \{t : u_C(t) \leqq b\}$. If the cone C is such that $\{t : 0 < u_C(t) \leqq b\}$ is a bounded set in $\mathbb{R}^n$, then using essentially the same proofs as in Theorems 1 and 2 we have that

$$
\int_{S_b} e^{2\pi i\langle z-t,\eta\rangle}\, d\eta \in \mathscr{D}_{L^q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p \leqq 2;
$$

and that

$$
(17)\qquad \left\langle U, \int_{S_b} e^{2\pi i\langle z-t,\eta\rangle}\, d\eta \right\rangle
$$

is analytic in $T^{O(C)}$. A more general boundedness condition than (8) can be obtained for (17) in which exponential factors appear. This more general setting will not change the essential content of our results concerning the boundary values of the generalized Cauchy integral; and the methods of proof are the same for arbitrary b as for $b = 0$. Thus to avoid making the additional assumption on the cone C that $\{t : 0 < u_C(t) \leqq b\}$ is a bounded set in $\mathbb{R}^n$, we shall prove our results for $b = 0$ and then note the corresponding results for arbitrary $b \geqq 0$.

We now introduce the generalized Poisson integral of an element $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Let z be an arbitrary but fixed point in $T^{O(C)}$, C being an open connected cone. We define the Poisson kernel related to $T^{O(C)}$ by

$$
(18)\qquad Q(z; t) = \frac{K(z - t)\overline{K(z - t)}}{K(2iy)},
$$

where $K(z - t)$ is the Cauchy kernel. Letting α be an arbitrary n-tuple of non-negative integers and using the generalized Leibnitz rule we have

$$D_t^\alpha Q(z;t) = \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_t^\beta K(z-t) D_t^\gamma \overline{K(z-t)}. \tag{19}$$

By Theorem 1, $D_t^\beta K(z - t)$ exists and is an element of L^q, $1/p + 1/q = 1, 1 < p \leqq 2$. By the same proof as in Theorem 1, $D_t^\gamma \overline{K(z - t)}$ exists for all n-tuples γ of non-negative integers; and by the same proof as in (5), $D_t^\gamma \overline{K(z - t)}$ is a bounded function of $t \in \mathbb{R}^n$ for fixed $z \in T^{O(C)}$. Thus for all β and γ such that $\beta + \gamma = \alpha$, $D_t^\beta K(z - t) \cdot D_t^\gamma \overline{K(z - t)}$ exists and is an element of L^q, $1/p + 1/q = 1$, $1 < p \leqq 2$. It follows from (19) that any derivative of $Q(z;t)$ exists and is in L^q. We have proved the following theorem.

THEOREM 3. *Let* $z \in T^{O(C)}$. *The Poisson kernel* $Q(z;t) \in \mathcal{D}_{L^q}$, $1/p + 1/q = 1$, $1 < p \leqq 2$, *as a function of* t.

If $T^{O(C)}$ is the upper half-plane in $\mathbb{C}^1$, then the Poisson kernel $Q(z;t)$ defined by (18) becomes

$$Q(z;t) = \frac{1}{\pi} \frac{y}{(t-x)^2 + y^2}, \qquad z = x + iy,$$

which is the classical Poisson kernel. We note that the Poisson kernel for general tubular radial domains $T^{O(C)}$ as defined in (18) was first introduced by Korányi [14] and Stein, Weiss and Weiss [15] and has been studied by these authors in connection with the Hardy $H^p(T^{O(C)})$-spaces.

Now let $U \in \mathcal{D}'_{L^p}$, $1 < p \leqq 2$; and put

$$P(U;z) = \langle U_t, Q(z;t)\rangle, \quad z \in T^{O(C)}, \tag{20}$$

C being an open connected cone. By Theorem 3, $P(U;z)$ is well-defined; and we call $P(U;z)$ the generalized Poisson integral of $U \in \mathcal{D}'_{L^p}$, $1 < p \leqq 2$. In Theorem 2 we saw that $C(U;z)$, the generalized Cauchy integral, is analytic in $T^{O(C)}$. In general, however, $P(U;z)$ is not an analytic function. In §§ 4 and 5 we shall obtain distributional boundary value results concerning both the generalized Cauchy and Poisson integrals.

In the remainder of this section we shall obtain results which will be needed in §§ 4 and 5. It is well known that if f and g are in L^2, then

$$\mathcal{F}^{-1}[\hat{f}\hat{g}] = f * g, \tag{21}$$

where $*$ denotes the usual convolution. Similarly if f and $\hat{g}$ are in L^p, $1 < p < 2$, and if $\hat{f}$ and g are in L^q, $1/p + 1/q = 1$, then (21) holds. Furthermore, if $f \in L^2$ and $g \in L^1$, then

$$\mathcal{F}[f * g] = \hat{f}\hat{g} \tag{22}$$

in the sense of L^2. It is obvious that under the above conditions both $f * g$ and $\hat{f}\hat{g}$ in (21) and (22) are elements of $\mathcal{S}'$.

LEMMA 1. *Let f and $\hat{g}$ be elements of L^p, $1 < p < 2$; and let $\hat{f}$ and g be elements of L^q, $1/p + 1/q = 1$. Then $\mathcal{F}[f * g] = \hat{f}\hat{g}$ in $\mathcal{S}'$.*

Proof. Let $\phi \in \mathcal{S}$. By (21), $\langle f * g, \phi\rangle = \langle \mathcal{F}^{-1}[\hat{f}\hat{g}], \phi\rangle$. Since the Fourier transform is a continuous, one-to-one mapping of $\mathcal{S}'$ onto $\mathcal{S}'$, we have $\langle \mathcal{F}[f * g], \phi\rangle = \langle \hat{f}\hat{g}, \phi\rangle$ as desired.

Using (21), (22), and exactly the same proof as in Lemma 1 we have the following lemma.

LEMMA 2. *Let* $f \in L^2$, *and let* $g \in L^1$ *or* $g \in L^2$. *Then* $\mathscr{F}[f * g] = \hat{f}\hat{g}$ *in* $\mathscr{S}'$.

LEMMA 3. *Let* $f \in L^p$, $1 < p \leqq 2$. *Let* $g \in L^q$, $1/p + 1/q = 1$; *and assume that* $\mathscr{F}^{-1}(g)$ *exists and belongs to* L^p. *Then*

$$\mathscr{F}^{-1}[f * g] = \mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g) \tag{23}$$

in $\mathscr{S}'$.

Proof. Since $f \in L^p$, $1 < p \leqq 2$, then $\mathscr{F}^{-1}(f)$ exists and is an element of L^q, $1/p + 1/q = 1$. Thus $\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g) \in \mathscr{S}'$. Also $f * g$ exists as a classical convolution, and it is known that $f * g \in L^r$, $1/r = 1/p + 1/q - 1$. Thus $f * g \in \mathscr{S}'$; hence $\mathscr{F}^{-1}[f * g] \in \mathscr{S}'$. Thus both sides of (23) are well-defined as elements of $\mathscr{S}'$. By a well-known result of Schwartz [10], the inverse Fourier transform in $\mathscr{S}'$ converts convolution into multiplication; and (23) is obtained.

Schwartz [10, p. 270] has shown that if $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$, and $V \in \mathscr{D}'_{L^q}$, $1 \leqq q \leqq 2$, then $\mathscr{F}[U * V] = \hat{U}\hat{V}$. We now obtain a variation of this result, where we let one of the distributions be an element of $\mathscr{D}'_{L^q}$ such that q does not have to lie in $1 \leqq q \leqq 2$.

THEOREM 4. *If* $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, *and* $V \in \mathscr{D}'_{L^q}$, $1/p + 1/q = 1$, *such that* $\hat{V} = \sum_{|\beta| \leqq r} (-1)^{|\beta|} x^\beta \hat{g}_\beta(x)$, $\hat{g}_\beta(x) \in L^p$, *then* $U * V \in \mathscr{S}'$ *and* $\mathscr{F}[U * V] = \hat{U}\hat{V}$ *in* $\mathscr{S}'$.

Proof. By the representation theorem of Schwartz, $U = \sum_{|\alpha| \leqq m} D^\alpha f_\alpha$, $f_\alpha \in L^p$, $1 < p \leqq 2$, and $V = \sum_{|\beta| \leqq r} D^\beta g_\beta$, $g_\beta \in L^q$, $1/p + 1/q = 1$. As in the proof of Lemma 3, $f_\alpha * g_\beta \in L^r$, $1/r = 1/p + 1/q - 1$. Thus $f_\alpha * g_\beta \in \mathscr{S}'$; hence the distributional derivative $D^{\alpha+\beta}(f_\alpha * g_\beta) \in \mathscr{S}'$. Let $\phi \in \mathscr{S}$. Then

$$\begin{aligned}
&\left\langle \sum_{|\alpha| \leqq m} \sum_{|\beta| \leqq r} D^{\alpha+\beta}(f_\alpha * g_\beta), \phi \right\rangle \\
&\qquad = \sum_{|\alpha| \leqq m} \sum_{|\beta| \leqq r} (-1)^{|\alpha|+|\beta|} \langle f_\alpha * g_\beta, D^{\alpha+\beta}\phi \rangle \\
&\qquad = \sum_{|\alpha| \leqq m} \sum_{|\beta| \leqq r} (-1)^{|\alpha|+|\beta|} \langle (f_\alpha)_\xi, \langle (g_\beta)_\eta, D^{\alpha+\beta}\phi(\xi + \eta)\rangle\rangle \\
&\qquad = \left\langle \sum_{|\alpha| \leqq m} D^\alpha (f_\alpha)_\xi, \left\langle \sum_{|\beta| \leqq r} D^\beta (g_\beta)_\eta, \phi(\xi + \eta) \right\rangle\right\rangle \\
&\qquad = \langle U * V, \phi \rangle.
\end{aligned} \tag{24}$$

Thus $U * V \in \mathscr{S}'$, and as a consequence $\mathscr{F}[U * V] \in \mathscr{S}'$. Using (24), the assumption that $\hat{g}_\beta(x) \in L^p$, and Lemmas 1 and 2 we have

$$\begin{aligned}
\langle \mathscr{F}[U * V], \phi(x) \rangle &= \left\langle \sum_{|\alpha| \leqq m} \sum_{|\beta| \leqq r} D^{\alpha+\beta}(f_\alpha * g_\beta), \hat{\phi}(t) \right\rangle \\
&= \sum_{|\alpha| \leqq m} \sum_{|\beta| \leqq r} (-1)^{|\alpha|+|\beta|} \langle \mathscr{F}[f_\alpha * g_\beta], x^{\alpha+\beta}\phi(x) \rangle \\
&= \left\langle \sum_{|\alpha| \leqq m} \sum_{|\beta| \leqq r} (-1)^{|\alpha|+|\beta|} x^{\alpha+\beta} \hat{f}_\alpha(x)\hat{g}_\beta(x), \phi(x) \right\rangle.
\end{aligned} \tag{25}$$

Now

$$\langle \hat{U}, \phi(x) \rangle = \left\langle \sum_{|\alpha| \leqq m} D^\alpha f_\alpha(t), \hat{\phi}(t) \right\rangle = \left\langle \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} x^\alpha \hat{f}_\alpha(x), \phi(x) \right\rangle, \tag{26}$$

where $\hat{f}_\alpha(x) \in L^q$, $1/p + 1/q = 1$, since $f_\alpha(t) \in L^p$, $1 < p \leqq 2$. By assumption, $\hat{V} = \sum_{|\beta| \leqq r} (-1)^{|\beta|} x^\beta \hat{g}_\beta(x)$, $\hat{g}_\beta(x) \in L^p$. Using this assumption and (26), we see from (25) that

$$(27) \qquad \langle \mathscr{F}[U * V], \phi \rangle = \langle \hat{U}\hat{V}, \phi \rangle$$

as desired.

We now obtain a converse result to Theorem 4. Recall that if $\hat{V} \in \mathscr{D}'_{L^q}$, then by the Schwartz representation theorem, $\hat{V} = \sum_{|\beta| \leqq r} D^\beta g_\beta$, $g_\beta \in L^q$.

THEOREM 5. *Let* $\hat{U} \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, *and* $\hat{V} \in \mathscr{D}'_{L^q}$, $1/p + 1/q = 1$, *such that* $V = \sum_{|\beta| \leqq r} x^\beta \mathscr{F}^{-1}[g_\beta(t); x]$, *where* $\mathscr{F}^{-1}[g_\beta(t); x]$ *is assumed to exist and belong to* L^p. *Then* $\mathscr{F}[UV] = \hat{U} * \hat{V}$ *in* $\mathscr{S}'$.

Proof. It suffices to prove that

$$(28) \qquad \mathscr{F}^{-1}[\hat{U} * \hat{V}] = UV$$

in $\mathscr{S}'$; for the desired result follows immediately by taking the Fourier transform of both sides of (28). It is evident that (28) can be proved by the same method as was used to obtain (27) in Theorem 4 where we use Lemma 3 instead of Lemmas 1 and 2. We leave the straightforward details to the interested reader.

We note that by using the same calculation as in (26), one can show that if $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$, then $\hat{U} \in \mathscr{S}'$ has the form $\hat{U} = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} x^\alpha \hat{f}_\alpha(x)$, where $\hat{f}_\alpha(x)$ is continuous and bounded if $p = 1$ and $\hat{f}_\alpha(x) \in L^q$, $1/p + 1/q = 1$, if $1 < p \leqq 2$. Schwartz [10, p. 256] first recognized this result. We note further that in the case $p = 2$ in Theorems 4 and 5, some of our assumptions are redundant. For example in Theorem 4 if $p = 2$, then $q = 2$. Thus the form of $\hat{V}$ can be proved as in (26), and $\hat{g}_\beta \in L^2$ by the Plancherel theory. In Theorem 5 the form of V can be proved if $p = 2$; and one does not have to assume that $\mathscr{F}^{-1}[g_\beta(t); x]$ exists, for it automatically does by the Plancherel theory and is an element of L^2. The stated assumptions are needed, however, for $1 < p < 2$.

Beltrami and Wohlers [5, Theorem 1.36, p. 43] have considered Theorems 4 and 5 for the case $p = 2$ and for one dimension. However, there is an error in their method of proof; for they have used the supposed fact that if f and g are in L^2 then $\mathscr{F}[f * g] = \hat{f}\hat{g}$ classically. This is incorrect since one can say only that $f * g \in L^\infty$ and is continuous. Such a function does not necessarily have a Fourier transform, and $\mathscr{F}[f * g]$ does not necessarily have meaning classically. The correct approach is as we have stated in Lemmas 1–3; that is, $\mathscr{F}[f * g] = \hat{f}\hat{g}$ and $\mathscr{F}^{-1}[f * g] = \mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g)$ as equalities in $\mathscr{S}'$.

In Carmichael [8] we have considered functions $f(z)$ which are analytic in T^C, C being an open connected cone, and which satisfy

$$(29) \qquad |f(z)| \leqq K(C')(1 + |z|)^N e^{2\pi(b+\sigma)|y|}, \qquad z = x + iy \in T^{C'},$$

for all $\sigma > 0$, where b is a nonnegative real number, N is any real number, and $K(C')$ is a constant depending on C', C' being an arbitrary compact subcone of C. We denote the set of all such functions $f(z)$ by G_C^b. In the present paper we shall need the following two theorems, the proofs of which can be found in [8].

THEOREM 6. *Let* $f(z) \in G_C^b$, *and let* $f(z) \to U$ *in the* $\mathscr{S}'$ *topology as* $y \to 0$, $y \in C' \subset C$. *Then* $U \in \mathscr{S}'$; *there exists an element* $V \in \mathscr{S}'$ *such that* $\operatorname{supp}(V) \subseteq S_b = \{t : u_C(t) \leqq b\}$ *and* $U = \hat{V}$; *and* $f(z) = \langle V, e^{2\pi i \langle z,t \rangle} \rangle$, $z \in T^{C'}$, $C' \subset C$.

THEOREM 7. *Let $V \in \mathscr{S}'$ and* $\operatorname{supp}(V) \subseteq S_b$ *for some fixed real number* $b \geqq 0$ *and some open connected cone C. Let* $\phi \in \mathscr{S}$ *and C′ be an arbitrary compact subcone of O(C). Then*

$$\langle\langle V, e^{2\pi i\langle z,t\rangle}\rangle, \phi\rangle = \langle V, e^{-2\pi\langle y,t\rangle}\hat{\phi}\rangle, \qquad z \in T^{C'}.$$

For special cases of Theorems 6 and 7 and other related results we refer to Carmichael [16]. We note that distributional boundary value results in $\mathscr{S}'$ using boundedness conditions similar to (29) are of importance in quantum field theory (see Streater and Wightman [17]).

We shall also use the following important result due to Vladimirov [12, Theorem 2, p. 239].

THEOREM 8 (Vladimirov). *Let C be an open cone and* $V \in \mathscr{S}'$ *with* $\operatorname{supp}(V) \subseteq S_b = \{t : u_C(t) \leqq b\}$, $b \geqq 0$. *Then* $\langle V, e^{2\pi i\langle z,t\rangle}\rangle$ *is analytic in* $T^{O(C)}$ *and satisfies*

$$(30) \qquad |D_z^\alpha\langle V, e^{2\pi i\langle z,t\rangle}\rangle| \leqq K(C')(1 + |z|)^N(1 + |y|^{-M}) \exp[2\pi b\rho_C|y|],$$

$z = x + iy \in T^{C'}$, *where C′ is an arbitrary compact subcone of O(C), K(C′) is a constant depending on C′, and M and N are nonnegative integers which do not depend on C′.*

Note that the cone C does not have to be connected in Theorem 8. Vladimirov first proved this result in [18], where he introduced the concept of distributional boundary values of functions analytic in tubular cones.

4. Distributional boundary values. In this section we shall show that the generalized Cauchy and Poisson integrals for tubular radial domains have boundary values in the distributional sense. We shall relate these generalized integrals to the space of functions G_C^b and shall obtain theorems in which an element $f(z) \in G_C^b$ has a distributional boundary value in $\mathscr{D}'_{L^p}$. It will be seen that our results concerning the space G_C^b are generalizations of classical H^p-space theorems. Throughout this section C will denote an open connected cone; and $C' \subset C$ or $C' \subset O(C)$ will denote that C′ is an arbitrary compact subcone of C or O(C), respectively.

Let $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$. Using the Schwartz characterization theorem for $\mathscr{D}'_{L^p}$ and a calculation as in (26) we have

$$(31) \qquad \mathscr{F}^{-1}(U) = \sum_{|\alpha| \leqq m} t^\alpha h_\alpha(t),$$

where $h_\alpha(t)$ is continuous and bounded if $p = 1$ and $h_\alpha(t) \in L^q$, $1/p + 1/q = 1$, if $1 < p \leqq 2$. This representation of $\mathscr{F}^{-1}(U)$ will be used frequently in this section and the next.

To obtain the boundary value result for the generalized Cauchy integral, we shall need two lemmas.

LEMMA 4. *Let* $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, *and* $\phi \in \mathscr{S}$. *For fixed* $y = \operatorname{Im}(z) \in O(C)$,

$$(32) \qquad \langle C(U; z), \phi(x)\rangle = \langle U, \langle K(z - t), \phi(x)\rangle\rangle.$$

Proof. By a proof as in Theorem 1 we have that $\int_{C^*} e^{2\pi i\langle z,\eta\rangle}\, d\eta \in \mathscr{D}_{L^q} \subset \mathscr{D}'_{L^q}$, $1/p + 1/q = 1$, as a function of x. Thus by a theorem of Schwartz [10, Theorem 26, p. 203], $U * \int_{C^*} e^{2\pi i\langle z,\eta\rangle}\, d\eta$ exists and is an element of $\mathscr{D}'_{L^r}$, $1/r = 1/p + 1/q = 1$. By a change of variable, we have

$$\langle K(z - t), \phi(x)\rangle = \left\langle \int_{C^*} e^{2\pi i\langle z,\eta\rangle}\, d\eta, \phi(x + t)\right\rangle;$$

so that

$$\langle U, \langle K(z - t), \phi(x)\rangle\rangle = \left\langle U * \int_{C^*} e^{2\pi i\langle z,\eta\rangle}\, d\eta, \phi\right\rangle. \tag{33}$$

Thus by (33), the right-hand side of (32) exists. Using the characterization theorem of Schwartz for U and a change of order of integration, we have

$$\begin{aligned}\langle U, \langle K(z - t), \phi(x)\rangle\rangle &= \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_\alpha(t) \int_{\mathbb{R}^n} D_t^\alpha K(z - t)\phi(x)\, dx\, dt \\ &= \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi(x) \int_{\mathbb{R}^n} f_\alpha(t) D_t^\alpha K(z - t)\, dt\, dx \\ &= \langle\langle U, K(z - t)\rangle, \phi(x)\rangle;\end{aligned}$$

and (32) is obtained.

LEMMA 5. *Let C be an open connected cone. Let $I_{C^*}(\eta)$ be the characteristic function of C^* and $\phi \in \mathscr{S}$. Then*

$$\langle K(z - t), \phi(x)\rangle \to \mathscr{F}^{-1}[I_{C^*}(\eta)\hat{\phi}(\eta); t]$$

in the topology of $\mathscr{D}_{L^q}$, $1/p + 1/q = 1$, $1 < p \leqq 2$, as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$.

Proof. For the present let y be a fixed point in $O(C)$. By a change of order of integration, we have

$$\langle K(z - t), \phi(x)\rangle = \int_{\mathbb{R}^n} I_{C^*}(\eta)\hat{\phi}(\eta)\, e^{-2\pi\langle y,\eta\rangle}\, e^{-2\pi i\langle t,\eta\rangle}\, d\eta;$$

and $\hat{\phi}(\eta) \in \mathscr{S}$. Thus for α being an arbitrary n-tuple of nonnegative integers,

$$\begin{aligned}&\|D_t^\alpha\langle K(z - t), \phi(x)\rangle - D_t^\alpha \mathscr{F}^{-1}[I_{C^*}(\eta)\hat{\phi}(\eta); t]\|_{L^q} \\ &\quad = \|\mathscr{F}^{-1}[(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha\hat{\phi}(\eta)\{e^{-2\pi\langle y,\eta\rangle} - 1\}; t]\|_{L^q};\end{aligned} \tag{34}$$

and because of the inequality

$$|I_{C^*}(\eta)\eta^\alpha\hat{\phi}(\eta)\{e^{-2\pi\langle y,\eta\rangle} - 1\}| \leqq 2|\eta^\alpha\hat{\phi}(\eta)| \tag{35}$$

and the fact that $\hat{\phi}(\eta) \in \mathscr{S}$, we have that $(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha\hat{\phi}(\eta)\{e^{-2\pi\langle y,\eta\rangle} - 1\} \in L^1 \cap L^p$, $1 < p \leqq 2$. Thus

$$\begin{aligned}&\|\mathscr{F}^{-1}[(-1)^{|\alpha|} I_{C^*}(\eta)\eta^\alpha\hat{\phi}(\eta)\{e^{-2\pi\langle y,\eta\rangle} - 1\}; t]\|_{L^q} \\ &\quad \leqq \|I_{C^*}(\eta)\eta^\alpha\hat{\phi}(\eta)\{e^{-2\pi\langle y,\eta\rangle} - 1\}\|_{L^p}.\end{aligned} \tag{36}$$

By (35), we can use the Lebesgue dominated convergence theorem to obtain

$$\lim_{\substack{y \to 0 \\ y \in O(C)}} \int_{\mathbb{R}^n} |I_{C^*}(\eta)\eta^\alpha\hat{\phi}(\eta)\{e^{-2\pi\langle y,\eta\rangle} - 1\}|^p\, d\eta = 0.$$

This fact combined with (34) and (36) proves the desired convergence.

We can now prove the following theorem.

THEOREM 9. *Let $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Then $C(U; z) \to \mathscr{F}[I_{C^*}(\eta)\mathscr{F}^{-1}(U)]$ in the topology of $\mathscr{D}'_{L^p}$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$.*

Proof. From Lemma 5 and the continuity of U we have for $\phi \in \mathscr{S}$ that

$$\langle U, \langle K(z - t), \phi(x)\rangle\rangle \to \langle U, \mathscr{F}^{-1}[I_{C^*}(\eta)\hat{\phi}(\eta); t]\rangle = \langle \mathscr{F}[I_{C^*}(\eta)\mathscr{F}^{-1}(U)], \phi(x)\rangle \tag{37}$$

as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$. The result is obtained by combining (32) and (37).

We shall now obtain conditions on $U \in \mathscr{D}'_{L^p}$ such that the boundary value of $C(U;z)$ is U. Under these conditions we shall see that $C(U;z)$ is identical in $\mathscr{S}'$ to the Fourier–Laplace transform of an element $V \in \mathscr{S}'$. The topology we use is that of $\mathscr{S}'$.

THEOREM 10. *Let* $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, *such that* $U = \hat{V}$, *where* $V \in \mathscr{S}'$ *and* $\operatorname{supp}(V) \subseteq C^*$, *C being an open connected cone. Then* $V = \sum_{|\alpha| \leqq m} t^\alpha h_\alpha(t)$, $h_\alpha(t) \in L^q$, $1/p + 1/q = 1$; *for any compact subcone* $C' \subset O(C)$ *we have*

$$\langle V, e^{2\pi i\langle z,t\rangle}\rangle = C(U;z), \qquad z \in T^{C'}, \tag{38}$$

as elements of $\mathscr{S}'$; *and* $C(U;z) \to U$ *in the topology of* $\mathscr{S}'$ *as* $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset O(C)$.

Proof. Since $\hat{V} = U$, then $V = \mathscr{F}^{-1}(U)$; and the representation of V follows as in (31). Let $\xi(\eta) \in \mathscr{E}$ such that $\xi(\eta) = 1$ for $\eta \geqq 0$, $\xi(\eta) = 0$ for $\eta \leqq -\varepsilon$, $\varepsilon > 0$; and $0 \leqq \xi(\eta) \leqq 1$. Put $\gamma(t) = \xi(\langle t, y\rangle)$, $y \in O(C)$. By Theorem 8, $\langle V, e^{2\pi i\langle z,t\rangle}\rangle = \langle V, \gamma(t)\, e^{2\pi i\langle z,t\rangle}\rangle$ is analytic in $T^{O(C)}$; and by Theorem 7 we have

$$\langle\langle V, e^{2\pi i\langle z,t\rangle}\rangle, \phi(x)\rangle = \langle V, \gamma(t)\, e^{-2\pi\langle y,t\rangle}\hat{\phi}(t)\rangle, \tag{39}$$

$z \in T^{C'}$, $C' \subset O(C)$, where $\phi \in \mathscr{S}$. Now $\operatorname{supp}(V) \subseteq C^*$; so that $\operatorname{supp}(h_\alpha(t)) \subseteq C^*$ for each element $h_\alpha(t)$ in the representation of V. Letting $I_{C^*}(t)$ denote the characteristic function of C^*, we obtain

$$\begin{aligned} \langle V, \gamma(t)\, e^{-2\pi\langle y,t\rangle}\hat{\phi}(t)\rangle &= \langle I_{C^*}(t)V, \gamma(t)\, e^{-2\pi\langle y,t\rangle}\hat{\phi}(t)\rangle \\ &= \langle \mathscr{F}[I_{C^*}(t)\gamma(t)\, e^{-2\pi\langle y,t\rangle}V], \phi(x)\rangle. \end{aligned} \tag{40}$$

By Theorem 5 and equations (32) and (33), we have

$$\begin{aligned} \langle \mathscr{F}[I_{C^*}(t)\gamma(t)\, e^{-2\pi\langle y,t\rangle}V], \phi(x)\rangle &= \langle \hat{V} * \mathscr{F}[I_{C^*}(t)\gamma(t)\, e^{-2\pi\langle y,t\rangle}; x], \phi(x)\rangle \\ &= \left\langle U * \int_{C^*} e^{2\pi i\langle z,t\rangle}\, dt, \phi(x)\right\rangle \\ &= \left\langle\left\langle U, \int_{C^*} e^{2\pi i\langle z-t,\eta\rangle}\, d\eta\right\rangle, \phi(x)\right\rangle. \end{aligned} \tag{41}$$

Combining (39), (40) and (41) we have (38). It is straightforward to show that $\gamma(t)\, e^{-2\pi\langle y,t\rangle}\hat{\phi}(t) \to \gamma(t)\hat{\phi}(t)$ in the topology of $\mathscr{S}$ as $y \to 0$, $y \in C' \subset O(C)$. By the continuity of V and (39), we have

$$\langle\langle V, e^{2\pi i\langle z,t\rangle}\rangle, \phi(x)\rangle \to \langle V, \gamma(t)\hat{\phi}(t)\rangle = \langle \hat{V}, \phi(x)\rangle \tag{42}$$

as $y \to 0$, $y \in C' \subset O(C)$. Since $U = \hat{V}$, then by (38) and (42), we have $C(U;z) \to U$ in $\mathscr{S}'$ as $y \to 0$, $y \in C' \subset O(C)$; and the proof is complete.

COROLLARY 1. *Let* $f(z) \in G^0_C$ *converge in the* $\mathscr{S}'$ *topology to* $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, *as* $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset C$. *Then* $f(z) = C(U;z)$, $z \in T^{C'}$, *as elements of* $\mathscr{S}'$; *and U is the Fourier transform of* $V = \sum_{|\alpha| \leqq m} t^\alpha h_\alpha(t) \in \mathscr{S}'$, $h_\alpha(t) \in L^q$, $1/p + 1/q = 1$, *such that* $\operatorname{supp}(V) \subseteq C^*$.

Proof. By Theorem 6, there exists an element $V \in \mathscr{S}'$ with $\operatorname{supp}(V) \subseteq C^*$ such that $U = \hat{V}$ and

$$f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle, \qquad z \in T^{C'}, \quad C' \subset C. \tag{43}$$

But for such a V, (38) holds. Thus by (38) and (43), we have $f(z) = C(U; z)$, $z \in T^{C'}$, as elements of $\mathscr{S}'$. The representation of V follows as in (31) since $V = \mathscr{F}^{-1}(U)$.

We now give necessary and sufficient conditions that an element $U \in \mathscr{D}'_{L^p}$ be the $\mathscr{S}'$ boundary value of a function $f(z) \in G_C^0$.

THEOREM 11. *Let $f(z) \in G_C^0$ converge in the $\mathscr{S}'$ topology to $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, as $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset C$. Then $C(U; z) = 0$, $z \in T^{\tilde{C}'}$, $\tilde{C}' \subset \tilde{C}$, where $\tilde{C}$ is any open connected cone such that $C^* \cap \{t : u_{\tilde{C}}(t) < 0\} = \varnothing$.*

Proof. $V = \mathscr{F}^{-1}(U)$ has the representation as in (31); and by Theorem 6, $\operatorname{supp}(V) \subseteq C^*$. Now let $\tilde{C}$ be any open, connected cone such that $C^* \cap \{t : u_{\tilde{C}}(t) < 0\} = \varnothing$. For each such $\tilde{C}$ let $\gamma_{\tilde{C}}(t) = \xi(\langle t, y \rangle)$, $y \in \tilde{C}$, where $\xi(\eta)$ is defined as in the proof of Theorem 10. Let $I_{\tilde{C}}(t)$ be the characteristic function of $\{t : u_{\tilde{C}}(t) < 0\}$. By exactly the same calculation as in obtaining (38), we have

$$C(U; z) = \langle I_{\tilde{C}}(t)V, \gamma_{\tilde{C}}(t)\, e^{2\pi i \langle z, t \rangle} \rangle, \qquad z \in T^{\tilde{C}'}, \quad \tilde{C}' \subset \tilde{C}. \tag{44}$$

But $\operatorname{supp}(V) \subseteq C^*$ and $\operatorname{supp}(I_{\tilde{C}}(t)\gamma_{\tilde{C}}(t)\, e^{2\pi i \langle z, t \rangle}) \subseteq \{t : u_{\tilde{C}}(t) < 0\}$. Thus $\operatorname{supp}(V) \cap \operatorname{supp}(I_{\tilde{C}}(t)\gamma_{\tilde{C}}(t)\, e^{2\pi i \langle z, t \rangle}) = \varnothing$, which implies that $\langle I_{\tilde{C}}(t)V, \gamma_{\tilde{C}}(t)\, e^{2\pi i \langle z, t \rangle} \rangle = 0$. This combined with (44) implies $C(U; z) = 0$, $z \in T^{\tilde{C}'}$, $\tilde{C}' \subset \tilde{C}$.

THEOREM 12. *Let $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$; and let C be an open connected cone. Suppose there exists a finite number of open connected cones C_j, $j = 1, \cdots, m$, such that*

$$\mathbb{R}^n \diagdown \left\{ \bigcup_{j=1}^{m} \{t : u_{C_j}(t) \leqq 0\} \cup C^* \right\}, \qquad C^* \cap \{t : u_{C_j}(t) \leqq 0\}, \qquad j = 1, \cdots, m,$$

and

$$\{t : u_{C_j}(t) \leqq 0\} \cap \{t : u_{C_k}(t) \leqq 0\}, \qquad j \neq k, \quad j = 1, \cdots, m, \quad k = 1, \cdots, m,$$

are sets of Lebesgue measure zero and such that $C(U; z) = 0$, $z \in T^{C'_j}$, $C'_j \subset C_j$, $j = 1, \cdots, m$. Then U is the $\mathscr{S}'$ boundary value of a function $f(z) \in G^0_{O(C)}$ as $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset O(C)$.

Proof. Let $I_{C^*}(t)$ be the characteristic function of C^*; and for each C_j, $j = 1, \cdots, m$, let $I_{C_j}(t)$ be the characteristic function of $\{t : u_{C_j}(t) \leqq 0\}$. $V = \mathscr{F}^{-1}(U)$ has the representation as in (31); and by hypothesis, we have

$$V = I_{C^*}(t)V + \sum_{j=1}^{m} I_{C_j}(t)V$$

almost everywhere in $\mathbb{R}^n$. Thus for $\phi \in \mathscr{S}$ we have

$$\langle U, \phi \rangle = \langle \mathscr{F}[I_{C^*}(t)V], \phi \rangle + \sum_{j=1}^{m} \langle \mathscr{F}[I_{C_j}(t)V], \phi \rangle. \tag{45}$$

By hypothesis and a calculation as in obtaining (38) we have for each C_j, $j = 1, \cdots, m$, that

$$0 = C(U; z) = \langle I_{C_j}(t)V, \gamma_{C_j}(t)\, e^{2\pi i \langle z, t \rangle} \rangle \tag{46}$$

in $\mathscr{S}'$ for $z \in T^{C'_j}$, $C'_j \subset C_j$, where $\gamma_{C_j}(t)$ is defined as in the proof of Theorem 10 for each C_j. Using the same proof as in (42) we have

$$\langle I_{C_j}(t)V, \gamma_{C_j}(t)\, e^{2\pi i \langle z, t \rangle} \rangle \to \mathscr{F}[I_{C_j}(t)V] \tag{47}$$

in $\mathscr{S}'$ as $y = \operatorname{Im}(z) \to 0$, $y \in C'_j \subset C_j$. By (46) and (47), we have that $\mathscr{F}[I_{C_j}(t)V] = 0$ for each C_j, $j = 1, \cdots, m$. Returning to (45) we thus have $\langle U, \phi \rangle = \langle \mathscr{F}[I_{C^*}(t)V]$,

$\phi\rangle$. Since the Fourier transform is a continuous, one-to-one mapping of $\mathscr{S}'$ onto $\mathscr{S}'$, $U = \hat{V} = \mathscr{F}[I_{C^*}(t)V]$ implies $V = I_{C^*}(t)V$. Thus $\operatorname{supp}(V) \subseteq C^*$; and we consider the function $f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in T^{O(C)}$. By Theorem 8, $f(z) \in G^0_{O(C)}$; and by a proof as in (42), $f(z) \to \hat{V} = U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, in the topology of $\mathscr{S}'$ as $y = \operatorname{Im}(z) \to 0$, $y \in C' \subset O(C)$. The proof is complete.

Recall now the discussion in § 3 in which we indicated that the generalized Cauchy integral can be defined in a more general setting if the cone C is such that $\{t : 0 < u_C(t) \leqq b\}$ is a bounded set in $\mathbb{R}^n$. In this setting Lemmas 4 and 5, Theorems 9 and 10, and Corollary 1 all hold with C^* replaced by $\{t : u_C(t) \leqq b\}$, $b \geqq 0$, and G^0_C replaced by G^b_C.

Tillmann [1] has shown that any $U \in \mathscr{D}'_{L^p}$, $1 < p < \infty$, can be decomposed into a sum of boundary values of functions analytic in the 2^n octants. We wish now to obtain a similar result for cones and $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$, using the techniques in the present paper. Our method of proof yields more information about the boundary values than Tillmann has obtained. Not only is each element in the decomposition the boundary value of an analytic function, it is also the Fourier transform of an element in $\mathscr{S}'$ having support in a specified subset of $\mathbb{R}^n$. We also include the case $p = 1$, which Tillmann has not done.

THEOREM 13. *Let $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$. Suppose there exist open connected cones C_j, $j = 1, \cdots, m$, such that*

$$\mathbb{R}^n \setminus \bigcup_{j=1}^{m} \{t : u_{C_j}(t) \leqq 0\}$$

and

$$\{t : u_{C_j}(t) \leqq 0\} \cap \{t : u_{C_k}(t) \leqq 0\}, \qquad j \neq k, \quad j = 1, \cdots, m, \quad k = 1, \cdots, m,$$

are sets of Lebesgue measure zero. Then $U = \sum_{j=1}^m W_j$ in $\mathscr{S}'$, where W_j, $j = 1, \cdots, m$, is the $\mathscr{S}'$ boundary value of a function $f_j(z) \in G^0_{O(C_j)}$ as $y = \operatorname{Im}(z) \to 0$, $y \in C'_j \subset O(C_j)$; and $W_j = \hat{V}_j$, where $V_j \in \mathscr{S}'$ and $\operatorname{supp}(V_j) \subseteq \{t : u_{C_j}(t) \leqq 0\}$.

Proof. $V = \mathscr{F}^{-1}(U)$ has the representation as in (31). For each $\{t : u_{C_j}(t) \leqq 0\}$ let $I_{C_j}(t)$ be the corresponding characteristic function. Let $\xi(\eta) \in \mathscr{E}$ such that $\xi(\eta) = 1$, $\eta \geqq 0$, $\xi(\eta) = 0$, $n \leqq -\varepsilon$, $\varepsilon > 0$; and $0 \leqq \xi(\eta) \leqq 1$. Put $\gamma_j(t) = \xi(\langle t, y\rangle)$, $y \in O(C_j)$, $j = 1, \cdots, m$; and let $V_j = I_{C_j}(t)V$, $j = 1, \cdots, m$. By the same method used to obtain (45), we have

$$\langle U, \phi\rangle = \left\langle \sum_{j=1}^{m} \hat{V}_j, \phi \right\rangle, \quad \phi \in \mathscr{S}. \tag{48}$$

Since $V_j \in \mathscr{S}'$ and $\operatorname{supp}(V_j) \subseteq \{t : u_{C_j}(t) \leqq 0\}$, then by Theorem 8, $f_j(z) = \langle V_j, \gamma_j(t)\, e^{2\pi i\langle z,t\rangle}\rangle$ is an element of $G^0_{O(C_j)}$ for $z \in T^{O(C_j)}$, $j = 1, \cdots, m$. (Note that if $f_j(z)$ satisfies (30), then $f_j(z)$ satisfies (29) for any $b \geqq 0$.) By the same proof as in (42), we have $f_j(z) \to \hat{V}_j$ in the topology of $\mathscr{S}'$ as $y = \operatorname{Im}(z) \to 0$, $y \in C'_j \subset O(C_j)$, $j = 1, \cdots, m$. Putting $W_j = \hat{V}_j$, $j = 1, \cdots, m$, and recalling (48), we have that $U = \sum_{j=1}^m W_j$, where the W_j are the desired $\mathscr{S}'$ boundary values of the corresponding analytic functions.

We turn our attention now to the generalized Poisson integral of $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Korányi [14, Prop. 3(c)] has stated that the Poisson integral of $f \in L^p$, $1 \leqq p < \infty$, corresponding to tube domains has f as boundary value in the L^p-norm topology. We shall show that the corresponding statement for the generalized Poisson integral of $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, holds where we use the topology of $\mathscr{D}'_{L^p}$. To do so we use several lemmas. Korányi [14] and Stein, Weiss and Weiss [15] have obtained the following lemma, which states that the Poisson kernel $Q(z; t)$ as defined in (18) is an approximate identity.

LEMMA 6. *Let C be an open connected cone; and let z be an arbitrary point in $T^{O(C)}$. Then*

$$Q(z; t) \geqq 0 \quad \text{for all } t \in \mathbb{R}^n; \tag{49}$$

$$\int_{\mathbb{R}^n} \frac{K(x+iy)\overline{K(x+iy)}}{K(2iy)}\, dx = 1, \qquad y \in O(C); \tag{50}$$

$$\text{if } \eta > 0, \quad \lim_{\substack{y \to 0 \\ y \in O(C)}} \int_{|x| \geqq \eta} \frac{K(x+iy)\overline{K(x+iy)}}{K(2iy)}\, dx = 0. \tag{51}$$

The following lemma was obtained by Korányi [14], who stated it without proof. We shall sketch the proof here, for the method used will be important in obtaining the boundary value result for the generalized Poisson integral.

LEMMA 7. *Let C be an open connected cone; and let $f \in L^p$, $1 \leqq p < \infty$. Then the Poisson integral of f converges to f in L^p as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$.*

Proof. Performing a change of variable gives

$$\int_{\mathbb{R}^n} f(t)Q(z; t)\, dt = \int_{\mathbb{R}^n} f(x-t)\frac{K(t+iy)\overline{K(t+iy)}}{K(2iy)}\, dt;$$

and by (50),

$$f(x) = \int_{\mathbb{R}^n} f(x)\frac{K(t+iy)\overline{K(t+iy)}}{K(2iy)}\, dt, \qquad y \in O(C).$$

We thus wish to show that

$$\begin{aligned} &\lim_{\substack{y \to 0 \\ y \in O(C)}} \left\| \int_{\mathbb{R}^n} f(t)Q(z; t)\, dt - f(x) \right\|_{L^p} \\ &= \lim_{\substack{y \to 0 \\ y \in O(C)}} \left\| \int_{\mathbb{R}^n} \{f(x-t) - f(x)\}\frac{K(t+iy)\overline{K(t+iy)}}{K(2iy)}\, dt \right\|_{L^p} = 0. \end{aligned} \tag{52}$$

Using Lemma 6, this can be done by exactly the same method of proof used in Hoffman [19, Theorem, pp. 17–19; see also Theorem, p. 32]. We leave the straightforward details to the interested reader.

LEMMA 8. *Let $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$; and let $\phi \in \mathscr{S}$. Then for fixed $y \in O(C)$, C being an open connected cone,*

$$\langle P(U; z), \phi(x) \rangle = \langle U, \langle Q(z; t), \phi(x) \rangle\rangle. \tag{53}$$

The proof of Lemma 8 is similar to that of Lemma 4 and will be omitted. We can now prove the following theorem.

THEOREM 14. *Let C be an open connected cone; and let $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Then $P(U; z) \to U$ in the topology of $\mathscr{D}'_{L^p}$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C)$.*

Proof. Let $\phi \in \mathscr{S}$. By a change of variable,

$$\int_{\mathbb{R}^n} Q(z; t)\phi(x)\, dx = \int_{\mathbb{R}^n} \phi(x+t)\frac{K(x+iy)\overline{K(x+iy)}}{K(2iy)}\, dx. \tag{54}$$

Let α be any n-tuple of nonnegative integers. Recalling that $D^{\alpha}\phi(t) = \psi(t) \in \mathscr{S}$ and using (54), we have

$$\left\| D_t^{\alpha} \int_{\mathbb{R}^n} Q(z;t)\phi(x)\,dx - D_t^{\alpha}\phi(t) \right\|_{L^q} = \left\| \int_{\mathbb{R}^n} \psi(x+t) \frac{K(x+iy)\overline{K(x+iy)}}{K(2iy)}\,dx - \psi(t) \right\|_{L^q}. \tag{55}$$

But $\psi \in \mathscr{S} \subset \mathscr{D}_{L^q} \subset L^q$, $1/p + 1/q = 1$, $1 < p \leqq 2$; and by exactly the same proof as in obtaining (52), we have that the right-hand side of (55) converges to zero as $y \to 0$, $y \in O(C)$. Thus by (55), $\langle Q(z;t), \phi(x)\rangle \to \phi(t)$ in $\mathscr{D}_{L^q}$, $1/p + 1/q = 1$, $1 < p \leqq 2$, as $y \to 0$, $y \in O(C)$. By the continuity of $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, it follows that $\langle U, \langle Q(z;t), \phi(x)\rangle\rangle \to \langle U, \phi\rangle$ as $y \to 0$, $y \in O(C)$. From this fact and (53) we obtain $P(U;z) \to U$ in $\mathscr{D}'_{L^p}$ as $y \to 0$, $y \in O(C)$; and the proof is complete.

A comparison of Theorems 9 and 14 shows that the generalized Cauchy and Poisson integrals have distinct properties. $C(U;z)$ attains a boundary value which depends on the cone C; while $P(U;z)$ always has U as its boundary value irrespective of C.

For the special case that T^C is an octant in $\mathbb{C}^n$, further results can be obtained for the generalized Poisson integral. For the octant $B_\delta = \{z \in \mathbb{C}^n : \delta_j(\operatorname{Im}(z_j)) > 0$, $\delta = (\delta_1, \cdots, \delta_n)$, $\delta_j = \pm 1$, $j = 1, \cdots, n\}$, the generalized Cauchy integral for $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, becomes

$$C(U; z \in B_\delta) = (2\pi i)^{-n} \left(\prod_{j=1}^{n} \operatorname{sgn} y_j \right) \left\langle U, \prod_{j=1}^{n} \frac{1}{t_j - z_j} \right\rangle,$$

where $\operatorname{sgn} y_j$ represents the sign of y_j, $j = 1, \cdots, n$. The generalized Poisson integral for U is

$$P(U; z \in B_\delta) = (\pi)^{-n} \left\langle U, \prod_{j=1}^{n} \frac{(\operatorname{sgn} y_j) y_j}{|t_j - z_j|^2} \right\rangle.$$

Since

$$\prod_{j=1}^{n} \frac{y_j}{\pi |t_j - z_j|^2} = \prod_{j=1}^{n} \frac{1}{2\pi i} \left\{ \frac{1}{t_j - z_j} - \frac{1}{t_j - \bar{z}_j} \right\},$$

then we have the relation

$$\begin{aligned} P(U; z \in B_\delta) = {} & C(U; z \in B_\delta) - C(U; (\bar{z}_1, z_2, \cdots, z_n), z \in B_\delta) + \cdots \\ & + (-1)^j C(U; (\bar{z}_1, \cdots, \bar{z}_j, z_{j+1}, \cdots, z_n), z \in B_\delta) + \cdots \\ & + (-1)^n C(U; (\bar{z}_1, \cdots, \bar{z}_n), z \in B_\delta), \end{aligned} \tag{56}$$

where there are 2^n elements in this sum. Using (56), a stronger result than Theorem 14 can be obtained for the octant; we can prove that $P(U; z \in B_\delta) \to U$ in the topology of $\mathscr{S}'$ as $\operatorname{Im}(z) \to 0$, $z \in B_\delta$. Further, consider the set of functions $f(z) \in G_\delta$ which are analytic in B_δ and satisfy

$$|f(z)| \leqq K_\gamma (1 + |z|)^N$$

in $\{z : \delta_j(\operatorname{Im}(z_j)) \geqq \gamma_j > 0,\ j = 1, \cdots, n\}$ where $\gamma = (\gamma_1, \cdots, \gamma_n)$. We can show that if $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, is the $\mathscr{S}'$ boundary value of $f(z) \in G_\delta$ for fixed δ, then there exists an element $V \in \mathscr{S}'$ with $\operatorname{supp}(V) \subseteq \{t : -\infty < \delta_j t_j \leqq 0, j = 1, \cdots, n\}$ such that

$$(57) \qquad \begin{aligned} f(z) &= \langle V, e^{-2\pi i\langle z,t\rangle}\rangle = C(U; z \in B_\delta) \\ &= P(U; z \in B_\delta), \qquad z \in B_\delta. \end{aligned}$$

Thus for the octant more can be said about the function $f(z)$ than in Corollary 1 of the present paper. Conversely, if we have that $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, is the Fourier transform of $V \in \mathscr{S}'$, $\operatorname{supp}(V) \subseteq \{t : -\infty < \delta_j t_j \leqq 0, j = 1, \cdots, n\}$, then U is the $\mathscr{S}'$ boundary value of a function $f(z) \in G_\delta$ such that (57) holds. Details of these results for the octant are contained in Carmichael [7].

Several of our results in this section are reminiscent of classical results in which analytic functions are related to their boundary values. Hille and Tamarkin [20], [21], [22] have obtained theorems concerning functions which are analytic in a half-plane and which obtain boundary values and have applied their results to the specific set of Hardy H^p-spaces. Theorems 10, 11 and 12 and Corollary 1 of the present paper have classical counterparts which have been obtained by Hille and Tamarkin (see [20, Theorems 1 and 3] and [21]). Zygmund [23], [24] extended the classical H^p-spaces to n dimensions by considering functions analytic in the polydisc $\{z \in \mathbb{C}^n : |z_j| < 1, j = 1, \cdots, n\}$ and in the octant $B_{(1,\cdots,1)}$. A further extension of H^p-spaces has been obtained by Korányi [14] and Stein, Weiss and Weiss [15] in which the domain of analyticity is a tubular radial domain. Our theorems concerning the space of functions G_C^0 and the results stated above concerning the space G_δ of functions analytic in an octant can be viewed as generalizations of H^p-space results. In our distributional setting the space of functions having boundary values and representable by Cauchy (Poisson) integrals has been enlarged from H^p to G_C^0 ($H^p(T^C) \subset G_C^0$), and the set of admissible boundary values has been enlarged from L^p-functions to $\mathscr{D}'_{L^p}$-distributions ($L^p \subset \mathscr{D}'_{L^p}$.) Further, our theorems concerning the boundary values of the generalized Cauchy and Poisson integrals generalize corresponding classical theorems. The distributional setting is obtained without altering the essential structure of the classical setting; that is, the relations between the analytic functions, the corresponding Cauchy (Poisson) integrals, and the boundary values are retained in the distributional setting. However, the topology employed in obtaining our distributional results is weaker than that of pointwise or norm convergence.

5. Functions analytic in tubular cones. Let C be an open cone which is not necessarily connected. In this section we shall obtain distributional boundary value results for functions analytic in tubular cones $T^C = \mathbb{R}^n + iC$. Some of the theorems in this section are generalizations of results contained in § 4.

Suppose C is an open cone which is the countable union of open connected cones, $C = \bigcup_\lambda C_\lambda$, $\lambda \in \Lambda$. Let $U \in \mathscr{S}'$ such that $\operatorname{supp}(U) \subseteq C^*$; and put $f(z) = \langle U, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in T^{O(C)}$. By Theorem 8, $f(z)$ is analytic in $T^{O(C)}$ and satisfies (30) for $b = 0$. Suppose now that y is restricted to C_λ, $\lambda \in \Lambda$; and put $f_\lambda(z) = f(z)$, $z \in T^{O(C_\lambda)}$. Since $f_\lambda(z)$ is analytic in $T^{O(C_\lambda)}$ and satisfies (30) for $b = 0$ in $T^{C'_\lambda}$, $C'_\lambda \subset O(C_\lambda)$, then by a theorem of Vladimirov [12, Theorem, p. 235] we have $f_\lambda(z) \to W_\lambda \in \mathscr{S}'$ in the topology of $\mathscr{S}'$ as $y \to 0$, $y \in C'_\lambda \subset O(C_\lambda)$, $\lambda \in \Lambda$. Assume now that $W_\lambda \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, $\lambda \in \Lambda$. Since $b = 0$, then by (30) we have $f_\lambda(z) \in G^0_{O(C_\lambda)}$, $\lambda \in \Lambda$. Applying Corollary 1 to each $f_\lambda(z)$, we obtain $f_\lambda(z) = C(W_\lambda; z)$, $z \in T^{C_\lambda}$, $C'_\lambda \subset O(C_\lambda)$; and W_λ is the Fourier transform of an element

$$V_\lambda = \sum_{|\beta| \leqq m} t^\beta h_{\lambda\beta}(t),$$

where $\operatorname{supp}(V_\lambda) \subseteq \{t : u_{O(C_\lambda)}(t) \leqq 0\}$. This proves the following generalization of Theorem 10.

THEOREM 15. *Let C be an open cone which is the countable union of open connected cones, $C = \bigcup_\lambda C_\lambda$, $\lambda \in \Lambda$. Let $U \in \mathscr{S}'$ with* $\operatorname{supp}(U) \subseteq C^*$; *and put $f(z) = \langle U, e^{2\pi i \langle z,t \rangle} \rangle$, $z \in T^{O(C)}$. Suppose the $\mathscr{S}'$ boundary values W_λ of $f_\lambda(z)$, which exist in the $\mathscr{S}'$ topology, are elements of $\mathscr{D}'_{L^p}$, $1 < p \leqq 2$. Then for all $\lambda \in \Lambda$,*

$$f_\lambda(z) = \langle V_\lambda, e^{2\pi i \langle z,t \rangle} \rangle = C(W_\lambda; z), \qquad z \in T^{C_\lambda}, \quad C'_\lambda \subset O(C_\lambda);$$

and W_λ is the Fourier transform of $V_\lambda = \sum_{|\beta| \leqq m} t^\beta h_{\lambda\beta}(t)$, $h_{\lambda\beta}(t) \in L^q$, $1/p + 1/q = 1$, where $\operatorname{supp}(V_\lambda) \subseteq \{t : u_{O(C_\lambda)}(t) \leqq 0\}$.

We now wish to extend the generalized Cauchy and Poisson integrals to tubular cones. In the proof of Theorem 1 one of the main objectives was to show that differentiation under the integral sign in $K(z - t)$ was justified. To do this we used Lemma 2, p. 223, of Vladimirov [12]. However, this lemma holds for any open cone C; C does not have to be connected. Thus using exactly the same proof as in Theorem 1, we have that $K(z - t) \in \mathscr{D}_{L^q}$, $1/p + 1/q = 1$, $1 < p \leqq 2$, for fixed $z \in T^{O(C)} \setminus \{z : y \in O(C),\ y_j = 0 \text{ for any } j = 1, \cdots, n\}$, where C is any open cone. But then using the same proof as in Theorem 3 we have for any open cone C that $Q(z; t) \in \mathscr{D}_{L^q}$, $1/p + 1/q = 1$, $1 < p \leqq 2$, for fixed $z \in T^{O(C)} \setminus \{z : y \in O(C),$ $y_j = 0 \text{ for any } j = 1, \cdots, n\}$. Using these facts and Theorems 9 and 14 we have the following two theorems.

THEOREM 16. *Let C be an open cone which is the countable union of open connected cones, $C = \bigcup_\lambda C_\lambda$, $\lambda \in \Lambda$. Then the generalized Cauchy integral $C(U; z)$ of $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, exists for $z \in T^{O(C)} \setminus \{z : y \in O(C),\ y_j = 0$ for any $j = 1, \cdots, n\}$; and for each $T^{O(C_\lambda)}$, $\lambda \in \Lambda$,*

$$\left\langle U, \int_{C^*_\lambda} e^{2\pi i \langle z - t, \eta \rangle}\, d\eta \right\rangle \to \mathscr{F}[I_{C^*_\lambda}(\eta) \mathscr{F}^{-1}(U)]$$

in the topology of $\mathscr{D}'_{L^p}$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C_\lambda)$.

THEOREM 17. *Let C be an open cone as in Theorem 16. Then the generalized Poisson integral $P(U; z)$ of $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, exists for $z \in T^{O(C)} \setminus \{z : y \in O(C),$ $y_j = 0$ for any $j = 1, \cdots, n\}$; and for each $T^{O(C_\lambda)}$, $\lambda \in \Lambda$,*

$$\left\langle U, \frac{K(z - t)\overline{K(z - t)}}{K(2iy)} \right\rangle \to U$$

in the topology of $\mathscr{D}'_{L^p}$ as $y = \operatorname{Im}(z) \to 0$, $y \in O(C_\lambda)$.

Theorems 16 and 17 point out the basic difference in the boundary values attained by the generalized Cauchy and Poisson integrals. The Cauchy integral corresponding to $T^{O(C_\lambda)}$ converges to a boundary value which depends on $\lambda \in \Lambda$; while the generalized Poisson integral corresponding to $T^{O(C_\lambda)}$, $\lambda \in \Lambda$, always has U as its boundary value.

For the remainder of this section we shall be concerned with open, disconnected cones C which are the union of a finite number of open, connected cones C_j, $j = 1, \cdots, m$. In the following theorem, which is a generalization of Theorem 13, the $f_j(z), j = 1, \cdots, m$, represent the analytic functions whose boundary values form the decomposition of U in Theorem 13.

THEOREM 18. *Let C be an open cone such that $C = \bigcup_{j=1}^m C_j$, where C_j, $j = 1, \cdots, m$, are open connected cones such that the assumptions of* Theorem 13 *hold. Let $U \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq 2$. Suppose the boundary values W_j, $j = 1, \cdots, m$, which form the decomposition of U, are equal. Then there exists an element $V \in \mathscr{S}'$, with* $\operatorname{supp}(V) \subseteq \{t : u_{O(C)}(t) \leqq 0\}$, *and a function $f(z)$, which is analytic in $T^{O(C)}$ and satisfies* (30) *for $b = 0$ and $z \in T^{C'}$, $C' \subset O(C)$, such that $mV = \mathscr{F}^{-1}(U)$ and $f(z) = f_j(z)$, $z \in T^{O(C_j)}$, $j = 1, \cdots, m$.*

Proof. By Theorem 13, $U = \sum_{j=1}^m W_j$, where W_j is the $\mathscr{S}'$ boundary value of a function $f_j(z) \in G^0_{O(C_j)}$, and $W_j = \hat{V}_j$ such that $\operatorname{supp}(V_j) \subseteq \{t : u_{C_j}(t) \leqq 0\}$. By hypothesis $W_1 = \cdots = W_m$, and we call this common value W. But $V_j = \mathscr{F}^{-1}(W_j)$, $j = 1, \cdots, m$. Thus $V_1 = \cdots = V_m$, and we call this common value V. Hence $W = \hat{V}$; and since $U = \sum_{j=1}^m \hat{V}_j$, then $mV = \mathscr{F}^{-1}(U)$. From $\operatorname{supp}(V_j)$, $j = 1, \cdots, m$, we see that V vanishes on $\bigcup_{j=1}^m \{t : u_{C_j}(t) > 0\}$. Now

$$u_C(t) = \max_{j=1,\cdots,m} u_{C_j}(t),$$

and from the definition of ρ_C we have $u_{O(C)}(t) \leqq \rho_C u_C(t)$. Thus

$$u_{O(C)}(t) \leqq \rho_C \max_{j=1,\cdots,m} u_{C_j}(t); \tag{58}$$

and by a lemma of Vladimirov [12, Lemma 3, p. 220], $1 \leqq \rho_C < +\infty$. Now consider the set $J = \{t : u_{O(C)}(t) > 0\}$. If $t \in J$, then by (58), $t \in \{t : \max_{j=1,\cdots,m} u_{C_j}(t) > 0\}$. Hence $t \in \bigcup_{j=1}^m \{t : u_{C_j}(t) > 0\}$, and on this set V vanishes. Thus V vanishes if $t \in J$ which implies that $\operatorname{supp}(V) \subseteq \{t : u_{O(C)}(t) \leqq 0\}$. Putting $f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in T^{O(C)}$, and applying Theorem 8, we obtain that $f(z)$ is analytic in $T^{O(C)}$ and satisfies (30) for $b = 0$. Further, since $V_j = V$, $j = 1, \cdots, m$, then by the definition of $f_j(z)$ from the proof of Theorem 13, we have $f(z) = f_j(z)$, $z \in T^{O(C_j)}$, $j = 1, \cdots, m$, The proof is complete.

THEOREM 19. *Let C be an open cone such that $C = \bigcup_{j=1}^m C_j$, where C_j, $j = 1, \cdots, m$, are open connected cones. Let $f(z)$ be analytic in the tubular cone T^C and satisfy* (29) *for $z \in T^{C'}$, $C' \subset C$. For each C_j, $j = 1, \cdots, m$, suppose that*

$$\lim_{\substack{y \to 0 \\ y \in C'_j \subset C_j}} f(z) = U_j \in \mathscr{D}'_{L^p}, \qquad 1 \leqq p \leqq \infty,$$

in the topology of $\mathscr{S}'$; and assume that $U_1 = U_2 = \cdots = U_m$. Then $f(z)$ is analytic in $\bigcup_{j=1}^m T^{O(C_j)}$ and satisfies (30) *for $z \in T^{C'}$, $C' \subset \bigcup_{j=1}^m O(C_j)$; and $f(z)$ has an analytic extension to $T^{O(C)}$.*

Proof. For each $j = 1, \cdots, m$, $f(z)$ is analytic in T^{C_j} and satisfies (29) for $z \in T^{C'_j}$, $C'_j \subset C_j$; and $f(z) \to U_j \in \mathscr{D}'_{L^p}$, $1 \leqq p \leqq \infty$, as $y \to 0$, $y \in C'_j \subset C_j$. Thus by Theorem 6, for each U_j there exists a $V_j \in \mathscr{S}'$ such that $\operatorname{supp}(V_j) \subseteq \{t : u_{C_j}(t) \leqq b\}$, $\hat{V}_j = U_j$; and

$$f(z) = \langle V_j, e^{2\pi i\langle z,t\rangle}\rangle, \qquad z \in T^{C'_j}, \quad C'_j \subset C'_j, \quad j = 1, \cdots, m. \tag{59}$$

By hypothesis, $U_1 = U_2 = \cdots = U_m = U$. As in the proof of Theorem 18, we obtain $V_1 = V_2 = \cdots = V_m = V$, where $\operatorname{supp}(V) \subseteq \{t : u_{O(C)}(t) \leqq b\rho_C\}$. Since $f(z)$ is analytic in T^{C_j}, $j = 1, \cdots, m$, we have by Bochner's theorem on analytic extension of tube domains [25, Chap. V] that $f(z)$ is analytic in $T^{O(C_j)}$, $j = 1, \cdots, m$. Thus $f(z)$ is analytic in $\bigcup_{j=1}^m T^{O(C_j)}$. By (59), Theorem 8, and the identity theorem for analytic functions, we have $f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z \in T^{O(C_j)}$, $j = 1, \cdots, m$; so that

$$f(z) = \langle V, e^{2\pi i\langle z,t\rangle}\rangle, \quad z \in \bigcup_{j=1}^{m} T^{O(C_j)}.$$

Applying Theorem 8, we see that $f(z)$ satisfies (30) for $z \in T^{C'}$, $C' \subset \bigcup_{j=1}^m O(C_j)$; and $\langle V, e^{2\pi i\langle z,t\rangle}\rangle$ is the analytic extension of $f(z)$ to $T^{O(C)}$.

We note that if p is restricted to $1 \leqq p \leqq 2$ in Theorem 19, then the exact form of V can be calculated as in (31).

6. Miscellaneous. In this section we briefly discuss some topics which will be considered in future research.

It is well known that the classical Poisson integral of a function $f \in L^p(\mathbb{R}^1)$, $1 \leqq p \leqq \infty$, is a harmonic function in the upper half-plane (see Hoffman [19, p. 123]). Korányi [14] has shown that the Poisson integral of $f \in L^p(\mathbb{R}^n)$, $1 \leqq p < \infty$, for tubular radial domains corresponding to homogeneous self-dual cones is a harmonic function with respect to a Laplace–Beltrami type operator. (See Hua [26] for the construction of such operators for the classical domains.) We have proved in Carmichael [7] that the generalized Poisson integral of $U \in \mathscr{D}'_{L^p}$, $1 < p \leqq 2$, corresponding to octants is an n-harmonic function with respect to the usual Laplace operator. It would be interesting to obtain a Laplace–Beltrami type operator such that the generalized Poisson integral for general tubular radial domains, as defined in this paper, is harmonic with respect to this operator.

In this paper we have related $U \in \mathscr{D}'_{L^p}$ to the space of functions G_C^b and have obtained generalizations of classical H^p-space theorems. To obtain our results it was necessary to consider only the spaces $\mathscr{D}'_{L^p}$, $1 < p \leqq 2$, and in most of our results $\mathscr{D}'_{L^p}$ was considered as a subset of $\mathscr{S}'$. We conjecture that the generalized Cauchy and Poisson integrals of $U \in \mathscr{D}'_{L^p}$, $1 < p < \infty$, exist for tubular radial domains and that results generalizing those of Tillmann [1] can be obtained using only the topology of $\mathscr{D}'_{L^p}$. We shall consider these problems in a future paper.

REFERENCES

[1] H. G. TILLMANN, *Distributionen als Randverteilungen analytischer Funktionen. II*, Math Z., 76 (1961), pp. 5–21.

[2] Z. LUSZCZKI AND Z. ZIELEŻNY, *Distributionen der Räume $\mathscr{D}'_{L^p}$ als Randverteilungen analytischer Funktionen*, Colloq. Math., 8 (1961), pp. 125–131.

[3] E. J. BELTRAMI AND M. R. WOHLERS, *Distributional boundary value theorems and Hilbert transforms*, Arch. Rational Mech. Anal., 18 (1965), pp. 304–309.

[4] ———, *Distributional boundary values of functions holomorphic in a half plane*, J. Math. Mech., 15 (1966), pp. 137–146.

[5] ———, *Distributions and the Boundary Values of Analytic Functions*, Academic Press, New York, 1966.

[6] ———, *The Cauchy integral of tempered distributions and some theorems on analytic continuation*, SIAM J. Appl. Math., 15 (1967), pp. 1077–1087.

[7] RICHARD D. CARMICHAEL, *Distributional boundary values in $\mathscr{D}'_{L^p}$. II*, Rend. Sem. Mat. Univ. Padova, 45 (1971), pp. 249–277.

[8] ———, *Distributional boundary values of functions analytic in tubular radial domains*, Indiana Univ. Math. J. (formerly J. Math. Mech.), 20 (1971), pp. 843–853.

[9] ———, *Distributional boundary values in $\mathscr{D}'_{L^p}$*, Rend. Sem. Mat. Univ. Padova, 43 (1970), pp. 35–53.

[10] L. SCHWARTZ, *Théorie des distributions*, Hermann, Paris, 1966.

[11] S. BOCHNER, *Group invariance of Cauchy's formula in several variables*, Ann. of Math., 45 (1944), pp. 686–707.

[12] V. S. VLADIMIROV, *Methods of the Theory of Functions of Several Complex Variables*, M.I.T. Press, Cambridge, Mass., 1966.

[13] L. SCHWARTZ, *Mathematics for the Physical Sciences*, Addison-Wesley, Reading, Mass., 1966.

[14] ADAM KORÁNYI, *A Poisson integral for homogeneous wedge domains*, J. Analyse Math., 14 (1965), pp. 275–284.

[15] E. M. STEIN, GUIDO WEISS AND MARY WEISS, *H^p classes of holomorphic functions in tube domains*, Proc. Nat. Acad. Sci. U.S.A., 52 (1964), pp. 1035–1039.

[16] RICHARD D. CARMICHAEL, *Functions analytic in an octant and boundary values of distributions*, J. Math. Anal. Appl., 33 (1971), pp. 616–626.

[17] R. F. STREATER AND A. S. WIGHTMAN, *PCT, Spin, and Statistics, and All That*, W. A. Benjamin, New York, 1964.

[18] V. S. VLADIMIROV, *On functions holomorphic in tubular cones*, Izv. Akad. Nauk SSSR Ser. Mat., 27 (1963), pp. 75–100 (in Russian).

[19] KENNETH HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[20] EINAR HILLE AND J. D. TAMARKIN, *On a theorem of Paley and Wiener*, Ann. of Math., 34 (1933), pp. 606–614.

[21] ———, *A remark on Fourier transforms and functions analytic in a half plane*, Compositio Math., 1 (1934), pp. 98–102.

[22] ———, *On the absolute integrability of Fourier transforms*, Fund. Math., 25 (1935), pp. 329–352.

[23] A. ZYGMUND, *On the boundary values of functions of several variables*, Ibid., 36 (1949), pp. 207–235.

[24] ———, *Trigonometric Series*, vols. I and II, 2nd. ed., Cambridge University Press, London, 1968.

[25] S. BOCHNER AND W. T. MARTIN, *Several Complex Variables*, Princeton University Press, Princeton, N.J., 1948.

[26] L. K. HUA, *Harmonic analysis of functions of several complex variables in the classical domains*, American Math. Soc. Translations of Mathematical Monographs, vol. 6, Providence, R.I., 1963.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

ON WIENER-HOPF OPERATORS ASSOCIATED WITH THE HANKEL TRANSFORM*

G. GREGORY STEPHEN†

Abstract. Asymptotic estimates of determinants of finite section Wiener–Hopf operators associated with the Fourier transform were obtained initially by Szegö and Kac. In 1966, Davis and Hirschman obtained similar results for operators associated with ultraspherical polynomials. In this paper we develop such estimates for determinants of finite section Wiener–Hopf integral operators associated with the Hankel transform. These estimates are continuous analogues of the Davis–Hirschman results.

1. Introduction. Let ν be a fixed positive number and set $d\mu(x) = k_\nu^{-1}x^{2\nu}\,dx$, where $k_\nu = 2^{\nu-1/2}\Gamma(\nu + 1/2)$. Let c be a function in $L^1((0, \infty); d\mu)$ whose Hankel transform $\hat{c}$ belongs to $L^1((0, \infty); d\mu)$. We denote by $c(x, y)$ the Hankel transform analogue of the translated function $c(x - y)$ (see § 2). For each $r > 0$, we define an operator T_r on $L^2([0, r]; d\mu)$ by

$$T_r f \cdot (x) = \int_0^r c(x, y) f(y)\, d\mu(y).$$

This is a nuclear operator (trace class). Let $\{\lambda_j(r)\}$ be a sequence consisting of all the nonzero eigenvalues of T_r enumerated in order of decreasing absolute value, each eigenvalue repeated according to its algebraic multiplicity. The determinant $D(r)$ of $I - T_r$ is defined by

$$D(r) = \Pi(1 - \lambda_j(r)).$$

Define

$$G = \exp\left(\frac{1}{\pi}\int_0^\infty \log(1 - \hat{c}(t))\,dt\right),$$

$$E = \exp\left(\frac{1}{2}\int_0^\infty x\left\{\frac{1}{2\pi}\int_{-\infty}^\infty \log(1 - \hat{c}(t))\,e^{-itx}\,dt\right\}^2 dx\right),$$

extending $\hat{c}$ as an even function. If $\int |c(t)|\,d\mu(t) < 1$, then G is finite and we shall show that

$$[D(r)]^{1/r} = G + o(1) \quad \text{as} \quad r \to \infty. \tag{1.1}$$

If we also have $\int_0^\infty t|c(t)|\,d\mu(t) < \infty$, then E is finite and

$$D(r)G^{-r} = [1 - \hat{c}(0)]^\alpha E + o(1) \quad \text{as} \quad r \to \infty, \tag{1.2}$$

where $\alpha = (1 - 2\nu)/4$. All logs and powers throughout the paper are principal values.

* Received by the editors June 11, 1971, and in revised form April 14, 1972.

† Department of Mathematical Studies, Southern Illinois University, Edwardsville, Illinois 62025. This paper is based on work contained in the author's doctoral dissertation at the University of New Mexico.

The condition $\int |c|\, d\mu < 1$ allows us to write

$$D(r) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}(n, r)\right),$$

where $\operatorname{tr}(n, r) = \operatorname{tr}(T_r^n)$ is the trace of $(T_r)^n$. In §§ 4 and 5 we develop asymptotic formulae for $\operatorname{tr}(n, r)$. In § 6, following Kac [10], we combine these to obtain the determinant estimates (1.1) and (1.2). Sections 2 and 3 describe the Hankel transform theory that is needed.

The prototypes of these formulae, for determinants of finite sections of Toeplitz matrices, were discovered by Szegö in 1915 and 1952; see [5]. In [10], Kac, Murdock and Szegö introduced the continuous analogue of the Toeplitz matrix, an integral operator with a displacement kernel, and proved a formula like (1.1). Then, in [11], Kac obtained a formula similar to (1.2).

More recently, Davis and Hirschman [2] developed these estimates in a new setting: for Toeplitz matrices associated with ultraspherical polynomials. Their work represents the first such results outside of a Fourier transform setting. Since the Hankel transform is the continuous analogue of the ultraspherical transform, and considering the parallel between the Szegö limit theorems for the discrete and continuous Fourier transform, one would expect estimates like (1.1) and (1.2).

The original Davis–Hirschman results have been improved [9] and extended to a wider class of Jacobi polynomials [1]. New proofs resulting in stronger theorems have been given in the Fourier case [8], [3], [6].

2. The Hankel transform. Let v be an arbitrary but fixed positive number and set

$$d\mu(x) = k_v^{-1} x^{2v}\, dx,$$

where $k_v = 2^{v-1/2}\Gamma(v + \frac{1}{2})$. We denote by $L_v^p(\Omega)$ the Banach space of complex-valued measurable functions f defined on $\Omega \subset [0, \infty)$ for which

$$\|f\|_p = \left[\int_\Omega |f(x)|^p\, d\mu(x)\right]^{1/p} < \infty.$$

We define

$$D(x, y, z) = \frac{2^{3v-5/2}\Gamma(v + \frac{1}{2})^2}{\Gamma(\frac{1}{2})\Gamma(v)} (x\, y\, z)^{1-2v} \Delta(x, y, z)^{2v-2},$$

where $\Delta(x, y, z)$ is the area of a triangle with side lengths x, y, z if there is such a triangle; otherwise set $D(x, y, z) = 0$. Finally, let

$$\mathbf{J}(x) = k_v x^{1/2-v} J_{v-1/2}(x),$$

where J_α is the Bessel function of the first kind of order α.

We have

$$|\mathbf{J}(x)| \leqq 1 \tag{2.1}$$

for all x, $0 \leqq x < \infty$; see [7]. Since

$$J_\alpha(x) = O(x^{-1/2}) \quad \text{as} \quad x \to \infty,$$

we have

$$\mathbf{J}(x) = O(x^{-\nu}) \quad \text{as} \quad x \to \infty. \tag{2.2}$$

Combining (2.1) and (2.2) we see that

$$|\mathbf{J}(x)| \leqq M(1 + x^{\nu})^{-1} \tag{2.3}$$

for all $x \geqq 0$, where M is a constant which depends only on ν.

The integral formula

$$\mathbf{J}(xt)\mathbf{J}(yt) = \int_0^\infty \mathbf{J}(zt)D(x, y, z)\, d\mu(z) \tag{2.4}$$

is valid for $x, y > 0$ and $t \geqq 0$; see [12, p. 367]. By setting $t = 0$ in (2.4) we obtain

$$\int_0^\infty D(x, y, z)\, d\mu(z) = 1.$$

Note that $D(x, y, z) \geqq 0$ and that $D(x, y, z)$ is symmetric in x, y and z.

If a function f is locally integrable (with respect to $d\mu$) and $y > 0$, we associate with f the "translated" function $f(x, y)$ defined for almost all x by

$$f(x, y) = \int f(z)D(x, y, z)\, d\mu(z).$$

If f is bounded, then $|f(x, y)| \leqq \|f\|_\infty$ and if $f \in L_\nu^p(0, \infty)$, then $\|f(\cdot, y)\|_p \leqq \|f\|_p$.

If $f, g \in L_\nu^1(0, \infty)$, the Hankel convolution of f and g is given by

$$(f \# g)(x) = \int f(y)g(x, y)\, d\mu(y).$$

This integral converges absolutely for almost all x, $0 < x < \infty$, and $\|f \# g\|_1 \leqq \|f\|_1 \|g\|_1$ (cf. Young's inequality).

The Hankel transform of $f \in L_\nu^1(0, \infty)$ is defined as

$$\hat{f}(x) = \int f(t)\mathbf{J}(xt)\, d\mu(t).$$

Among the basic properties of this transform are the following:

(a) $(f \# g)\hat{} = \hat{f}\hat{g}$;

(b) $\hat{f}$ is bounded and continuous on $[0, \infty)$;

(c) if $\hat{f} \in L_\nu^1(0, \infty)$, then $(\hat{f})\hat{} = f$ a.e.

These properties of the Hankel transform and convolution are developed in detail in [7].

PROPOSITION 2.1. *Let $f \in L_\nu^1(0, \infty)$. Then*

$$\int_0^\infty f(t, y)\mathbf{J}(tx)\, d\mu(t) = \hat{f}(x)\mathbf{J}(xy),$$

and if $g = \hat{f}$, then

$$g(x, y) = \int_0^\infty f(t)\mathbf{J}(xt)\mathbf{J}(yt)\, d\mu(t).$$

These are proved in the same way as their Fourier transform analogues with identity (2.4) playing the role of the group character property $\chi(u)\chi(v) = \chi(u + v)$.

We shall frequently work with functions $f \in L^1_\nu(0, \infty)$ for which $\hat{f} \in L^1_\nu(0, \infty)$. Such functions are essentially bounded and hence are integrable on $(0, \infty)$ with respect to ordinary Lebesgue measure.

3. A relation between the Hankel and Fourier transforms. For $x > 0$, we define

$$E(h, x) = \begin{cases} b_\nu x^{1-2\nu}(x^2 - h^2)^{\nu-1} & \text{if } 0 < |h| < x, \\ 0 & \text{if } 0 < x < |h|, \end{cases}$$

where $b_\nu = \Gamma(\nu + 1/2)/(\Gamma(1/2)\Gamma(\nu))$.

LEMMA 3.1 (a) $E(h, x) \geqq 0$,

(b) $\int_{-\infty}^{\infty} E(h, x)\, dh = 1$,

(c) $\int_{-\infty}^{\infty} |h|^\alpha E(h, x)\, dh \leqq x^\alpha$, $\alpha \geqq 0$,

(d) $\int_{-\infty}^{\infty} E(h, x)\, e^{iht}\, dh = \mathbf{J}(tx)$.

Proof. By evaluating the integral we obtain (b); (c) is a consequence of (b) and the finite support of E; identity (d) is a variant of the Lommel integral representation of the Bessel function.

For $f \in L^1_\nu(0, \infty)$ and $t > 0$, set

$$Sf \cdot (t) = \int_0^\infty E(t, x) f(x)\, d\mu(x).$$

This formula defines Sf a.e. in $(-\infty, \infty)$ and, moreover,

$$\int_{-\infty}^{\infty} |Sf \cdot (t)|\, dt \leqq \int_0^\infty |f(x)|\, d\mu(x).$$

Also

$$\int_{-\infty}^{\infty} (Sf)(t)\, e^{-ith}\, dt = \int_0^\infty f(x)\mathbf{J}(hx)\, d\mu(x).$$

Thus Sf is an even function in $L^1(-\infty, \infty)$ whose Fourier transform is equal to the Hankel transform of f.

The next two propositions indicate how we shall encounter the mapping S. Both will be used in § 4.

PROPOSITION 3.2. *Let x, y, z be fixed real numbers, $z > 0$. Then*

$$\lim_{r \to \infty} k_\nu^{-1}(r + x)^\nu (r + y)^\nu D(r + x, r + y, z) = E(x - y, z).$$

Proof. If u, v, w are the sides of a triangle and $\Delta(u, v, w)$ is the area of this triangle, then

$$\Delta(u, v, w)^2 = s(s - u)(s - v)(s - w),$$

where $s = (u + v + w)/2$. Thus for $|x - y| < z$ and r large,

$$k_\nu^{-1}(r + x)^\nu (r + y)^\nu D(r + x, r + y, z)$$
$$= 2^{2-2\nu} b_\nu \left\{\frac{(2r + x + y)^2 - z^2}{(r + x)(r + y)}\right\}^{\nu-1} z^{1-2\nu}[z^2 - (x - y)^2]^{\nu-1}.$$

It is now evident that the limit of this expression as $r \to \infty$ is $E(x-y,z)$. When $|x-y| \geqq z$, $D(r+x, r+y, z)$ is zero for all r.

Formally, the next proposition follows from the preceding material. We give a direct proof.

PROPOSITION 3.3. *If c and its Hankel transform $\hat{c}$ are in $L_\nu^1(0,\infty)$, then*

$$\lim_{r\to\infty} k_\nu^{-1}(r+x)^\nu(r+y)^\nu c(r+x, r+y) = (Sc)\cdot(x-y).$$

Proof. Set

$$I(r) = k_\nu^{-1}(r+x)^\nu(r+y)^\nu c(r+x, r+y).$$

Then, by Proposition 2.1,

$$I(r) = \int_0^\infty \hat{c}(t)[(r+x)t]^{1/2}J_{\nu-1/2}[(r+x)t][(r+y)t]^{1/2}J_{\nu-1/2}[(r+y)t]\,dt.$$

Since

$$z^{1/2}J_{\nu-1/2}(z) = (2/\pi)^{1/2}\cos\left(z - \frac{\nu\pi}{2}\right) + O(z^{-1})$$

as $z \to \infty$, and $z^{1/2}J_{\nu-1/2}(z)$ is bounded, we have

$$z^{1/2}J_{\nu-1/2}(z) = (2/\pi)^{1/2}\cos\left(z - \frac{\nu\pi}{2}\right) + R(z),$$

where $|R(z)| \leqq M(1+z)^{-1}$ for all $z \geqq 0$ and M is a constant depending only on ν. Thus

$$\begin{aligned}
&[(r+x)t]^{1/2}J_{\nu-1/2}[(r+x)t][(r+y)t]^{1/2}J_{\nu-1/2}[(r+y)t] \\
&\quad= \frac{1}{\pi}\cos[t(x-y)] + \frac{1}{\pi}\cos[2rt + (x+y)t - \nu\pi] \\
&\qquad+ (2/\pi)^{1/2}\cos\left[(r+x)t - \frac{\nu\pi}{2}\right]R[(r+y)t]\cdot \\
&\qquad+ (2/\pi)^{1/2}\cos\left[(r+y)t - \frac{\nu\pi}{2}\right]R[(r+x)t] \\
&\qquad+ R[(r+x)t]R[(r+y)t],
\end{aligned}$$

so that

$$\begin{aligned}
I(r) &= \frac{1}{\pi}\int_0^\infty \hat{c}(t)\cos[t(x-y)]\,dt \\
&\quad+ \frac{1}{\pi}\int_0^\infty \hat{c}(t)\cos[2rt + (x+y)t - \nu\pi]\,dt \\
&\quad+ (2/\pi)^{1/2}\int_0^\infty \hat{c}(t)\cos\left[(r+x)t - \frac{\nu\pi}{2}\right]R[(r+y)t]\,dt
\end{aligned}$$

(cont.)

$$+ (2/\pi)^{1/2} \int_0^\infty \hat{c}(t) \cos\left[(r + y)t - \frac{\nu\pi}{2}\right] R[(r + x)t]\, dt$$

$$+ \int_0^\infty \hat{c}(t) R[(r + x)t] R[(r + y)t]\, dt.$$

It is clear that the last three integrals go to zero as $r \to \infty$. By the Riemann–Lebesgue lemma,

$$\int_0^\infty \hat{c}(t) \cos[2rt + (x + y)t - \nu\pi]\, dt \to 0$$

as $r \to \infty$.

4. Trace estimates. Let $c \in L_\nu^1(0, \infty)$ and let $\operatorname{tr}(n, r)$ denote the trace of the operator $(T_r)^n$. For $n \geqq 1$,

$$\operatorname{tr}(n, r) = \int \cdots \int c(x_1, x_2) c(x_2, x_3) \cdots c(x_n, x_1)\, d\mu(x_1) \cdots d\mu(x_n),$$

where the integration is over the cube $[0, r]^n$ in R^n. We shall decompose this integral into two parts and estimate each.

Let

$$H(n, r) = \int_0^r \int_0^\infty \cdots \int_0^\infty c(x_1, x_2) c(x_2, x_3) \cdots c(x_n, x_1)\, d\mu(x_1) \cdots d\mu(x_n).$$

Using an induction argument based on Proposition 2.1, one easily proves that for $n \geqq 2$,

$$[\hat{c}(x)]^{n-1} \mathbf{J}(xt) = \int \overset{\infty}{\underset{0}{\cdots}} \int c(t, x_2) c(x_2, x_3) \cdots c(x_{n-1}, x_n) \mathbf{J}(x x_n)\, d\mu(x_n) \cdots d\mu(x_2).$$

Thus if $\hat{c} \in L_\nu^1(0, \infty)$, it follows from the same proposition that

$$H(n, r) = \int_0^\infty [\hat{c}(x)]^n \left[\int_0^r \mathbf{J}^2(xt)\, d\mu(t)\right] d\mu(x).$$

Asymptotic expansions of this integral (as $r \to \infty$) are developed in the next section.

The difference $H(n, r) - \operatorname{tr}(n, r)$ is the integral

$$I(n, r) = \int \cdots \int c(x_1, x_2) c(x_2, x_3) \cdots c(x_n, x_1)\, d\mu(x_1) \cdots d\mu(x_n),$$

the integration being over the region $\Omega(r)$ in R^n consisting of those points $(x_1, \cdots, x_n)$ for which $0 \leqq x_1 \leqq r$, $x_j \geqq 0$ for all j, and $x_j > r$ for at least one j, $2 \leqq j \leqq n$.

THEOREM 4.1. *If c and its Hankel transform $\hat{c}$ are in $L_\nu^1(0, \infty)$, then $\operatorname{tr}(n, r) = H(n, r) + o(r)$ as $r \to \infty$.*

Proof. We can write

$$I(n,r) = \int_{\Omega(r)} \cdots \int \left\{ \int_0^\infty \cdots \int c(z_1)c(z_2) \cdots c(z_{n-1}) D(x_1, x_2, z_1) D(x_2, x_3, z_2) \cdots \right.$$

$$\left. D(x_{n-1}, x_n, z_{n-1})\, d\mu(z_1) \cdots d\mu(z_{n-1}) \right\} c(x_n, x_1)\, d\mu(x_n) \cdots d\mu(x_1),$$

or

$$I(n,r) = \int_0^\infty \cdots \int c(z_1)c(z_2) \cdots c(z_{n-1}) \psi(\mathbf{z}, r)\, d\mu(z_1) \cdots d\mu(z_{n-1}), \tag{4.1}$$

where $\mathbf{z} = (z_1, \cdots, z_{n-1})$ and

$$\psi(\mathbf{z}, r) = \int_{\Omega(r)} \cdots \int D(x_1, x_2, z_1) D(x_2, x_3, z_2) \cdots$$

$$\cdot D(x_{n-1}, x_n, z_{n-1}) c(x_n, x_1)\, d\mu(x_n) \cdots d\mu(x_1).$$

Since c is essentially bounded, the transition to (4.1) is easily justified.

We shall show that $r^{-1}\psi(\mathbf{z}, r)$ is bounded independent of $\mathbf{z}$ and r and that $\psi(\mathbf{z}, r) = o(r)$ as $r \to \infty$ for each $\mathbf{z}$. It will then follow from (4.1) and the dominated convergence theorem that $I(n, r) = o(r)$ as $r \to \infty$.

Set

$$\beta(\mathbf{z}, r; x) = \int_{A(n,r)} \cdots \int D(x, x_1, z_1) D(x_1, x_2, z_2) \cdots$$

$$D(x_{n-2}, x_{n-1}, z_{n-1}) c(x_{n-1}, x)\, d\mu(x_{n-1}) \cdots d\mu(x_1),$$

where $A(n, r) = \{(x_1, \cdots, x_{n-1}) \in R^{n-1} : x_j \geqq 0$ for $j = 1, \cdots, n-1$ and $x_j > r$ for at least one of $j = 1, \cdots, n-1\}$. Then

$$\psi(\mathbf{z}, r) = \int_0^r \beta(\mathbf{z}, r; x)\, d\mu(x).$$

We claim that

$$|r^\nu x^\nu \beta(\mathbf{z}, r; x)| \leqq (n-1)M, \tag{4.2}$$

where M depends only on ν and c. To prove this inequality, we shall decompose the region $A(n, r)$ into $n - 1$ disjoint subregions and obtain a common bound for the integral over each subregion.

Set $A_1(2, r) = A(2, r)$ and let $A_k(n, r) = \{(x_1, \cdots, x_{n-1}) \in A(n, r) : 0 \leqq x_1 \leqq r$ and $(x_2, \cdots, x_{n-1}) \in A_k(n-1, r)\}$ for $k = 1, \cdots, n-2$ and $A_{n-1}(n, r) = \{(x_1, \cdots, x_{n-1}) \in A(n, r) : x_1 > r\}$. To illustrate, we give explicit descriptions of the four subregions of $A(5, r)$:

$A_1(5, r)$:	$0 \leqq x_1 \leqq r,$	$A_2(5, r)$:	$0 \leqq x_1 \leqq r,$
	$0 \leqq x_2 \leqq r,$		$0 \leqq x_2 \leqq r,$
	$0 \leqq x_3 \leqq r,$		$r < x_3 < \infty,$
	$r < x_4 < \infty,$		$0 \leqq x_4 < \infty,$

$$A_3(5, r): \quad 0 \leqq x_1 \leqq r, \qquad A_4(5, r): \quad r < x_1 < \infty,$$
$$r < x_2 < \infty, \qquad 0 \leqq x_2 < \infty,$$
$$0 \leqq x_3 < \infty, \qquad 0 \leqq x_3 < \infty,$$
$$0 \leqq x_4 < \infty, \qquad 0 \leqq x_4 < \infty.$$

For the argument which follows, the important fact is that $x_{n-k} > r$ in the subregion $A_k(n, r)$.

Now,

$$\beta(\mathbf{z}, r; x) = \sum_{k=1}^{n-1} \int_{A_k(n,r)} \cdots \int D(x, x_1, z_1) \cdots \cdot D(x_{n-2}, x_{n-1}, z_{n-1}) c(x_{n-1}, x) \, d\mu(x_{n-1}) \cdots d\mu(x_1). \tag{4.3}$$

Since $\hat{c} \in L^1_\nu(0, \infty)$, $|x^\nu c(x, y) y^\nu| \leqq M$ for all nonnegative x, y, where M depends only on ν and $\int |\hat{c}(t)| \, dt$ (combine (2.3) and the Proposition of § 2). Thus

$$\left| \int_{A_1(n,r)} \cdots \int D(x, x_1, z_1) \cdots D(x_{n-2}, x_{n-1}, z_{n-1}) c(x_{n-1}, x) \, d\mu(x_{n-1}) \cdots d\mu(x_1) \right|$$
$$= \left| \int_0^r \cdots \int_0^r \int_r^\infty D(x, x_1, z_1) \cdots D(x_{n-2}, x_{n-1}, z_{n-1}) c(x_{n-1}, x) \, d\mu(x_{n-1}) \cdots d\mu(x_1) \right|$$
$$\leqq M x^{-\nu} \int_0^r \cdots \int_0^r \int_r^\infty D(x, x_1, z_1) \cdots D(x_{n-2}, x_{n-1}, z_{n-1}) x_{n-1}^{-\nu} \, d\mu(x_{n-1}) \cdots d\mu(x_1)$$
$$\leqq M x^{-\nu} r^{-\nu} \int_0^\infty \cdots \int D(x, x_1, z_1) \cdots D(x_{n-2}, x_{n-1}, z_{n-1}) \, d\mu(x_{n-1}) \cdots d\mu(x_1)$$
$$= M x^{-\nu} r^{-\nu}.$$

For $k > 1$,

$$\int_{A_k(n,r)} \cdots \int D(x, x_1, z_1) \cdots D(x_{n-2}, x_{n-1}, z_{n-1}) c(x_{n-1}, x) \, d\mu(x_{n-1}) \cdots d\mu(x_1)$$
$$= \int_0^r \cdots \int_0^r \int_r^\infty \int_0^\infty \cdots \int_0^\infty D(x, x_1, z_1) \cdots \cdot D(x_{n-2}, x_{n-1}, z_{n-1}) c(x_{n-1}, x) \, d\mu(x_{n-1}) \cdots d\mu(x_1),$$

where $r < x_{n-k} < \infty$. We shall exhibit the method of obtaining a bound for an integral of this type by considering the particular case $n = 5$, $k = 3$. The integral is

$$\int_0^r \int_r^\infty \int_0^\infty \int_0^\infty D(x, x_1, z_1) D(x_1, x_2, z_2) D(x_2, x_3, z_3) \cdot D(x_3, x_4, z_4) c(x_4, x) \, d\mu(x_4) \cdots d\mu(x_1). \tag{4.4}$$

Note that since $\hat{c} \in L^1_\nu(0, \infty)$,

$$\int c(x, w)D(x, y, z)\, d\mu(x) = \int \left\{ \int \hat{c}(t)\mathbf{J}(xt)\mathbf{J}(wt)\, d\mu(t) \right\} D(x, y, z)\, d\mu(x)$$

$$= \int \hat{c}(t)\mathbf{J}(wt) \left\{ \int \mathbf{J}(xt)D(x, y, z)\, d\mu(x) \right\} d\mu(t)$$

$$= \int \hat{c}(t)\mathbf{J}(wt)\mathbf{J}(yt)\mathbf{J}(zt)\, d\mu(t).$$

Thus (4.4) can be written as

$$\int_0^\infty \hat{c}(t)\mathbf{J}(xt)\mathbf{J}(z_3 t)\mathbf{J}(z_4 t)\, d\mu(t) \int_0^r D(x, x_1, z_1)\, d\mu(x_1)$$
(4.5)
$$\int_r^\infty \mathbf{J}(x_2 t)D(x_1, x_2, z_2)\, d\mu(x_2).$$

Since $|\mathbf{J}(x)| \leqq 1$ and $\mathbf{J}(x) = O(x^{-\nu})$ as $x \to \infty$, (4.5) is bounded by

$$Mx^{-\nu} \int_0^\infty |\hat{c}(t)|\, dt \int_0^r D(x, x_1, z_1)\, d\mu(x_1) \int_r^\infty x_2^{-\nu} D(x_1, x_2, z_2)\, d\mu(x_2),$$

where M depends only on $\nu \cdots |x^\nu y^\nu \mathbf{J}(x)\mathbf{J}(y)| \leqq M$. This expression is no larger than

$$Mx^{-\nu}r^{-\nu} \int_0^\infty |\hat{c}(t)|\, dt \int_0^\infty D(x, x_1, z_1)\, d\mu(x_1) \int_0^\infty D(x_1, x_2, z_2)\, d\mu(x_2)$$

$$= Mx^{-\nu}r^{-\nu} \int_0^\infty |\hat{c}(t)|\, dt.$$

Hence each term in the sum (4.3) is bounded by $Mx^{-\nu}r^{-\nu}$, where M depends only on ν and $\int |\hat{c}(t)|\, dt$, and (4.2) is proved.

We now have

$$|\Psi(\mathbf{z}, r)| \leqq \int_0^r |\beta(\mathbf{z}, r; x)|\, d\mu(x)$$

$$\leqq (n - 1)M \int_0^r (x/r)^\nu\, dx,$$

so that

(4.6) $$|\Psi(\mathbf{z}, r)| \leqq (n - 1)Mr$$

for all $\mathbf{z}$.

Fix x, $0 \leqq x < \infty$, and assume that $(x_1, \cdots, x_{n-1}) \in A(n, r)$ is such that the integrand of $\beta(\mathbf{z}, r; x)$ is nonzero. Then, from the definition of $D(a, b, c)$, we must have

$$x_1 < x + z_1,$$

$$x_2 < x_1 + z_2,$$

$$\vdots$$

$$x_{n-1} < x_{n-2} + z_{n-1}.$$

Since at least one of $x_1, \cdots, x_{n-1}$ is greater than r, it follows that

$$r < x + \sum_{k=1}^{n-1} z_k$$

or

$$r - \sum_{k=1}^{n-1} z_k < x.$$

Thus the support of $\beta(\mathbf{z}, r; x)$ as a function of x is contained in the interval $r - \sum z_k \leqq x < \infty$. Therefore,

$$|\Psi(\mathbf{z}, r)| = \left|\int_{r-\Sigma z_k}^{r} \beta(\mathbf{z}, r; x)\, d\mu(x)\right| \leqq (n-1)M \sum_{k=1}^{n-1} z_k, \tag{4.7}$$

so that $\Psi(\mathbf{z}, r) = o(r)$ as $r \to \infty$ for each $\mathbf{z}$.

For future reference, we state the following.

LEMMA 4.2. *If c and its Hankel transform $\hat{c}$ are in $L_\nu^1(0, \infty)$, then*

$$|\mathrm{tr}\,(n, r)| \leqq nrM\|c\|^n,$$

where M is independent of n and r. If, in addition, $\int x|c(x)|\, d\mu(x) < \infty$, then

$$|\mathrm{tr}\,(n, r) - H(n, r)| \leqq n^2 M\|c\|^n.$$

Proof. In the following, M is a constant independent of n and r. It will vary from step to step. From (2.3),

$$|H(n, r)| = \left|\int [\hat{c}(t)]^n \left[\int_0^r \mathbf{J}^2(xt)\, d\mu(x)\right] d\mu(t)\right| \leqq rM \int |\hat{c}(t)|^n\, dt,$$

while from (4.1) and (4.6), $|I(n, r)| \leqq nrM\|c\|^{n-1}$. These inequalities along with $|\hat{c}(t)| \leqq \|c\|$ yield the first part of the lemma. The second part follows from (4.1) and (4.7).

The estimate of Theorem 3.1 can be improved. For each $\mathbf{z}$,

$$\lim_{r\to\infty} \Psi(\mathbf{z}, r) = \int_{-\infty}^{\infty} \cdots \int \max(0, s_1, \cdots, s_{n-1}) E(h_1, z_1) \cdots \cdot E(h_{n-1}, z_{n-1}) f(h_1 + \cdots + h_{n-1})\, dh_1 \cdots dh_{n-1}, \tag{4.8}$$

where $s_k = h_1 + \cdots + h_k$ and $f = Sc$. We shall verify (4.8) for $n = 4$ as this case contains all the essential features of a general proof.

When $n = 4$,

$$\Psi(\mathbf{z}, r) = \int_{\Omega(r)} \cdots \int D(x_1, x_2, z_1) D(x_2, x_3, z_2) D(x_3, x_4, z_3) c(x_4, x_1)\, d\mu(x_4) \cdots du(x_1).$$

We begin by setting

$$\begin{aligned} x_1 &= r + x, \\ x_2 &= r + x + h_1, \\ x_3 &= r + x + h_1 + h_2, \\ x_4 &= r + x + h_1 + h_2 + h_3. \end{aligned}$$

This transformation is a combination of a linear transformation with Jacobian equal to one and a translation. Thus if we let $s_k = h_1 + \cdots + h_k$, then

$$
\begin{aligned}
\Psi(\mathbf{z}, r) = k_v^{-4} \int_{B(r)} \cdots \int & D(x + r, x + r + s_1, z_1) D(x + r + s_1, x + r + s_2, z_2) \\
& \cdot D(x + r + s_2, x + r + s_3, z_3) c(x + r + s_3, x + r)(x + r)^{2v} \\
& \cdot (x + r + s_1)^{2v}(x + r + s_2)^{2v}(x + r + s_3)^{2v} \, dx \, dh_1 \, dh_2 \, dh_3,
\end{aligned} \tag{4.9}
$$

where $B(r)$ is the region in R^4 consisting of those points (x, h_1, h_2, h_3) for which (i) $-r \leqq x \leqq 0$, (ii) $0 < x + \max(0, s_1, s_2, s_3)$, and (iii) $x + s_k \geqq -r$ for $k = 1, 2, 3$.

In order that $D(u, v, w)$ be nonzero it is necessary that $|u - v| < w$. Thus if the integrand of (4.9) is nonzero, then $|h_k| \leqq z_k$ for $k = 1, 2, 3$.

Let $U(\mathbf{z})$ denote the set of points (x, h_1, h_2, h_3) in R^4 for which $-\max(0, s_1, s_2, s_3) < x \leqq 0$ and $|h_k| \leqq z_k$ for $k = 1, 2, 3$. If we let $F(\mathbf{z}, r)$ represent the support of the integrand of (4.9), then when $r > 2(z_1 + z_2 + z_3)$ we have $U(\mathbf{z}) \cap F(\mathbf{z}, r) = B(r) \cap F(\mathbf{z}, r)$ so that $B(r)$ in (4.9) may be replaced by $U(\mathbf{z})$.

For the remainder of this discussion, M will denote a constant depending only on v and c. It may vary from step to step.

If u, v, w are the lengths of sides of a triangle, then

$$D(u, v, w) = M(uvw)^{1-2v}\Delta(u, v, w)^{2v-2},$$

where $\Delta(u, v, w)^2 = [(u + v)^2 - w^2][w^2 - (u - v)^2]/16$. Thus

$$D(x + r + s_{k-1}, x + r + s_k, z_k) = M \frac{[(2x + 2r + 2s_{k-1} + h_k)^2 - z_k^2]^{v-1}}{(x + r + s_{k-1})^{2v-1}(x + r + s_k)^{2v-1}} E(h_k, z_k)$$

if it is nonzero.

When $r > 3(z_1 + z_2 + z_3)$ and $(x, h_1, h_2, h_3) \in U(\mathbf{z})$, we have

$$
\begin{aligned}
0 < (x + r + s_{k-1})(x + r + s_k) &\leqq (2x + 2r + 2s_{k-1} + h_k)^2 - z_k^2 \\
&\leqq 8(x + r + s_{k-1})(x + r + s_k)
\end{aligned}
$$

and therefore,

$$D(x + r + s_{k-1}, x + r + s_k, z_k) \leqq M(x + r + s_{k-1})^{-v}(x + r + s_k)^{-v}E(h_k, z_k).$$

Hence, for r sufficiently large, the integrand in (4.9) is dominated by

$$ME(h_1, z_1)E(h_2, z_2)E(h_3, z_3)|(x + r + s_3)^v c(x + r + s_3, x + r)(x + r)^v|$$

and since $|x^v c(x, y) y^v| \leqq M$ for all $x, y \geqq 0$, this is no larger than $ME(h_1, z_1) E(h_2, z_2)E(h_3, z_3)$.

From Propositions 3.2 and 3.3, the pointwise limit of the integrand in (4.9) as $r \to \infty$ is

$$E(h_1, z_1)E(h_2, z_2)E(h_3, z_3)f(h_1 + h_2 + h_3),$$

where $f = Sc$. Thus, by the dominated convergence theorem,

$$\lim_{r\to\infty} \Psi_4(\mathbf{z}, r)$$

$$= \int_{U(\mathbf{z})} \cdots \int E(h_1, z_1)E(h_2, z_2)E(h_3, z_3)f(h_1 + h_2 + h_3)\,dx\,dh_1\,dh_2\,dh_3$$

$$= \iiint_{-\infty}^{\infty} \max(0, s_1, s_2, s_3)E(h_1, z_1)E(h_2, z_2)E(h_3, z_3)f(h_1 + h_2 + h_3)\,dh_1\,dh_2\,dh_3,$$

and (4.8) is proved.

By combining (4.7) and this last result, one can easily prove the following theorem.

THEOREM 4.3. *If c and its Hankel transform $\hat{c}$ are in $L^1_\nu(0, \infty)$ and if $\int x|c(x)|\,d\mu(x) < \infty$, then for $n \geqq 2$,*

$$\operatorname{tr}(n, r) = H(n, r)$$

$$- \int_{-\infty}^{\infty} \cdots \int \max(0, s_1, \cdots, s_{n-1})f(x_1)\cdots f(x_{n-1})f(s_{n-1})\,dx_1 \cdots dx_{n-1}$$

$$+ o(1)$$

as $r \to \infty$, where $s_k = x_1 + x_2 + \cdots + x_k$ and $f = Sc$.

5. Trace estimates (conclusion). The next theorem provides an estimate for

$$H(n, r) = \int_0^\infty [\hat{c}(x)]^n \left\{ \int_0^r \mathbf{J}^2(xt)\,d\mu(t) \right\} d\mu(x)$$

as $r \to \infty$.

THEOREM 5.1. *If $f \in L^1(0, \infty)$, then*

$$\int f(x) \left\{ \int_0^r \mathbf{J}^2(xt)\,d\mu(t) \right\} d\mu(x) = \frac{r}{\pi} \int_0^\infty f(x)\,dx + o(r)$$

as $r \to \infty$. If, in addition, $|f(x) - f(0)|/x$ is in $L^1(0, \varepsilon)$ for some $\varepsilon > 0$, then

$$\int f(x) \left\{ \int_0^r \mathbf{J}^2(xt)\,d\mu(t) \right\} d\mu(x) = \frac{r}{\pi} \int_0^\infty f(x)\,dx + f(0)(1 - 2\nu)/4 + o(1)$$

as $r \to \infty$.

Proof. Let $\phi(x, r) = k_\nu^{-1} x^{2\nu} \int_0^r \mathbf{J}^2(xt)\,d\mu(t)$. From Watson [12, p. 135],

$$\phi(x, r) = \frac{xr^2}{2}\{J^2_{\nu-1/2}(xr) - J_{\nu-3/2}(xr)J_{\nu+1/2}(xr)\}. \tag{5.1}$$

Using the usual asymptotic expansion for J,

$$J_\alpha(z) = (\pi z/2)^{-1/2}\left[\cos(z - \alpha\pi/2 - \pi/4) - \frac{(4\alpha^2 - 1)}{8z}\sin(z - \alpha\pi/2 - \pi/4) + O(z^{-2})\right],$$

we find that

$$\phi(x, r) = (r/\pi)\left[1 + \frac{\alpha(\nu)}{xr}\sin(2xr - \nu\pi) + O((xr)^{-2})\right] \tag{5.2}$$

as $(xr) \to \infty$, where $\alpha(\nu)$ depends only on ν.

Therefore,

$$\lim_{r\to\infty}\left\{\phi(x, r) - \frac{r}{\pi} - \frac{\alpha(\nu)}{x\pi}\sin(2xr - \nu\pi)\right\} = 0 \tag{5.3}$$

for each $x > 0$. Also

$$\lim_{r\to\infty}\left(r^{-1}\phi(x, r) - \frac{1}{\pi}\right) = 0 \tag{5.4}$$

for each $x > 0$. From (2.3) it follows that $\phi(x, r) \leqq Mr$ for all $r \geqq 0$, where M depends only on ν. Hence,

$$\left|\phi(x, r) - \frac{r}{\pi}\right| \leqq Mr \tag{5.5}$$

for all $x \geqq 0$. The first part of the theorem follows from (5.4), (5.5), and the dominated convergence theorem.

Let N be a fixed large number. If $xr > N$, then from (5.2) we have $|\phi(x, r) - r/\pi| \leqq M/x$, where M depends only on ν and N. If $xr < N$, then using (5.5) we obtain

$$|\phi(x, r) - r/\pi| \leqq M/x. \tag{5.6}$$

Thus (5.6) is valid for all $x > 0$ and all $r \geqq 0$.

If $f \in L^1(0, \infty)$, then from (5.3) and (5.6) we find that

$$\int_1^\infty f(x)\left[\phi(x, r) - \frac{r}{\pi} - \frac{\alpha(\nu)}{\pi x}\sin(2xr - \nu\pi)\right] dx \to 0$$

as $r \to \infty$. But

$$\int_1^\infty f(x)x^{-1}\sin(2xr - \nu\pi)\, dx \to 0$$

as $r \to \infty$, by the Riemann–Lebesgue lemma. Hence,

$$\int_1^\infty f(x)[\phi(x, r) - r/\pi]\, dx \to 0 \quad \text{as} \quad r \to \infty. \tag{5.7}$$

If $|f(x) - f(0)|/x$ is in $L^1[0, \varepsilon]$ for some $\varepsilon > 0$, then by the same argument,

$$\int_0^1 [f(x) - f(0)][\phi(x, r) - r/\pi]\, dx \to 0 \quad \text{as} \quad r \to \infty. \tag{5.8}$$

Now we investigate $\int_0^1 [\phi(x, r) - r/\pi]\, dx$. By combining (5.1) and the basic Bessel function recurrence relation we obtain

$$\phi(x, r) = \frac{xr^2}{2}[J^2_{\nu-1/2}(xr) + J^2_{\nu+1/2}(xr)] - r(\nu - \tfrac{1}{2})J_{\nu+1/2}(xr)J_{\nu-1/2}(xr).$$

Hence,

$$\int_0^1 \phi(x,r)\,dx = (r^2/4)[J^2_{\nu-1/2}(r) + J^2_{\nu+1/2}(r)]$$
$$- (r^2/4)[J_{\nu-3/2}(r)J_{\nu+1/2}(r) + J_{\nu-1/2}(r)J_{\nu+3/2}(r)]$$
$$- (\nu - \tfrac{1}{2})\int_0^r J_{\nu-1/2}(x)J_{\nu+1/2}(x)\,dx.$$

Applying the asymptotic formula for J and using the fact that

$$\lim_{r\to\infty}\int_0^r J_{\nu-1/2}(x)J_{\nu+1/2}(x)\,dx = \tfrac{1}{2}$$

(Watson [12, p. 404]), we arrive at

$$\int_0^1 [\phi(x,r) - r/\pi]\,dx = (1-2\nu)/4 + o(1)$$

as $r \to \infty$. This coupled with (5.7) and (5.8) yields the second part of the theorem.

Thus, if c and its Hankel transform $\hat{c}$ are in $L^1_\nu(0,\infty)$, then for $n \geqq 1$,

$$\text{(5.9)} \qquad \operatorname{tr}(n,r) = \frac{r}{\pi}\int_0^\infty [\hat{c}(t)]^n\,dt + o(r) \quad \text{as} \quad r\to\infty.$$

If, in addition, $\int x|c(x)|\,d\mu(x) < \infty$, then $\hat{c}$ is differentiable at 0. (This follows from the corresponding Fourier transform result. The even extension of $\hat{c}$ is the Fourier transform of Sc and, by Lemma 3.1(c), $\int |tSc\cdot(t)|\,dt \leqq \int x|c(x)|\,d\mu(x)$.) Therefore,

$$\operatorname{tr}(n,r) = \frac{r}{\pi}\int_0^\infty [\hat{c}(t)]^n\,dt + [\hat{c}(0)]^n(1-2\nu)/4$$
$$-\int_{-\infty}^\infty \cdots \int \max(0, s_1, \cdots, s_{n-1}) f(x_1)\cdots f(x_{n-1}) f(s_{n-1})\,dx_1 \cdots dx_{n-1} + o(1)$$

as $r\to\infty$, where $s_k = x_1 + \cdots + x_k$ and $f = Sc$.

The trace estimate (5.9) can be used to prove the following distribution theorem of Szegö [5, p. 141].

THEOREM 5.2. *Let c be a real-valued function in $L^1_\nu(0,\infty)$ and let $N_r(\alpha,\beta)$ denote the number of eigenvalues of T_r which lie in the interval $[\alpha,\beta]$. If $[\alpha,\beta]$ does not contain* 0 *and if the set $\{x\in R: \hat{c}(x)=\alpha$ or $\hat{c}(x)=\beta\}$ has Lebesgue measure* 0, *then*

$$\lim_{r\to\infty}\frac{1}{r}N_r(\alpha,\beta) = \frac{1}{\pi}\Omega(\alpha,\beta),$$

where $\Omega(\alpha,\beta)$ is the Lebesgue measure of the set $\{x\in R: \alpha \leqq \hat{c}(x) \leqq \beta\}$.

Proof. Let A be the set of real-valued functions in $L^1_\nu(0,\infty)$ whose Hankel transforms are in $L^1_\nu(0,\infty)$. For $c\in A$, the proof given for the analogous theorem in [11] can be used. This requires (5.9). Otherwise, there is a sequence $\{c_n\}$ in A converging to c in $L^1_\nu(0,\infty)$. Then $\|T^{(n)}_r - T_r\| \to 0$ uniformly in r as $n\to\infty$ ($T^{(n)}_r$ is the operator corresponding to c_n) and an application of a theorem of the Courant–Weyl type (e.g., [4, p. 1091]) will conclude the proof.

6. Determinant estimates. Suppose that $c \in L^1_\nu(0, \infty)$ is real-valued. If its Hankel transform $\hat{c}$ is in $L^1_\nu(0, \infty)$, then $c(x, y)$ is the difference of two positive definite, continuous kernels

$$c_1(x, y) = \int (\hat{c}_+)(t)\mathbf{J}(xt)\mathbf{J}(yt)\, d\mu(t),$$

$$c_2(x, y) = \int (\hat{c}_-)(t)\mathbf{J}(xt)\mathbf{J}(yt)\, d\mu(t),$$

where $\hat{c}_+$ and $\hat{c}_-$ are the positive and negative parts of $\hat{c}$. Thus each operator T_r is nuclear (or of trace class) since it is the difference of two such operators, and the determinant definition of § 1 is valid. More generally, if c is complex-valued and c, $\hat{c} \in L^1_\nu(0, \infty)$, each T_r is nuclear since it can be written as a linear combination of such operators.

If -1 is not in the spectrum of T_r, then

$$D(r) = \exp(\operatorname{tr}[\log(I - T_r)])$$

and, in particular, if $\|T_r\| < 1$, then

$$D(r) = \exp\left(-\sum \frac{1}{n} \operatorname{tr}(n, r)\right)$$

(see Dunford and Schwartz [4, Chap. XI, § 9]). The restriction $\|c\| < 1$ will allow us to use this representation since, as is easily verified, $\|T_r\| \leqq \|c\|$ for all r.

THEOREM 6.1. *If c and its Hankel transform $\hat{c}$ are in $L^1_\nu(0, \infty)$ and if $\|c\| < 1$, then*

$$[D(r)]^{1/r} = G + o(1) \quad \text{as} \quad r \to \infty,$$

where

$$G = \exp\left\{\frac{1}{\pi}\int_0^\infty \log(1 - \hat{c}(t))\, dt\right\}.$$

Proof. From Lemma 4.2, $|r^{-1} \operatorname{tr}(n, r)| \leqq nM\|c\|^n$, where M is independent of r and n, and hence, by the dominated convergence theorem and our first trace estimate (5.9),

$$\frac{1}{r}\sum \frac{1}{n} \operatorname{tr}(n, r) \to \sum \frac{1}{n\pi}\int [\hat{c}(t)]^n\, dt$$

as $r \to \infty$. Since $|\hat{c}(t)| \leqq \|c\| < 1$, we can interchange the summation and integration in this last expression and write it as

$$-\frac{1}{\pi}\int \log(1 - \hat{c}(t))\, dt.$$

THEOREM 6.2. *If*
(i) *c and its Hankel transform* $\hat{c}$ *are in* $L_\nu^1(0, \infty)$,
(ii) $\|c\| < 1$,
(iii) $\int x|c(x)|\,d\mu(x) < \infty$,
then

$$D(r)G^{-r} = [1 - \hat{c}(0)]^\alpha E + o(1) \quad as \quad r \to \infty,$$

where

$$E = \exp\left(\frac{1}{2}\int_0^\infty x\left\{\frac{1}{2\pi}\int_{-\infty}^\infty \log(1 - \hat{c}(t))e^{-itx}\,dt\right\}^2 dx\right)$$

(*extend* $\hat{c}$ *as an even function*) *and* $\alpha = (1 - 2\nu)/4$.

Proof. Let

$$G_r = \exp\left(\frac{1}{r}\int \log(1 - \hat{c}(t))\left\{\int_0^r \mathbf{J}^2(xt)\,d\mu(x)\right\} d\mu(t)\right).$$

Since $|\hat{c}(x)| \leqq \|c\| < 1$, we have

$$[G_r]^r = \exp\left\{-\sum_{n\geqq 1}\frac{1}{n}H(n, r)\right\}$$

and hence

$$D(r)[G_r]^{-r} = \exp\left\{\sum \frac{1}{n}(H(n, r) - \operatorname{tr}(n, r))\right\}.$$

By combining Lemma 4.2, the dominated convergence theorem, and Theorem 4.3, we obtain

$$\lim_{r\to\infty} D(r)[G_r]^{-r} = \exp\left(\sum_{n\geqq 2}\frac{1}{n}\int\cdots\int \max(0, s_1, \cdots, s_{n-1}) \cdot f(x_1)\cdots f(x_{n-1})f(s_{n-1})\,dx_1\cdots dx_{n-1}\right),$$

where $f = Sc$ and $s_k = x_1 + \cdots + x_k$. The second and third hypotheses and the properties of the mapping S allow us to apply an identity of Kac [10] to reduce the right side of this expression to E. Finally, we use Theorem 5.1 to obtain

$$\lim_{r\to\infty} [G_r/G]^r = \exp\left\{\frac{1 - 2\nu}{4}\log(1 - \hat{c}(0))\right\}.$$

Acknowledgment. The author wishes to thank Professor Jeffrey Davis for suggesting this problem.

REFERENCES

[1] R. Askey and S. Wainger, *A dual convolution structure for Jacobi polynomials*, Orthogonal Expansions and Their Continuous Analogues, D. T. Haimo, ed., Southern Illinois University Press, Edwardsville, 1968, pp. 25–36.

[2] J. R. Davis and I. I. Hirschman, Jr., *Toeplitz forms and ultra-spherical polynomials*, Pacific J. Math., 18 (1966), pp. 73–95.

[3] A. Devinatz, *On Wiener-Hopf operators*, Functional Analysis, B. R. Gelbaum, ed., Thompson Book Company, Washington, 1967, pp. 81–118.

[4] N. Dunford and J. T. Schwartz, *Linear Operators, Part II*, Interscience, New York, 1963.

[5] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, University of California Press, Berkeley and Los Angeles, 1958.

[6] R. E. Hartwig and M. E. Fisher, *Asymptotic behavior of Toeplitz matrices and determinants*, Arch. Rational Mech. Anal., 32 (1969), pp. 190–225.

[7] I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. Analyse Math., 8 (1960–61), pp. 307–336.

[8] ———, *On the formula of Kac and Achiezer*, J. Math. Mech., 16 (1966), pp. 167–196.

[9] ———, *The strong Szegö limit theorem for Toeplitz determinants*, Amer. J. Math., 88 (1966), pp. 577–614.

[10] M. Kac, *Toeplitz matrices, translation kernels, and a related problem in probability theory*, Duke Math. J., 21 (1954), pp. 501–509.

[11] M. Kac, W. Murdock and G. Szegö, *On the eigenvalues of certain Hermitian forms*, J. Rational Mech. Anal., 2 (1953), pp. 767–800.

[12] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1962.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

FINITE MELLIN CONVOLUTION EQUATIONS*

WILLIAM L. PERRY†

Abstract. Sufficient conditions on $f(x)$ and $k(u)$ are given so that the integral equation of the first kind

$$\int_a^b k(xy)\phi(y)\,dy = f(x), \qquad x \in (a, b),$$

with $-\infty < a \leqq 0 \leqq b < +\infty$, can be solved by means of the (left-sided) Laplace transform.

1. Introduction. An integral equation of the first kind of the form

$$\int_D k(xy)\phi(y)\,dy = f(x), \qquad x \in D, \tag{1.1}$$

where $D \subseteq (-\infty, \infty)$, may be called an integral equation of the *Mellin convolution type*, because the equation in (1.1) can be solved (under appropriate conditions) by means of the Mellin transform when $D = (0, \infty)$ [8] or $D = (-\infty, \infty)$ [4]. When $D = (0, \infty)$, one may sometimes use the results of Fox [3].

When D is a finite interval, we may distinguish between two cases: $0 \in [a, b]$ or $0 \notin [a, b]$. In the latter case, dilatory and exponential changes of variable show (1.1) to be equivalent to an integral equation of the form

$$\int_{-1}^{1} k_1(x - y)\phi_1(y) = f_1(x), \qquad x \in [-1, 1], \tag{1.2}$$

which can be solved in principle by the general method of Wiener and Hopf developed by Shinbrot [6]. Regarding the former case, no integral transform techniques for solving (1.1) have appeared. Thus, in this paper, we exhibit an integral transform technique, valid under conditions to be specified, for solving (1.1) with $D = [a, b]$, $0 \in [a, b]$. To show that there actually are equations to which this technique is applicable, we shall supply an example.

2. The problem. The problem under consideration is to find, for a given kernel $k(xy)$ and given free term $f(x)$, a measurable function $\phi(y)$ such that the integrals, $\int_a^b k(xy)\phi(y)\,dy$, exist for almost all x in $[a, b]$ and such that for almost all x in $[a, b]$,

$$\int_a^b k(xy)\phi(y)\,dy = f(x), \tag{2.1}$$

where a and b are fixed constants, $a \leqq 0 \leqq b$.

To simplify the presentation, we start with the equation,

$$\int_0^b k(xy)\phi(y)\,dy = f(x), \qquad x \in [0, b]. \tag{2.2}$$

---

* Received by the editors December 28, 1971, and in revised form March 16, 1972.

† Department of Mathematics, Texas A & M University, College Station, Texas 77843. This work forms a portion of the author's doctoral dissertation at the University of Illinois, Urbana.

That is, the origin is an endpoint of the domain of integration. We now assume that the given kernel and free term satisfy the following conditions:

(2.3) There exists a real number $d > 0$, such that the integrals

$$\int_0^b x^{-2d+1}(f(x))^2\,dx, \qquad \int_0^{b^2} x^{-2d+1}(k(x))^2\,dx,$$

are finite.

(2.4) The complex-valued function of the complex variable s,

$$\frac{\int_0^b x^{s-1} f(x)\,dx}{\int_0^{b^2} x^{s-1} k(x)\,dx},$$

which is well-defined by virtue of (2.3), is analytic in the half-plane $\operatorname{Re} s < \sigma_0$, $-d < \sigma_0$, is of order $O(|s|^{-k})$, as $|s| \to \infty$ where $k > 0$, and the only singularities of this function are poles.

The condition in (2.3) is not too restrictive; it essentially means that $f(x)$ and $k(x)$ are of the order $O(x^{d+\varepsilon-1})$, $x \to 0^+$, $\varepsilon > 0$. More restrictive is condition (2.4) which is imposed to ensure the validity of the method of solution exhibited below.

3. An equivalent integral equation and the application of the left-sided Laplace transform. Assume for the moment that there exists a measurable function $\phi(y)$ such that $\int_0^b k(xy)\phi(y)\,dy$ exists for almost all x in $[0, b]$ and that equation (2.2) holds for almost all x in $[0, b]$. To derive the desired equation, we make two changes of variable. First, let $x_1 = (1/b)x$ and $y_1 = (1/b)y$ to obtain for almost all x_1 in $[0, 1]$, the equation

$$\int_0^1 k(b^2 x_1 y_1) b\phi(by_1)\,dy_1 = f(bx_1).$$

Next set $x_1 = e^u$, $y_1 = e^v$, so that for almost u in $(-\infty, 0) \equiv R^-$,

$$\int_{-\infty}^0 k(b^2\, e^{u+v}) b\, e^v \phi(b\, e^v)\,dv = f(b\, e^u).$$

Thus, defining

$$k(b^2\, e^{u+v}) = k_1(u+v), \quad b\, e^v\phi(b\, e^v) = \phi_1(v), \quad f(b\, e^u) = f_1(u),$$

we may rewrite the preceding integral equation in the form

$$\int_{-\infty}^0 k_1(x+y)\phi_1(y)\,dy = f_1(x), \qquad x \in R^-. \tag{3.1}$$

Since the changes of variable are reversible, (2.2) and (3.1) are equivalent in the sense that a solution of (3.1) leads to a solution of (2.2) and conversely. Thus, we may restrict our attention to (3.1).

The *left-sided Laplace transform*, $F(s)$, of a real-valued function $f(x)$ is defined by

$$(L(f))(s) \equiv F(s) \equiv \int_{-\infty}^0 f(x)\, e^{-sx}\,dx, \tag{3.2}$$

whenever the improper integral on the right-hand side of (3.2) exists. If $L(f)$ exists for all s with $\operatorname{Re} s = c$, then it exists for all s with $\operatorname{Re} s < c$, and it is analytic in that half-plane. The number,

$$\alpha(f) = \sup \left\{ \alpha \,\middle|\, \int_{-\infty}^{0} e^{-\alpha x} f(x)\, dx < \infty, \alpha \in R \right\}$$

is called the abscissa of convergence of $L(f)$ and the line $\operatorname{Re} s = \alpha(f)$ is called the axis of convergence of $L(f)$.

The convolution theorem for the left-sided Laplace transform states that if $L(k) = K, L(\phi) = \Phi$, and

$$\alpha(k) > 0 \quad \text{and} \quad \alpha(\phi) > 0; \tag{3.3}$$

or

$$\alpha(k) < 0 \quad \text{and} \quad \alpha(\phi) > 0, \quad \text{with} \quad -\alpha(\phi) < \alpha(k); \tag{3.4}$$

or

$$\alpha(k) > 0 \quad \text{and} \quad \alpha(\phi) < 0, \quad \text{with} \quad -\alpha(\phi) < \alpha(k); \tag{3.5}$$

then [7, pp. 30–31 with a simple change of variable],

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)\Phi(-s)\, e^{xs}\, ds = \int_{-\infty}^{\min(0,x)} k(y)\phi(y - x)\, dy, \tag{3.6}$$

where c is a real number in the interval

$$\{\sigma | -\alpha(\phi) < \sigma < \min(\alpha(k), \alpha(\phi))\}, \tag{3.3'}$$

or

$$\{\sigma | -\alpha(\phi) < \sigma < \alpha(k)\}, \tag{3.4'}$$

or

$$\{\sigma | -\alpha(\phi) < \sigma < \alpha(k)\}, \tag{3.5'}$$

according to which of the cases (3.3)–(3.5) holds. For fixed $x < 0$, we may put $y = u + x$ in (3.6) so that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)\Phi(-s)\, e^{xs}\, ds = \int_{-\infty}^{0} k(x + u)\phi(u)\, du. \tag{3.7}$$

Taking the left-sided Laplace transform of both sides of (3.7) we obtain

$$L\left(\int_{-\infty}^{0} k(x + y)\phi(y)\, dy \right) = K(s)\Phi(-s) \tag{3.8}$$

for all s in the respective vertical strips with horizontal section given in (3.3′)–(3.5′).

Now suppose that there exists a measurable function $\phi_1(y)$ having a left-sided Laplace transform such that (3.1) holds for almost all $x < 0$, and that $\alpha(k_1)$ and $\alpha(\phi_1)$ satisfy (3.3). It follows from the conditions on k and f in (2.3) that the functions k_1 and f_1 have left-sided Laplace transforms, $K_1(s)$ and $F_1(s)$, respectively, and their abscissae of convergence, $\alpha(k_1)$ and $\alpha(f_1)$, are both positive. In fact,

$\alpha(f_1)$ and $\alpha(k_1)$ are both greater than or equal to d. Since $\Phi_1(s)$ exists and (3.3) holds, we have from the equality in (3.8) that

$$K_1(s)\Phi_1(-s) = F_1(s) \tag{3.9}$$

for all s in the vertical strip $\{s|-\alpha(\phi_1) < \operatorname{Re} s < \min(d, \alpha(\phi_1))\}$. Similar results are valid if (3.4) or (3.5) hold.

4. Construction of solution. The reasoning above is now reversed to construct a left-sided Laplace transformable, measurable function $\phi_1(x)$ which solves (3.1). More precisely, we define a function $\phi_1(y)$ as the inverse Laplace transform of the function $F_1(-s)/K_1(-s)$ and show that the function so defined solves (3.1).

To this end, we first note that $F_1(-s)/K_1(-s)$ is analytic in the half-plane $\operatorname{Re} s < \sigma_0$, $\sigma_0 > -d$, and is of order $O(|s|^{-k})$ as $|s| \to \infty$, for some $k > 0$. Hence, $F_1(-s)/K_1(-s)$ has an inverse Laplace transform, $L^{-1}(F_1/K_1)$,

$$\phi_1(x) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{F_1(-s)}{K_1(-s)} e^{xs}\, ds, \qquad x < 0, \tag{4.1}$$

where $-d < c < \sigma_0$. Thus, for all s in the vertical strip, $\{s|-d < \operatorname{Re} s < \sigma_0\}$, we have $(L(\phi_1))(s) \equiv \Phi_1(s) = F_1(-s)/K_1(-s)$. This means that for all s with $-\sigma_0 < \operatorname{Re} s < d$, $K_1(s)\Phi_1(-s) = F_1(s)$, and since $-\alpha(\phi_1) = -\sigma_0$ and $\alpha(k_1) = d$, it follows from the convolution theorem that

$$\int_{-\infty}^{0} k_1(x + y)\phi_1(y)\, dy = f_1(x), \qquad x < 0.$$

That is, ϕ_1 defined in (4.1) solves (3.1).

By virtue of condition (2.4), $F_1(-s)/K_1(-s)$ has only a finite number of poles in the plane. Thus, using the residue calculus, it is not hard to verify that $\phi_1(x)$ is continuous on R^-. Also, the solution is unique as a consequence of the uniqueness theorem for the Laplace transform. What has been shown above establishes the following theorem.

THEOREM 1. *If the given kernel k and free term f satisfy the conditions* (2.3) *and* (2.4), *then the integral equation* (2.2) *has a unique solution which is continuous on* $(0, b)$.

5. Example. Consider the equation

$$\int_0^1 \frac{(xy)(1 - 2a\log(xy))}{(-\log(xy))^{1/2}}\phi(y)\, dy = x(-\log x)^{1/2}, \tag{5.1}$$

where $x \in (0, 1)$ and $a < 0$. By making the exponential change of variable, we obtain as equivalent to (5.1) the equation:

$$\int_{-\infty}^{0} \frac{\exp(x + y)(1 - 2a(x + y))}{(-(x + y))^{1/2}}[e^y\phi(e^y)]\, dy = e^x\sqrt{-x}, \tag{5.2}$$

$x < 0$, where $a < 0$. Applying the left-sided Laplace transform, we obtain for K_1 and F_1 (see [2]),

$$K_1(s) = \frac{(\pi)^{1/2}(-s + a + 1)}{(-s + 1)^{3/2}}, \qquad \mathrm{Re}\, s < 1,$$

$$F_1(s) = \frac{\Gamma(3/2)}{(-s + 1)^{3/2}}, \qquad \mathrm{Re}\, s < 1.$$

Hence,

$$\Phi_1(s) \equiv \frac{F_1(-s)}{K_1(-s)} = \frac{1}{2(s + a + 1)}, \qquad \mathrm{Re}\, s < -1 - a,$$

is analytic in the half-plane $\mathrm{Re}\, s < -1 - a = \sigma_0$, $\sigma_0 > -1 = d$, and is of order $O(|s|^{-1})$ as $|s| \to \infty$. In the strip $\{s | 1 + a < \mathrm{Re}\, s < 1\}$, $K_1(s)\Phi_1(-s) = F_1(s)$. In addition,

$$\phi_1(x) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{2(s + a + 1)}\, e^{xs}\, ds = -\tfrac{1}{2}\exp\left((-1 - a)x\right), \tag{5.3}$$

where $-1 < c < -1 - a$ and $x < 0$. Thus, the convolution theorem implies that $\phi_1(x)$ defined in (5.3) is a solution to (5.2); this may also be verified by a table of integrals. Reversing the exponential change of variable, we see that the solution to (5.1) is

$$\phi(x) = -1/(2x^{2+a}), \qquad x \in (0, 1),$$

which is continuous on (0, 1).

6. Zero in (a, b). We now extend the preceding analysis to equation (4.1) with $a < 0 < b$. To this end, we define

$$\phi^+(x) \equiv \begin{cases} \phi(x), & x \in (0, b), \\ 0, & x \in (a, 0), \end{cases} \qquad \phi^-(x) \equiv \begin{cases} 0, & x \in (0, b), \\ \phi(x), & x \in (a, 0), \end{cases} \tag{6.1}$$

and $f^+(x)$ and $f^-(x)$ in a completely similar way. Using these definitions, (2.1) may be written as the system of equations:

$$\begin{aligned} &\int_a^0 k(xy)\phi^-(y)\, dy + \int_0^b k(xy)\phi^+(y)\, dy = f^+(x), \qquad x \in (0, b), \\ &\int_a^0 k(xy)\phi^-(y)\, dy + \int_0^b k(xy)\phi^+(y)\, dy = f^-(x), \qquad x \in (a, 0). \end{aligned} \tag{6.2}$$

Again making dilatory and exponential changes of variables it is easy to see that, in obvious notation, (2.1) is equivalent to the system of integral equations:

$$\sum_{j=1}^{2} \int_{-\infty}^{0} k_{ij}(x + y)\phi_j(y)\, dy = f_i(x), \qquad i = 1, 2, \quad x < 0. \tag{6.3}$$

Under assumptions on $\alpha(k_{ij})$ and $\alpha(\phi_j)$ similar to those which have been used above, namely, that $\alpha(k) \equiv \min \alpha(k_{ij}) > 0$ and $\alpha(\phi) \equiv \min \{\alpha(\phi_j)\}$ (the minimum taken over $i, j = 1, 2$) satisfy one of the inequalities in (3.3′)–(3.5′), we see that a solution pair (ϕ_1, ϕ_2) to the system (6.3) must satisfy the equations

$$\sum_{j=1}^{2} K_{ij}(s)\Phi_j(-s) = F_i(s), \qquad i = 1, 2, \tag{6.4}$$

for all s in a nonempty vertical strip Γ determined by $\alpha(k_{ij})$, $\alpha(\phi_j)$, and $\alpha(f_i)$.

Defining functions $J_1(s)$ and $J_2(s)$ by

$$J_1(s) \equiv \frac{F_1(-s)K_{22}(-s) - F_2(-s)K_{12}(-s)}{K_{11}(-s)K_{22}(-s) - K_{12}(-s)K_{21}(-s)},$$

$$J_2(s) \equiv \frac{F_2(-s)K_{11}(-s) - F_1(-s)K_{21}(-s)}{K_{11}(-s)K_{22}(-s) - K_{12}(-s)K_{21}(-s)},$$

we may state a result analogous to the preceding theorem as follows.

THEOREM 2. *If there is a real number $d > 0$ such that all the integrals*

$$\int_0^b x^{-2d+1} f^2(x)\,dx, \quad \int_a^0 |x|^{-2d+1} f^2(x)\,dx,$$

$$\int_0^{a^2} x^{-2d+1} k^2(x)\,dx, \quad \int_{ab}^0 |x|^{-2d+1} k^2(x)\,dx,$$

are finite and if the functions $J_1(s)$ and $J_2(s)$ are analytic in a half-plane $\operatorname{Re} s < \sigma_0$, $-d < \sigma_0$, *are of order $O(|s|^{-k})$, $|s| \to \infty$, $k > 0$, and have only poles as singularities, then* (2.1) *with $a < 0 < b$ has a unique solution, continuous on (a, b) with the possible exception of a discontinuity at zero. The solution is formed from the solution pair (ϕ_1, ϕ_2) of* (6.3), *obtained as $\phi_1 = L^{-1}(J_1)$, $\phi_2 = L^{-1}(J_2)$.*

7. Remarks. If in the example of § 5, we let $a = 0$, then the method is no longer applicable, due to the fact that the equality $K(s)\Phi(-s) = F(s)$ no longer holds in a vertical strip in the complex plane.

The conditions given in (2.4) may be relaxed, and the method will still work, provided the residue integrals involving the necessary inverse Laplace transforms can still be computed. The properties of the solution may be changed by the change in condition (2.4).

REFERENCES

[1] V. G. ANTIPOV, *A singular integral equation with sum kernel*, Izv. Vyssh. Uchebn. Zaved. Matematika, 13 (1959), no. 6, pp. 9–13. (In Russian.)

[2] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Table of Integral Transforms*, vol. I, McGraw-Hill, New York, 1954.

[3] G. FOX, *Solving integral equations by L and L^{-1} operators*, Proc. Amer. Math. Soc., 29 (1971), pp. 299–306.

[4] W. PERRY, *Integral equations with product-type kernels*, Doctoral dissertation, University of Illinois, Urbana, 1971.

[5] E. J. SCOTT, *Transform Calculus*, Harper and Row, New York, 1955.

[6] M. SHINBROT, *The solution of some integral equations of Wiener-Hopf type*, Quart. Appl. Math., 28 (1970), pp. 15–36.

[7] I. N. SNEDDON, *Fourier Transforms*, McGraw-Hill, New York, 1951.

[8] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London, 1937.

[9] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

GENERIC PROPERTIES OF DIFFERENTIAL EQUATIONS*

THOMAS COSTELLO†

Abstract. Property P is said to be *generic* for a class of equations, E, if P is satisfied by each equation in $E - A$, where A is a set of the first category in E. It is shown that uniqueness of solutions is generic for the functional differential equations $\dot{x}(t) = f(t, x_t)$, and existence, uniqueness and continuous dependence of solutions are generic for the characteristic initial value problem for $u_{xy} = f(x, y, u, u_x, u_y)$.

1. Introduction. The identification of properties generic for a particular class of differential equations was begun by Orlicz [6]. He showed that uniqueness of solutions is generic for $y' = f(x, y)$. Using similar techniques, Alexiewicz and Orlicz [1] extended this result to the characteristic initial value problem for

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{1.1}$$

Lasota and Yorke [5] employed a different approach to show that existence of solutions is generic for $y' = f(x, y)$ in Banach space.

In § 3 we show uniqueness of solutions is generic for the functional differential equation

$$\dot{x}(t) = f(t, x_t). \tag{1.2}$$

In § 4 existence and continuous dependence of solutions are proved to be generic properties for the class of equations studied in [1].

2. Preliminaries. A property is said to be *generic* if the set of continuous real-valued functions f for which problem (1.1) (or problem (1.2)) does not possess that property is a set of the first category in the space of continuous functions with the topology of uniform convergence, that is, if it is a "small" set in the sense of the Baire category theorem.

Let a, b and q be finite positive real numbers. We denote by C_H the set of all real-valued, bounded, continuous functions defined on $[0, a] \times [0, b] \times R^3$. C^q denotes the set of continuous functions mapping $[-q, 0]$ into R^n and C_F is the set of bounded continuous functions defined on $[0, \infty) \times C^q$ with values in R^n. For $x \in C^q$, $\|x(\cdot)\| = \sup_{\theta \in [-q, 0]} |x(\theta)|$; C_H and C_F are given the supremum norm.

We let (H) and (F) denote the following problems:

$$(H)\qquad \begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y) \quad \text{for} \quad 0 \leqq x \leqq a \quad \text{and} && 0 \leqq y \leqq b, \\ &u(x, 0) = \tau(x), && 0 \leqq x \leqq a, \\ &u(0, y) = \sigma(y), && 0 \leqq y \leqq b, \end{aligned}$$

where $f \in C_H$, $\tau(\cdot)$, $\sigma(\cdot)$ are continuously differentiable functions;

$$(F)\qquad \begin{aligned} &\dot{x}(t) = f(t, x_t), && t \geqq t_0, \\ &x_{t_0}(\theta) = \phi(\theta), && -q \leqq \theta \leqq 0, \end{aligned}$$

* Received by the editors June 24, 1971, and in revised form April 3, 1972.

† Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland. Now at Raytheon Company, MSD, Bedford, Massachusetts 01730. This research was supported in part by the National Science Foundation under Grant GP-27284.

where $f \in C_F$, $\phi \in C^q$.

If $x(\cdot)$ is continuous on $[t_0 - q, t_0]$, then x_{t_0} is that function in C^q defined by

$$x_{t_0}(\theta) = x(t_0 + \theta), \quad -q \leqq \theta \leqq 0.$$

(H,f) and (F,f) denote problems (H) and (F) with the right-hand side f.

Let E_1 and E_2 be Banach spaces with $U_1 \subset E_1$. We shall say $f: U_1 \to E_2$ is *locally Lipschitz* if for each $p \in U_1$ there is an open set O_p with $p \in O_p \subset U_1$ and an $L_p > 0$ such that $\|f(x) - f(y)\|_2 \leqq L_p\|x - y\|_1$ for all $x, y \in O_p$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the norms on E_1 and E_2.

3. Functional differential equations. Consider the set X defined by $X = \{f \in C_F | (F,f) \text{ has nonunique solutions}\}$. X consists of all functions in C_F for which (F) has at least two solutions.

THEOREM 3.1. *The set X is of the first category in C_F.*

Recall that a set is said to be of the first category if it is the union of a countable collection of nowhere dense sets.

Before proving this result we state a lemma that will be used in the proof.

LEMMA 3.2. *Let U be an open subset of $[0, \infty) \times C_q$ and $f: U \to R^n$ be continuous. Let $\delta > 0$ be given. Then there exists a locally Lipschitz function $g: U \to R^n$ such that*

$$|f(t, \phi) - g(t, \phi)| < \delta$$

for all $(t, \phi) \in U$.

For a proof of this lemma the reader is referred to Lasota and Yorke [5] where a more general statement is proved.

Proof of the theorem. The hypotheses guarantee that solutions exist for (F,f). Furthermore, there is a number $\omega > t_0$ such that each solution is defined at least on $[t_0, \omega]$. This result is well known (see, for example, a proof in [3]). Define

$$\gamma(f) = \limsup_{t \in [t_0, \omega]} \{\|x_t^1 - x_t^2\| : x^1(\cdot), x^2(\cdot) \text{ solutions of } (F,f)\},$$

and

$$T_n = \left\{f \in C_F | \gamma(f) \geqq \frac{1}{n}\right\}.$$

Note that we have

$$X = \bigcup_{n=1}^{\infty} T_n.$$

LEMMA 3.3. *Each T_n is a nowhere dense set.*

Proof. Choose $f \in T_n$ arbitrarily. Let $\varepsilon > 0$ be arbitrary and consider a neighborhood of f of radius ε. Lemma 3.2 guarantees that there is a locally Lipschitz function g_ε such that $g_\varepsilon \in N_\varepsilon(f)$. This implies each neighborhood of f contains a locally Lipschitz function. However, (F, g_ε) has unique solutions since g_ε is locally Lipschitz. Hence no point of T_n has a neighborhood contained in T_n, that is, T_n is a nowhere dense set in C_F.

Therefore X is a set of the first category in C_F, since each T_n is a closed set. This is a consequence of the convergence of solutions of (F,f_n) to solutions of (F, h) when $f_n \to h$. So X is an "F_σ-set", i.e., the countable union of closed sets.

4. Hyperbolic partial differential equations. For the remainder of the paper we shall deal with generic properties of (H). Hartman and Wintner [4] studied (H) and obtained sufficient conditions for existence and uniqueness of solutions. Furthermore, they demonstrated, by examples, that certain hypotheses could not be dropped without possible loss of the related property. That is, there exist continuous functions f such that (H,f) has no solution whatever; there exist continuous functions f which are Lipschitz in the last two variables but for which (H,f) has nonunique solutions. A natural question arises: Are these examples in some sense "a small fraction" of the cases, or do they indicate a general behavior for (H,f). Orlicz and Alexiewicz [1] demonstrated that "uniqueness of solutions" for problem (H) is a generic property in a more restrictive class of functions than C_H. We shall show that existence and continuous dependence are also generic properties for (H). In addition, a shorter proof is given for the result of Orlicz and Alexiewicz.

DEFINITION. A function $u(\cdot,\cdot)$ is said to be a *solution of* (H,f) if u is defined on $R_a^b = [0,a] \times [0,b]$ and

(i) $u(\cdot,\cdot)$ is continuously differentiable on R_a^b,

(ii) $u(x,y)$ satisfies (1.1) for each $(x,y) \in R_a^b$,

(iii) $u(x,0) = \tau(x)$ for $0 \leqq x \leqq a$,
$u(0,y) = \sigma(y)$ for $0 \leqq y \leqq b$.

Notice that a solution of (H,f) is not necessarily a C^2-function on R_a^b.

4.1. Existence. The following result, which appears in [1], will be needed later in this section. Let $\|\cdot\|_{C_1}$ be the usual supremum norm on the space of continuously differentiable functions.

THEOREM 4.1. *Let $f \in C_H$ and be locally Lipschitz in (u_x, u_y). Assume (H,f) has a unique solution. Let $\|f_n - f\| \to 0$ as $n \to \infty$ and (H,f_n) have a solution $u_n(\cdot,\cdot)$ for each n. Then*

$$\|u_n - u\|_{C_1} \to 0 \quad as \quad n \to \infty.$$

THEOREM 4.2. *Let $f \in C_H$ and set*

$$\mathsf{N} = \{f \in C_H | (H,f) \text{ has no solution on } R_a^b\}.$$

Then N *is a set of the first category in C_H.*

Proof. Define a function $V(\cdot)$ on C_H by

$$V(f) = \limsup_{f_1,f_2 \to f} \{\|u_{f_1} - u_{f_2}\|_{C_1} : u_{f_i} \text{ solves } (H,f_i)\},$$

where f_1 and f_2 are restricted to be functions in C_H which have at least one solution of (H,f_i) defined on R_a^b. The density of the Lipschitz functions in C_H assures that this can be done. $V(f)$ is defined and finite for all $f \in C_H$. If $f \in C_H$ and f is locally-Lipschitz in (u, u_x, u_y), then $V(f) = 0$. As, for such functions, the problem (H,f) has a unique solution, and Theorem 4.1 implies $\{u_{f_i}(\cdot)\}$ converges to that solution.

Assume f is such that $V(f) = 0$. Then $\lim_{f_1,f_2 \to f} \{\|u_{f_1} - u_{f_2}\|_{C_1}\}$ exists and implies $\lim_{f_n \to f} u_{f_n}(x,y)$ exists uniformly on R_a^b for some sequence $\{u_{f_n}\}$. The u_{f_n} are chosen so that for any two consecutive terms u_{f_n}, $u_{f_{n+1}}$ in the sequence we have $\|u_{f_n} - u_{f_{n+1}}\|_{C_1} < 2^{-2n}$. This generates a Cauchy sequence in $C^1(R_a^b)$. For details the reader is referred to Buck [2, p. 46]. Denote the limit by $u(x,y)$. Using the integral form of (H) it can be seen that u satisfies (H,f).

Therefore, if (H,f) has no solution on R_a^b we must have $V(f) > 1/n$ for some n. Hence, letting $N_n = \{f \in C_H | V(f) > 1/n)\}$, we have $\mathsf{N} = \bigcup_{n=1}^{\infty} N_n$.

Each N_n is a nowhere dense set, since for any $f \in N_n$ there is a locally Lipschitz function g for which $V(g) = 0$ and g may be chosen arbitrarily close to f. Clearly $g \notin N_n$. The following result shows $g \notin \overline{N_n}$.

LEMMA 4.3. *If* $f_n \in C_H$ *and* $f_n \to g$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} V(f_n) = 0$.

Proof. Assume this is not the case. Then there exists a sequence $\{f_n\}$ and an $\varepsilon > 0$ such that $f_n \to g$ as $n \to \infty$ and $V(f_n) > \varepsilon$ for each n. We may choose f_n^1, f_n^2 within $1/n$ of f_n and such that

$$\|u_{f_n^1} - u_{f_n^2}\|_{C_1} > \varepsilon - \frac{1}{n}.$$

This implies $V(g) > 0$ which contradicts $V(g) = 0$ for a locally Lipschitz function g.

Therefore N is a set of the first category.

4.2. Continuous dependence. Before proceeding to the next result we define what we mean by "continuous dependence on f". When we assume continuous dependence on f we shall mean that the following property is satisfied for $f \in C_H$:

(CD) Let $\{f_n\} \subset C_H, f \in C_H, f_n \to f$ uniformly. Consider the fixed initial condition $(\tau(\cdot), \sigma(\cdot))$. When (H,f) and (H,f_n) have at least one solution u, u_n, respectively, defined on R_a^b, then $\lim_{n\to\infty} u_n(x, y) = u(x, y)$ uniformly on R_a^b.
Condition (CD) is basically the conclusion of the Alexiewicz–Orlicz Theorem 4.1. Notice that if $f(x, y, u, p, q)$ satisfies a local Lipschitz condition in (u, p, q) and $f \in C_H$, then (CD) holds for f and (H,f) has a solution.

THEOREM 4.4. *Let* $\mathsf{G} = \{f \in C_H | \text{(CD) does not hold for } (H,f)\}$. *Then* G *is a set of the first category in* C_H.

Proof. The proof is essentially the same as for the previous result. Let f be chosen so that (H,f) has a solution. Let

$$V(f) = \limsup_{f_n \to f} \{\|u_n - u\|_{C_1}\},$$

where $u(\cdot,\cdot)$ is a solution of (H,f). f satisfies (CD) if and only if $V(f) = 0$. This follows in a manner similar to the method used in the proof of Theorem 4.2.

Define: $N_n = \{f \in C_H | V(f) > 1/n\}$. $\mathsf{G} = \bigcup_{n=1}^{\infty} N_n$ and each N_n is a nowhere dense set in C_H. To see this is true, let n be arbitrary and choose any $h \in N_n$. There exists a sequence $\{h_m\} \subset C_H$, where h_m satisfies all the conditions of Theorem 4.1 and (H, h_m) has a unique solution for each m. As a result of this choice, $V(h_m) = 0$ implies $h_m \notin N_n$. Thus, every neighborhood of h contains points which are not in N_n. From this it follows that N_n is a nowhere dense set, and thus that G is a set of the first category in C_H.

4.3. Uniqueness. The following theorem is proved in [1]. The proof we give is shorter and simpler than the proof of Orlicz and Alexiewicz.

THEOREM 4.5. *Let* $\mathsf{U} = \{f \in C_H | (H, f) \text{ has nonunique solutions}\}$. *Then* U *is a set of the first category in* C_H.

Proof. We repeat the arguments of the preceding proofs using

$$V(f) = \limsup \{\|u_1 - u_2\|_{C_1}\},$$

where the lim sup is taken over all solutions u_i of (H,f), and

$$N_n = \left\{ f \in C_H | V(f) \geqq \frac{1}{n} \right\}.$$

We leave the details to the reader.

Remark. If one were to define (CD) in a different way, keeping the convergence notion, it seems as though our techniques would still work. T. Langan of the University of Maryland has proved a continuous dependence result, for hyperbolic operators, similar to the one we present in [3] for (F). It seems likely his result could be substituted for (CD) with no effect on the conclusions. However, we concentrate on the simpler case.

Acknowledgment. The author wishes to express his gratitude to Professor James A. Yorke of the University of Maryland for his assistance during the preparation of this paper.

REFERENCES

[1] A. ALEXIEWICZ AND M. W. ORLICZ, *Some remarks on the existence and uniqueness of solutions of the hyperbolic equation* $\partial^2 z/\partial x \partial y = f(x, y, z, \partial z/\partial x, \partial z/\partial y)$, Studia Math., 15 (1952), pp. 201–215.

[2] R. C. BUCK, *Advanced Calculus*, 2nd ed., McGraw-Hill, New York, 1965.

[3] T. M. COSTELLO, *Fundamental theory of differential and integral equations*, Doctoral thesis, University of Maryland, College Park, 1971.

[4] P. HARTMAN AND A. WINTNER, *On hyperbolic partial differential equations*, Amer. J. Math., 74 (1952), pp. 834–864.

[5] A. LASOTA AND J. A. YORKE, *The generic property of existence of solutions of differential equations in Banach space*, IFDAM Tech. Note BN-655, University of Maryland, College Park, 1970.

[6] M. W. ORLICZ, *Zur Theorie der Differentialgleichung* $y' = f(x, y)$, Acad. des Sciences, Bull. Inter. Sci. Math., (1932A), pp. 221–228.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

ON THE UNIQUENESS OF BOUNDED SOLUTIONS TO $u'(t) = A(t)u(t)$ AND $u''(t) = A(t)u(t)$ IN HILBERT SPACE*

HOWARD A. LEVINE†

Abstract. Let $A:D_A \to H$ be a symmetric linear operator. If 0 is not an eigenvalue of A, then every solution u to $u'(t) = Au(t)$, $-\infty < t < \infty$, is either identically zero or satisfies $\sup_{-\infty<t<\infty} \|u(t)\| = +\infty$. This result is proved via an elementary argument and then extended in two directions: (i) $A = A(t)$, $t \in (-\infty, \infty)$, and (ii) $A = A_+ + A_-$, where A_+ is symmetric, zero is not an eigenvalue of A_+, A_- is skew symmetric and $\operatorname{Re}(A_+x, A_-x) > -\|A_+x\|^2$ for all $x \in D_A, x \neq 0$. This inequality is sharp. A similar analysis is carried out for $u''(t) = A(t)u(t)$. A number of examples from partial differential equations are given.

1. Introduction. In this paper we prove, via elementary considerations, an extension of the following theorem of S. Zaidman [6], who based his proof on the spectral theorem for self-adjoint operators. We also prove an analogous result for certain abstract equations of the form $d^2u/dt^2 = A(t)u$ in the case that the symmetric part of $A(t)$ is "positive".

THEOREM. *Let H be a Hilbert space and suppose that A is a self-adjoint operator defined on a dense domain $D \subseteq H$. Suppose that zero is not an eigenvalue of A. Let $u:(-\infty, +\infty) \to D$ be a (strongly) continuously differentiable solution to*

$$du/dt = Au, \qquad -\infty < t < +\infty.$$

Then either

$$\sup_{-\infty<t<\infty} \int_t^{t+1} \|u(\eta)\|^2 \, d\eta = +\infty$$

or else $u \equiv 0$.

Our proof of this result has the advantage that it applies to a wider class of operators, that it is elementary and that it even allows us to have A "time-dependent". In general, A can be somewhat worse than symmetric, provided the skew symmetric part is not "too big" relative to the symmetric part and that zero is not an eigenvalue of A. (As is well known, id/dx has no self-adjoint extension when thought of as an operator on $C_0^\infty(0, \infty)$.) Moreover, it is very difficult to prove that partial differential operators which are symmetric have self-adjoint extensions (unless they have real coefficients or are semibounded). Nevertheless, we can apply our results to certain partial differential equations of mixed or even no type.

Zaidman [7] has extended his result to the case of evolutionary equations in a Banach space if A is the infinitesimal generator of a one-parameter semigroup. Since we rely heavily on the Hilbert space structure, we cannot extend our results in this direction.

* Received by the editors December 10, 1971, and in revised form April 11, 1972.

† School of Mathematics, Institute of Technology, University of Minnesota, Minneapolis, Minnesota 55455. This research was supported in part by the National Science Foundation under Contract GP 7041X at the University of Minnesota and in part by the Battelle Institute, Advanced Studies Center of Geneva, Switzerland.

2. The equation $u'(t) = A(t)u(t)$**.** We prove the following theorem.

THEOREM 1. *Let H be a Hilbert space, real or complex, and let $D \subseteq H$ be a dense linear subspace. Let, for each $t \in (-\infty, \infty)$, $A(t)$ be a symmetric operator defined on D with 0 not a eigenvalue of $A(t)$. Let $A(t)x$ be strongly differentiable for each $x \in D$ and suppose that $(\dot{A}(t)x, x) \geqq -\gamma(t)\|A(t)x\|^2$ for all $x \in D$ and $t \in (-\infty, \infty)$, where, for each t, $\gamma(t)$ is continuous and $\gamma(t) < 2$. Let $u:(-\infty, \infty) \to D$ be a continuously differentiable (in the strong sense) solution of $du/dt = A(t)u(t)$. Then either*

$$M = \sup_{-\infty < t < \infty} \int_t^{t+1} \|u(\eta)\|^2 \, d\eta = +\infty \tag{α}$$

or

$$u \equiv 0. \tag{β}$$

Proof. Assume that $M < \infty$, and suppose that $u \not\equiv 0$. Let

$$F(t) \equiv \int_t^{t+1} \|u(\eta)\|^2 \, d\eta. \tag{1}$$

Then

$$\begin{aligned} F'(t) &= \|u(t+1)\|^2 - \|u(t)\|^2 \\ &= \int_t^{t+1} \frac{d}{d\eta}\|u(\eta)\|^2 \, d\eta \\ &= 2\int_t^{t+1} \operatorname{Re}(u_\eta, u) \, d\eta \qquad (u_\eta \equiv du/d\eta) \\ &= 2\int_t^{t+1} (u(\eta), A(\eta)u(\eta)) \, d\eta. \end{aligned}$$

Moreover,[1]

$$\begin{aligned} F''(t) &= 2(u, Au)(t+1) - 2(u, Au)(t) \\ &= 2\int_t^{t+1} \frac{d}{d\eta}(u, Au)(\eta) \, d\eta \\ &= 4\int_t^{t+1} \operatorname{Re}(u_\eta, Au) \, d\eta + 2\int_t^{t+1} (u, \dot{A}u) \, d\eta. \end{aligned}$$

Thus

$$F''(t) \geqq 2\int_t^{t+1} (2 - \gamma(\eta))\|A(\eta)u(\eta)\|^2 \, d\eta. \tag{2}$$

Now, since $u(t_0) \neq 0$ for some t_0, $A(t_0)u(t_0) \neq 0$. Also $\|A(t)u(t)\|^2$ is continuous, as it is the same as $\|u'(t)\|^2$. Therefore,

$$F''(t_0 - \tfrac{1}{2}) > 0.$$

[1] Agmon [1] has shown that the indicated differentiation of (u, Au) is justified for u, u' strongly continuous, $A(t)$ is symmetric and $A(t)x$ is strongly differentiable for each x.

Thus F is a nonconstant, convex function of t, which is bounded ($0 \leqq F(t) \leqq M < \infty$). However, as is well known, there are no such functions. (For if F were such a function and $F'(\tilde{t}_0) > 0$, say, for some $\tilde{t}_0$, then for $t > \tilde{t}_0$, $F'(t) \geqq F'(\tilde{t}_0)$ ($F''(t) \geqq 0$) so that $F(t) \geqq (t - \tilde{t}_0)F'(\tilde{t}_0) + F(\tilde{t}_0)$ and $F(t) \to +\infty$ as $t \to +\infty$. If $F'(\tilde{t}_0) < 0$, we find that $F'(t) \leqq F'(\tilde{t}_0)$ for $t < \tilde{t}_0$ so that $F(t) \geqq (t - \tilde{t}_0)F'(\tilde{t}_0) + F(\tilde{t}_0)$ for $t < \tilde{t}_0$ and $F(t) \to +\infty$ as $t \to -\infty$. Thus $F'(t) \equiv 0$ and F is a constant. Thus we have a contradiction and $u \equiv 0$.)

As a first example, consider the equation

$$\frac{\partial u}{\partial t} = i\frac{\partial u}{\partial x} \quad \text{in} \quad [0, \infty) \times (-\infty, \infty).$$

Here

$$H = \mathscr{L}^2(0, \infty), \quad A = id/dx \quad \text{and} \quad D_A \subseteq \{f \mid f \in H, f' \in H\}.$$

If

$$D_A \subseteq \{f \in H \mid f' \in H \text{ and } f(0) = 0\},$$

then A will be symmetric. However it is well known that A has deficiency indices (0, 1) so that it has *no* self-adjoint extension [3]. Nevertheless, 0 is not an eigenvalue of A and therefore we have that solutions to this equation satisfy

$$\sup_{-\infty < t < \infty} \int_0^\infty |u(x, t)|^2 \, dx = +\infty$$

or else $u \equiv 0$.

Remark 1. Note that Theorem 1 remains true if the operator domain D depends on t under the following conditions. Denote by

$$f(t) \equiv d(u(t), A(t)u(t))/dt - 2 \operatorname{Re}(u_t, A(t)u)$$

and require that $f(t) > -\gamma(t)\|A(t)u(t)\|^2$, where $\gamma(t) < 2$ and is continuous.

Remark 2. Condition (α) may be replaced by

$$(\alpha') \qquad \sup_{-\infty < t < \infty} \|u(t)\| = +\infty.$$

Remark 3. Suppose $A(t) = A$ for all t and that zero is an eigenvalue of A. Let $N = \{x \in D \mid Ax = 0\}$. Then we have the result that either

$$u(t) \equiv \text{const.}$$

or

$$\sup_{-\infty < t < \infty} \int_t^{t+1} \|u(\eta)\|^2 \, d\eta = +\infty.$$

To prove this we write $u(t) = v(t) + w(t)$, where for each t, $(v(t), w(t)) = 0$ and v, w take values in $D \cap N$ and $D \cap N^\perp$ respectively. Then $u'(t) = v'(t) + w'(t) = Aw(t)$. A limiting argument shows that $v'(t)$ and $w'(t)$ take values in N and $N^\perp$

respectively. Assuming $v'(t) \in D$ also, we have $(v', v') + (v', w') = (v', Aw) = 0$ by the symmetry of A so that $(v', v') = 0$ and thus $v \equiv v_0$, a constant. Hence, $u = v_0 + w$ and $u'(t) = w'(t) = Aw$. Therefore, as before, we see that either $w \equiv 0$ or $\sup_{-\infty<t<\infty} \int_t^{t+1} \|w\|^2 \, d\eta = \infty$ by reasoning with $F(t) = (w(t), w(t))$. (Then $F''(t) = 4\|Aw(t)\|^2$.) Since

$$\int_t^{t+1} \|u(\eta)\|^2 \, d\eta = \|v_0\|^2 + \int_t^{t+1} \|w(\eta)\|^2 \, d\eta,$$

the result follows.

Remark 4. Suppose again that A is independent of t. Then it is sometimes possible to have the result of Theorem 1 if $A = A_+ + A_-$, where A_+ is symmetric and A_- is skew symmetric. Suppose that zero is not an eigenvalue of A_+. Letting $F(t)$ be as in the theorem, we have

$$\begin{aligned} F'(t) &= 2 \int_t^{t+1} \operatorname{Re}(u_\eta, u) \, d\eta \\ &= 2 \int_t^{t+1} [\operatorname{Re}(u, A_+u) + \operatorname{Re}(u, A_-u)] \, d\eta \\ &= 2 \int_t^{t+1} (u, A_+u) \, d\eta \end{aligned}$$

so that

$$\begin{aligned} F''(t) &= 4 \int_t^{t+1} \operatorname{Re}(u_\eta, A_+u) \, d\eta \\ &= 4 \int_t^{t+1} \|A_+u\|^2 \, d\eta + 4 \int_t^{t+1} \operatorname{Re}(A_-u, A_+u) \, d\eta. \end{aligned}$$

Suppose now that $A_- = i\lambda A_+ + B$, where λ is a real-valued, measurable, locally integrable function ($\lambda(t) \equiv 0$ in the case of real Hilbert space) and B is a skew symmetric operator satisfying

$$\operatorname{Re}(Bu, A_+u) \geqq -\gamma(t)\|A_+u\|^2$$

for some nonnegative, continuous function γ such that $\gamma(t) < 1$ for all t. Then

$$F''(t) > 4 \int_t^{t+1} (1 - \gamma(\eta))\|A_+u(\eta)\|^2 \, d\eta$$

and we may finish the argument as before. This result fails if $\operatorname{Re}(Bx, A_+x) = -\|A_+x\|^2$ for some $x \in D$ ($x \neq 0$) as the following example shows: Let $H = L^2(-\infty, \infty)$ and let

$$\phi(x) = \begin{cases} \frac{1}{2}(n - x)^{-1} & \text{for } \; n - 1 < x < n, \quad n = 1, 2, 3, \cdots, \\ 0 & \text{for } \; x = 0, \pm 1, \pm 2, \cdots, \\ -\frac{1}{2}(n + x)^{-1} & \text{for } \; -n < x < -n + 1, \quad n = 1, 2, 3, \cdots. \end{cases}$$

Let

$$D = \{f \in H \mid f' \in H \text{ and } \phi \cdot f \in H\}.$$

(Note that this says that if $f \in D$, then f is "small" to the right of every integer point. Note also that D is dense in H.) Now consider the partial differential equation

$$\frac{\partial u}{\partial t} = \phi(x)u + \frac{\partial u}{\partial x}$$

in the (x, t)-plane. One can easily verify the following:

(i) $A_+ : D \to H$ defined by $A_+ f = \phi \cdot f$ is a symmetric operator and zero is not an eigenvalue of A_+.

(ii) $A_- : D \to H$ given by $A_- f = f'$ is skew symmetric $(\operatorname{Re}(f, f') = 0)$.

(iii) $\phi'(x) = 2\phi^2(x)$ except when x is an integer.

Let

$$\psi(x) = \begin{cases} \sqrt{n - x} & \text{for } \; n - 1 \leqq x < n, \qquad n = 1, 2, 3, \cdots, \\ 1 & \text{for } \; x = 0, \\ \sqrt{n + x} & \text{for } \; -n < x \leqq -n + 1, \qquad n = 1, 2, 3, \cdots, \end{cases}$$

and note that $\sup_{-\infty < x < \infty} |\psi(x)| \leqq 1$.

Let $f_0 \in D$ with $\|f_0\| \neq 0$ and f_0 vanishing in a small neighborhood of each integer point. Let

$$u(x, t) = f_0(x + t)\psi(x).$$

One can easily verify that u is a solution to the differential equation in the plane except on the lines $x = n, n = 0, \pm 1, \pm 2, \cdots$, and that the evolutionary equation

$$\frac{du}{dt} = Au(t) = (A_+ + A_-)u(t)$$

is likewise satisfied. (The solution is found by formally applying the method of characteristics to the equation and obtaining the formal solution $u(x, t) = f_0(x + t) \cdot \exp\left(-\int_0^x \phi(\eta)\, d\eta\right)$.) We see that for all t,

$$\begin{aligned} \|u(\cdot, t)\|^2 &= \int_{-\infty}^{\infty} |f_0(x + t)|^2 |\psi(x)|^2\, dx \\ &\leqq \int_{-\infty}^{\infty} |f_0(x + t)|^2\, dx = \|f_0\|^2, \end{aligned}$$

while

$$\begin{aligned} \operatorname{Re}(A_+ f_0, A_- f_0) &= \operatorname{Re} \int_{-\infty}^{\infty} \phi(x) f_0(x) \overline{f_0'(x)}\, dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \phi(x) \frac{d}{dx} |f_0(x)|^2\, dx \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} \phi'(x) |f_0(x)|^2\, dx \end{aligned}$$

(cont.)

$$= -\int_{-\infty}^{\infty} \phi^2(x)|f_0(x)|^2\,dx$$
$$= -\|A_+ f_0\|^2.$$

Thus $u(\cdot, t)$ is a bounded nontrivial solution to $u_t = Au$ while $\mathrm{Re}\,(A_+ f_0, A_- f_0) = \|A_+ f_0\|^2$. Therefore the inequality $\gamma(t) < 1$ must be strict.

As a simple example, let $H = \mathscr{L}^2(-\pi, \pi)$ and $D = \{f \in H | f, f'$ are absolutely continuous, $f'' \in H$ and $f(-\pi) = f(\pi) = 0\}$. Let $a(x, t)$ be a real-valued, continuous function such that for each real t, $\partial a/\partial x$ and $\partial a/\partial t$ are bounded in x, $\partial a/\partial t \leqq 0$ almost everywhere in x and $\int_{-\pi}^{\pi} 1/a(x, t)\,dx$ exists and is not zero. Let

$$[A(t)f](x) \equiv \frac{\partial}{\partial x}\left(a(x, t)\frac{\partial f}{\partial x}(x)\right).$$

One easily checks that $(A(t)f)(x) = 0$ a.e. implies that $f(x) = 0$ a.e. Moreover,

$$(f, \dot{A}(t)f) = -\int_{-\pi}^{\pi} \frac{\partial a}{\partial t}(x, t)|f'(x)|^2\,dx \geqq 0.$$

Thus, there are no nontrivial bounded solutions to

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x, t)\frac{\partial u}{\partial x}(x, t)\right)$$

in $(-\pi, \pi) \times (-\infty, \infty)$ with $u(-\pi, t) = u(\pi, t) = 0$ for all t. (Bounded here means that

$$\sup_{-\infty < t < \infty} \int_{-\pi}^{\pi} |u(x, t)|^2\,dx < \infty.)$$

As another example, consider the Blackstock equation for δ, c given real constants and $\delta \neq 0$;

$$u_t = \delta u_{xx} + c u_x, \qquad -\infty < x, t < \infty.$$

With $D = \{f \in L^2(-\infty, \infty) | f', f'' \in L^2(-\infty, \infty)\}$, let $A_+ = \delta\, d^2/dx^2$, $A_- = c\, d/dx$. Since, as is well known, $C_0^\infty(R^1)$ is dense in D (D being nothing more than $H^2(R^1)$), one easily checks that

(i) A_+ is symmetric on D and 0 is not an eigenvalue of A_+,
(ii) A_- is skew symmetric on D,
(iii)

$$\mathrm{Re}\,(A_+ f, A_- f) = \mathrm{Re}\int_{-\infty}^{\infty} \delta c f''(x) f'(x)\,dx = \frac{\delta c}{2}\int_{-\infty}^{\infty} \frac{d}{dx}|f'(x)|^2\,dx = 0.$$

Therefore, for any solution $u(x, t)$ of Blackstock's equation,

$$\sup_{-\infty < t < \infty} \int_{-\infty}^{\infty} |u(x, t)|^2\,dx = +\infty$$

or

$$u \equiv 0.$$

If $c = \alpha + i\beta$ where $\beta \neq 0$, we take

$$A_+ = \delta\frac{d^2}{dx^2} + i\beta\frac{d}{dx}$$

and

$$A_- = \alpha\frac{d}{dx}.$$

One easily checks that for $f \in D$, $A_+ f = 0$ implies that $f'(x) + (i\beta/\delta)f(x) = a_1$ and consequently, $f(x) = a_2\delta/i\beta + a_2 \exp(-i\beta x/\delta)$, where a_1 and a_2 are constants so that $f \in D$ if and only if $a_1 = a_2 = 0$ and thus zero is not an eigenvalue of A_+. Moreover, $\operatorname{Re}(A_+ f, A_- f) = \int_{-\infty}^{\infty} \delta\alpha f'' f' \, dx = 0$ so that the above result remains true in this somewhat more general case as well. This same result can also be proved by means of Fourier transforms, using the well-known fact that the Fourier transform is unitary.

3. The equation $u''(t) = A(t)u(t)$. Now let us consider the equation $u''(t) = A(t)u(t)$. In fact we shall consider the more general equation

$$Pu_{tt} = A(t)u + \mathscr{F}(t, u, u_t),$$

where $\mathscr{F}(t, \cdot, \cdot): D_{A(t)} \times H \to H$, and prove two theorems, one for the case $\mathscr{F} \equiv 0$ and the other when $\mathscr{F} \not\equiv 0$.

THEOREM 2. *Suppose that either set of hypotheses given below is satisfied and that* $\mathscr{F} \equiv 0$. *Then either*

$$(\alpha) \qquad \sup_{-\infty < t < \infty} (u(t), Pu(t)) = +\infty$$

or

$$(\beta) \qquad u \equiv \text{const.}$$

for every twice continuously differentiable solution $u: (-\infty, \infty) \to H$ *of the equation*

$$P\frac{d^2u}{dt^2} = A(t)u$$

such that $D_{A(t)} \subseteq D_P$, $u(t), u'(t), u''(t) \in D_P$ *and* $u(t) \in D_{A(t)}$ *for all* t. *Moreover, the constant in* (β) *is zero if* I-P, I-A_1 *and* I-A_2 *below hold*:

I-P. *P is symmetric and, for all* $x \in D_P$, $(x, Px) \geqq 0$.

I-A_1. $A(t) = A_1(t) + A_2(t)$, *where, for each* t, $A_1(t)$ *is symmetric and* A_2 *is skew symmetric.*

I-A_2. *For each* t, $(x, A_1(t)x) > 0$ *for* $x \in D_{A(t)}$, $x \neq 0$.

II-P. *P is symmetric and* $(x, Px) > 0$ *for all* $x \in D_P$ *and* $x \neq 0$.

II-A. *Same as* I-A *above except that* $(x, A_1x) \geqq 0$ *for* $x \in D_P$.

Proof. The proof is straightforward. Let

$$F(t) = (u, Pu).$$

Then, we have, successively,

$$F'(t) = 2\,\mathrm{Re}\,(u_t, Pu),$$

$$F''(t) = 2\,\mathrm{Re}\,(Pu_{tt}, u) + 2(u_t, Pu_t)$$

$$= 2(A_1(t)u, u) + 2(u_t, Pu_t).$$

Suppose that (β) fails. Using I-P, I-A_1 and I-A_2, there is a t_0 with $F''(t_0) \geqq 2(u(t_0), A_1(t_0)u(t_0)) > 0$ so that (α) holds, by the remarks in the proof of the first theorem. If (β) holds, then $F'' \equiv 0$ and hence $(u, A_1 u) \equiv 0$ so that $u \equiv 0$. If the second set of hypotheses holds and (β) fails, then $F''(t) \geqq (u_t, Pu_t) > 0$ as u not constant implies $u_t \not\equiv 0$ and thus (α) holds.

Remark 5. Note that $A_2(t)$ played almost no role in the proof and that $A_1(t)$ was not required to be differentiable.

Remark 6. Condition (α) may be replaced by

$$(\alpha') \qquad \sup_{-\infty < t < \infty} \int_t^{t+1} (u(\eta), Pu(\eta))\, d\eta = +\infty.$$

We also have the following theorem.

THEOREM 3. *Let u be a twice continuously differentiable solution to*

$$Pu_{tt} = A(t)u + \mathscr{F}(t, u, u_t).$$

Suppose that P and A satisfy I-P *and* I-A_1 *above and*

(i) *there is a constant* $\lambda > 0$ *such that for all* $x \in D_{A(t)}$ *and all* t, $(x, A_1(t)x) \geqq \lambda(x, x)$;

(ii) *there is a constant* μ *(which may depend upon the solution u) such that for all t,*

$$\|\mathscr{F}(t, u, u_t)\|^2 \leqq \mu[(u_t, Pu_t) + (u, A_1(t)u)];$$

(iii) $0 < \mu < \lambda$.

Then either

$$(\alpha) \qquad \sup_{-\infty < t < \infty} (u(t), Pu(t)) = +\infty$$

or

$$(\beta) \qquad u \equiv 0.$$

Proof. As in the preceding theorem, let $F(t) = (u(t), Pu(t))$. We find that for any $\alpha > 0$,

$$F''(t) = 2(u, A_1 u) + 2(u_t, Pu_t) + 2(u, \mathscr{F})$$

$$\geqq (2 - \mu\alpha - 1/\alpha\lambda)(u, A_1 u) + (2 - \mu\alpha)(u_t, Pu_t),$$

where we have used the estimates

$$|(u, \mathscr{F})| \leqq \|u\|\,\|\mathscr{F}\| \leqq (1/2\alpha)\|u\|^2 + (\alpha/2)\|\mathscr{F}\|^2$$

$$\leqq (1/2\lambda\alpha)(u, A_1 u) + (\alpha/2)\|\mathscr{F}\|^2$$

and condition (ii) of the theorem. We want to choose $\alpha > 0$ such that $(2 - \mu\alpha) \geqq 0$ and $(2 - \mu\alpha - 1/\alpha\lambda) > 0$. This will be possible provided the intersection of the intervals

$$\{\alpha | 0 < \mu\alpha < 2\}$$

and

$$\{\alpha | 1 - (1 - \mu/\lambda)^{1/2} < \mu\alpha < 1 + (1 - \mu/\lambda)^{1/2}\}$$

is not empty. Since $0 < \mu/\lambda < 1$, this is the case. Therefore $F''(t) \geqq C(\mu, \lambda)(u, A_1 u)$ for some positive constant $C(\mu, \lambda)$. The remainder of the argument is now routine.

From Theorem 2 we can deduce a sort of Liouville theorem. Let **n** be a fixed direction in R^n, which, since $\Delta_n = \sum \partial^2/\partial x_i^2$ is invariant under rotations, we may take to be $(0, \cdots, 0, 1)$. Let $D \subseteq \mathscr{L}^2(R^{n-1})$ be the usual space $H^2(R^{n-1})$ consisting of (the completion of) the set of functions f such that f, f_{x_i} and $f_{x_i x_j}$ are square integrable for all $i, j = 1, \cdots, n - 1$. Let u be a solution of

$$u_{x_n x_n} = -\Delta_{n-1} u, \qquad \Delta_n u = 0,$$

in R^n and suppose, for each x_n, $u(\cdot, x_n) \in D$. Then if

$$\sup_{-\infty < x_n < \infty} \int_{R^{n-1}} |u(x_1, \cdots, x_{n-1}, x_n)|^2 \, dx_1 \cdots dx_{n-1} < \infty,$$

we have $u \equiv 0$. (The same result, we note in passing, can be proved via Fourier transform arguments.)

Since the skew symmetric part of A played no role in the determination of a positive lower bound for F'' in Theorems 2 and 3 we could equally well make an analogous statement for the equation

$$\frac{\partial^2 u}{\partial x_n^2} = -\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n-1} b_i \frac{\partial u}{\partial x_i},$$

where the b_i's are constants. If the b_i's are not constants but functions of $x_i, \cdots, x_n$, the second sum should be replaced by

$$\sum_{i=1}^{n-1} bi \frac{\partial u}{\partial x_i} + \left(\frac{1}{2} \sum_{i=1}^{n-1} \frac{\partial b_i}{\partial x_i}\right) u.$$

In addition, one can even apply the conclusion of Theorem 2 to elliptic equations of the form (with $a_{ij} = a_{ji}$)

$$\sum_{i=1}^{n-1} b_i \frac{\partial u}{\partial x_i} + \left(\frac{1}{2} \sum_{i=1}^{n-1} \frac{\partial b_i}{\partial x_i}\right) u.$$

as long as one has, for each x_n, a constant $M = M(x_n) > 0$,

$$M \sum_{i=1}^{n-1} \xi_i^2 \geqq \sum_{i,j=1}^{n-1} a_{ij}(\mathbf{x}, x_n) \xi_i \xi_j \geqq 0$$

for all real $\xi_1, \cdots, \xi_{n-1}$ and $\mathbf{x} \in R^{n-1}$. (In this case the appropriate operator domain is

$$D = D(x_n) = \left\{ f \in H^1(R^{n-1}) \,\middle|\, \int_{R^{n-1}} \left| \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial f}{\partial x_j}\right) \right|^2 dx_1 \cdots dx_{n-1} < \infty \right\}.)$$

Finally, we remark that our results can be applied to

$$\left[b(\mathbf{x}) - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(b_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}\right)\right]\frac{\partial^2 u}{\partial t^2} + \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j}\right)$$

$$= \sum_{i,j,k=1}^{n-1} C_{ijk}\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}$$

under the appropriate "ellipticity" conditions on the matrices (a_{ij}) and (b_{ij}). Here $b(\mathbf{x}) \geqq 0$ for all $\mathbf{x} \in R^{n-1}$ and we take the C_{ijk} to be constants. In the interest of brevity, we omit the details.

In conclusion, we note that similar results can be obtained for initial boundary value problems for the preceding equations which have solutions in the entire space-time cylinder $\Omega \times (-\infty, \infty)$, where $\Omega \subseteq R^{n-1}$ is a bounded domain with a "nice" boundary. It simplifies matters to consider the case $u = 0$ on $\partial\Omega \times (-\infty, \infty)$ although certainly similar results follow with "Neuman" data prescribed on part of $\partial\Omega$. Again, we omit details.

Acknowledgments. The author would like to thank the referee for suggesting the improved and simplified version of the result contained in Remark 3. He would also like to acknowledge the helpful comments of Professor S. Zaidman concerning format and style.

REFERENCES

[1] S. Agmon, *Unicité et convexité dans les problèmes différentiels*, Sem. Math. Sup. (1965), University of Montreal Press, 1966.

[2] S. Agmon and L. Nirenberg, *Lower bounds and uniqueness theorems for solutions of differential inequalities in a Hilbert space*, Comm. Pure Appl. Math., 20 (1967), pp. 207–229.

[3] N. I. Ahiezer and I. M. Glazeman, *Theory of Linear Operators in Hilbert Space*, vol. II, Frederick Ungar, New York, 1963.

[4] H. A. Levine, *Logarithmic convexity, first order differential inequalities and some applications*, Trans. Amer. Math. Soc., 152 (1970), pp. 229–320.

[5] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, New York, 1961.

[6] S. Zaidman, *Uniqueness of bounded solutions for some abstract differential equations*, Ann. Univ. Ferrara. Sez. VII (N.S.), 14 (1969), pp. 101–104.

[7] ———, *Bounded solutions of some abstract differential equations*, Proc. Amer. Math. Soc., 23 (1969), pp. 340–342.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

ASYMPTOTICALLY NEUTRAL FAMILIES IN E^3*

D. T. PIELE†

Abstract. Consider a bounded, open, connected region D in E^3 with connected complement. For a sufficiently smooth Lyapunov boundary surface S, we construct an *asymptotically neutral family* $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}$, $n = n_j \to \infty$, of points on S which, by definition, have the property that the sum of the potentials due to unit charges placed at $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}$ converge (modulo constants C_n) to zero as $n = n_j \to \infty$. Specifically, $\sum_{k=1}^{n} (1/\|x - y_{n,k}\|) + C_n \to 0$ uniformly on every compact subset $K \subset D$. The fields corresponding to *asymptotically neutral families* tend to zero uniformly on every compact subset $K \subset D$, $\sum_{k=1}^{n} \nabla[(1/\|x - y_{n,k}\|)] \to 0$. In the course of the construction we examine: (i) the equilibrium distribution μ on S, $\int_s (\mu(y)/\|x - y\|)\, d\sigma(y) = C$, and how the Hölder continuous differentiability of μ is related to that of S; (ii) a proof of the strict positivity of μ using a result of E. Hopf; (iii) an approximation to the integral $\int_s (\mu(y)/\|x - y\|)\, d\sigma(y)$ by a sum of plane integrals each of which is further approximated by a Gauss-type numerical integration rule. The construction of asymptotically neutral families for bounded simply connected regions in E^2 has been done by Korevaar. New techniques are developed in this paper to extend the results to E^n, $n \geqq 3$.

Introduction. Let D be a bounded, open, connected region in E^3 with connected complement. For a sufficiently smooth boundary S (specifically, if it has a local parametric representation with Hölder continuous third partial derivatives) we construct an *asymptotically neutral family* $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}$, $n = n_j \to \infty$, of points on S which by definition (see [6]) have the property that the sum of the potentials due to unit charges placed at $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}$ converge (modulo constants C_n) to zero as $n = n_j \to \infty$. Specifically,

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to 0$$

uniformly on every compact subset $K \subset D$. We note that the fields generated by the asymptotically neutral family tend to zero as $n = n_j \to \infty$ in D.

The construction begins by taking the well-established equilibrium distribution μ on S (see [4]) and normalizing it on S. Hence,

$$\int_S \mu(y)\, d\sigma(y) = 1 \quad \text{and} \quad \int_S \frac{\mu(y)}{\|x - y\|}\, d\sigma(y) = C$$

for all $x \in D$.

We next approximate the integral

$$n^2 \int_S \frac{\mu(y)}{\|x - y\|}\, d\sigma(y) = n^2 C$$

by a sum of potentials of unit charges

$$\sum \frac{1}{\|x - y_j\|}, \qquad y_j \in S,$$

* Received by the editors November 2, 1971, and in revised form April 14, 1972.

† College of Science and Society, The University of Wisconsin, Parkside, Kenosha, Wisconsin 53140.

by: (i) a careful decomposition of the surface S into patches S_j of equal charge; (ii) transforming the integral over each patch into an appropriate plane integral; (iii) approximating each plane integral by appropriate values of the integrand using a Gauss-type numerical integral rule. As necessary steps in this procedure, we apply a result of E. Hopf (see [3]) to prove that $\mu(y) > 0$ everywhere on S. Also, we use results of N. M. Günter (see [2]) to demonstrate how the Hölder continuous differentiability of μ is related to the same type of smoothness of S.

Asymptotically neutral families are useful in proving theorems which provide approximations to harmonic functions in D by sums of potentials of unit charges. For example, let D be a bounded, open, connected region with connected complement and boundary S which has a well-defined tangent plane at each point. If S contains an asymptotically neutral family, then for every harmonic function f in D we can find a family of finite sequences $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}$, $n = n_j \to \infty$, of points on S such that

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to f(x) \quad \text{as} \quad n \to \infty$$

uniformly on every compact subset of D. This result is proved in [8]. We note as a consequence, every field (gradient of a harmonic function) can be uniformly approximated by fields due to unit charges on the boundary.

Results of this type in E^2 are by-products of much deeper theorems by J. Korevaar (see [5]) where D is a bounded simply connected region with no restrictions on the boundary. New techniques are developed in this paper to construct asymptotically neutral families on appropriately smooth boundaries S of D in E^3. The methods used here are adaptable to E^n, $n > 3$.

1. Notations. We restrict our consideration to E^3, the points of which are denoted by x, y. The Euclidean distance between x and y is denoted by $\|x - y\|$. Integrals over 2-dimensional surfaces are denoted by $\int (\cdot)\, d\sigma$, $d\sigma$ being the surface element. Integrals over 3-dimensional regions are denoted by $\int (\cdot)\, dx$.

A Lyapunov surface in E^3 is a closed bounded 2-dimensional surface S satisfying the following conditions:

(i) At each point of the surface there exists a well-defined tangent plane, and hence a well-defined normal.

(ii) There exist constants A and λ, $0 < \lambda \leqq 1$, such that if θ is the angle between the normals at any two points x and y of S, then θ satisfies a Hölder condition $\theta \leqq A\|x - y\|^{\lambda}$.

(iii) There is a constant d such that for all points y of S, the portion of the surface inside a sphere of radius d about y intersects lines parallel to the normal at y in at most one point.

From condition (i) we can construct, at each point y of a Lyapunov surface, a rectangular coordinate system (ξ, η, ζ) with the ζ-axis along the normal to the surface at y. From condition (iii), the subregion of S contained in a Lyapunov sphere about y can be represented by a function $\Phi(\xi, \eta)$ over a region Λ in the (ξ, η)-plane.

Lyapunov regions are regions bounded by Lyapunov surfaces. For interesting properties of Lyapunov regions, see Günter [2].

Let $f(\xi,\eta)$, defined in a region $\Lambda \subset E^2$, be bounded and possess bounded, continuous derivatives up to order k,

$$\left|\frac{\partial^t f}{\partial\xi^{t_1}\partial\eta^{t_2}}\right| < A,$$

$t_1 + t_2 = t$, $t = 0, 1, 2, \cdots, k$, such that the derivatives of order k are λ-Hölder continuous with the same constant A (see [2]). The class of such functions is denoted by $H_k(A, \lambda)$.

The surface S belongs to the class $L_k(A, \lambda)$, if $\Phi(\xi, \eta) \in H_k(A, \lambda)$, where A and λ are independent of the choice of y on S. Note Lyapunov surfaces belong to the class $L_1(A, \lambda)$.

Let μ be a function defined on S. If (ξ, η, ζ) are the coordinates of a point y of S, we may define μ on a region Λ in the (ξ, η)-plane by putting $\mu(\xi, \eta) = \mu(\Phi(\xi, \eta)) = \mu(y)$. A function μ defined on S belongs to the class $H_k(A, \lambda)$ if $\mu(\xi, \eta) \in H_k(A, \lambda)$ on Λ, where A and λ are independent of the choice of y.

2. Equilibrium distribution. Let S be a Lyapunov surface, which separates the bounded region D_i and the unbounded region D_e. For a continuous function μ on S the single layer potential

$$U(x) = \int_S \frac{\mu(y)}{\|x - y\|}\, d\sigma(y)$$

and its normal derivatives at the boundary satisfy well-known properties (see [2]). Specifically,

$$\frac{\partial U_e}{\partial N_{y'}} - \frac{\partial U_i}{\partial N_{y'}} = -4\pi\mu(y'), \tag{2.1}$$

where $\partial U_i/\partial N_{y'}$ and $\partial U_e/\partial N_{y'}$ denote the limits of the directional derivative $\partial U(x)/\partial N_{y'}$, as x approaches $y' \in S$ from the interior $(x \in D_i)$ and the exterior $(x \in D_e)$ respectively, and

$$\frac{\partial U_i}{\partial N_{y'}} = 2\pi\mu(y') - \int_S \mu(y)\frac{\cos(y' - y, N_{y'})}{\|y' - y\|^2}\, d\sigma(y), \tag{2.2}$$

where $N_{y'}$ is the free unit vector in the direction of the outward normal to S at y' and $(y' - y, N_{y'})$ is the angle between $N_{y'}$ and the vector $y' - y$.

The equilibrium distribution μ, defined by the condition

$$\int_S \frac{\mu(y)}{\|x - y\|}\, d\sigma(y) = C$$

for $x \in D_i$, is established by setting $\partial U_i/\partial N_{y'} = 0$ in (2.2) and solving the resulting homogeneous integral equation

$$\mu(y') - \int_S \mu(y)K(y, y')\, d\sigma(y) = 0, \tag{2.3}$$

where

$$K(y, y') = \frac{\cos(y' - y, N_{y'})}{2\pi\|y' - y\|^2}.$$

(The condition $\partial U_i/\partial N_{y'} = 0$ for all $y' \in S$ is sufficient to imply that U is a constant in D_i; see [2]). A continuous solution to (2.3) is given in Kellogg [4], where S has a continuous curvature, and in Günter [2], where S satisfies the Lyapunov conditions.

It is natural to expect that the continuous equilibrium distribution μ will reflect in some manner the boundary S. For example, if S is the unit sphere, we would expect, and indeed it is the case, that the normalized distribution is the uniform distribution $\mu(y) = \pi/4$. Generally, if S is a sufficiently smooth surface, it is reasonable to expect that $\mu(y)$ will inherit corresponding smoothness properties. One of the first results of this type is due to Korn [7].

LEMMA 1. *If S is a Lyapunov surface and ϕ a continuous function on S, then*

$$\Phi(x) = \int_S \phi(y)K(y, x)\, d\sigma(y)$$

satisfies a uniform Hölder condition on S.

Korn further proved the following lemma.

LEMMA 2. *If S is a Lyapunov surface and ϕ is a Hölder continuous function on S, then the derivative of Φ is Hölder continuous.*

Applying these two results to the continuous solution μ of (2.3) shows that μ is in fact Hölder continuously differentiable. Günter [2] shows, more generally, that the following lemma holds.

LEMMA 3. *If $S \in L_k(B, \lambda)$ and $\phi \in H_{k-2}(A, \lambda)$ on S, then $\Phi \in H_{k-1}(cA, \lambda')$ on S, where λ' is an arbitrary positive number satisfying $\lambda' < \lambda$ and c depends only on B and λ'.*

Applying a "boot-strap" operation further, we obtain the following relationship between the Hölder continuous differentiability of S and μ.

THEOREM 1. *If $S \in L_k(B, \lambda)$ and $\mu(y)$ is the (continuous) equilibrium distribution on S, then $\mu \in H_{k-1}(A', \lambda')$ for any $\lambda' < \lambda$.*

To establish the strict positivity of the *equilibrium distribution* μ, we shall apply the following theorem of E. Hopf [3]—specialized to our situation.

Let R denote a connected open set in E^n, $n \geqq 3$, and y denote a point on the boundary ∂R. Assume that ∂R has the property that R contains a hypersphere H centered in R and touching ∂R only at y.

THEOREM 2. *Suppose u is harmonic and $u \leqq 0$ in D_e with $\lim u(x) = u(y) = 0$ as $x \to y$ along the normal ($x \in D_e$). Then either the normal lower derivative (directed inward)*

$$\frac{\partial u}{\partial N} = \underline{\lim} \frac{u(y) - u(x)}{y - x} > 0$$

(where $x \to y$ as before) or $u \equiv 0$ in D_e.

We shall need the following.

LEMMA 4. *The equilibrium distribution μ is strictly positive.*

Proof. By definition, the equilibrium potential U satisfies $\partial U_i/\partial N_y = 0$, from which we derive (see [2]) $\partial U_e/\partial N_y = -4\pi\mu(y)$. Hence, it is sufficient to show $\partial U_e/\partial N_y < 0$ for $y \in S$. We assume, without loss of generality, $U(x) = c > 0$ for $x \in D_i$. Apply the maximum principle to the exterior region D_e to observe $U(x) \leqq c$ for $x \in D_e$. Define $u = U - c$. We have $u = 0$ on S and $u \leqq 0$ in D_e. Let y be a point in the boundary surface S. The Lyapunov conditions on S are sufficient conditions for S to apply Theorem 2. Differentiating $u = U - c$ in the opposite (outward normal) direction,

$$\frac{\partial U}{\partial N_y} = \frac{\partial u}{\partial N_y} < 0.$$

Since $\partial U_e/\partial N_y$ exist for all $y \in S$, a simple application of the mean value theorem establishes the equality

$$\frac{\partial U}{\partial N_y} = \lim_{x \to y} \frac{U(x) - U(y)}{x - y} = \lim \frac{\partial U}{\partial N_y}(x') = \lim_{x \to y} \frac{\partial U}{\partial N_y}(x) = \frac{\partial U_e}{\partial N_y},$$

where $x \to y$ along the normal N_y and x' lies on N_y between x and y. We conclude $\partial U_e/\partial N_y < 0$ and hence $\mu > 0$.

3. Partitioning the boundary. We now discuss the partitioning of the surface S into patches of equal charge. The standing assumptions are: (i) S is a Lyapunov surface with associated sphere of radius d, where $Ad^\lambda < \frac{1}{2}$ (ii) the equilibrium distribution μ is continuous and normalized such that $\int_S \mu \, d\sigma = 1$.

Denote the intersection of the compact surface S and a finite covering of S with open spheres of radius $d/2$ by $S_1, S_2, \cdots, S_m$. The surface patch S_j can be represented by $\zeta = \Phi_j(\xi, \eta)$ on a domain Λ_j. In fact, Φ_j may be defined in a disk about the origin of radius $(7/9)d$, while Λ_j is contained in the disk of radius $d/2$ (see [2, p. 3]). Redefine the surface patches S_j, $j = 1, 2, \cdots, m$, to make them nonoverlapping and share at most a common boundary. Let c_j be the charge on each patch S_j, $c_j = \int_{S_j} \mu \, d\sigma$. We have

$$\sum_{j=1}^{m} c_j = \int_S \mu \, d\sigma = 1.$$

Since μ is continuous on S and we have the freedom to expand or contract our domain of definition of S_j, we may assume without loss of generality that c_j is rational for $j = 1, 2, \cdots, m$. Write $c_j = p_j/l$, where l is the least common multiple of the denominators of c_j, $j = 1, 2, \cdots, m$. The charge on each patch S_j is given by

$$p_j/l = \int_{S_j} \mu \, d\sigma = \int_{\Lambda_j} \mu(\xi, \eta)\sqrt{1 + \Phi_{j,1}^2 + \Phi_{j,2}^2} \, d\xi \, d\eta, \tag{3.1}$$

where $\Phi_{j,1}$ and $\Phi_{j,2}$ denote the partial derivatives of Φ_j with respect to ξ and η, respectively. We introduce a transformation Υ from the (ξ, η)-plane to the (u, v)-plane given by

$$T: \begin{cases} v = \eta, \\ u(\xi, \eta) = \displaystyle\int_0^\xi \sqrt{1 + \Phi_{j,1}^2(x, \eta) + \Phi_{j,2}^2(x, \eta)}\mu(x, \eta) \, dx. \end{cases}$$

The Jacobian of this transformation has the value:

$$\frac{\partial(u, v)}{\partial(\xi, \eta)} = u_\xi v_\eta - u_\eta v_\xi = \sqrt{1 + \Phi_{j,1}^2 + \Phi_{j,2}^2}\mu(\xi, \eta).$$

We can express (3.1) as $\int_{\Upsilon\Lambda_j} du\, dv$, where $\Upsilon\Lambda_j$ is the image of Λ_j under the transformation Υ. Let $A(\Upsilon\Lambda_j)$ denote the area $\Upsilon\Lambda_j$. We then have $A(\Upsilon\Lambda_j) = p_j/l$.

Notice that the transformation Υ is nonsingular since $\mu > 0$ (Lemma 3). If we divide $\Upsilon\Lambda_j$ into regions of equal area in the (u, v)-plane and map these regions onto the surface S via the map $\Phi_j \circ \Upsilon^{-1}$, we shall obtain a decomposition of the surface patch S_j into regions of equal charge, exactly as desired.

For each integer n we divide $\Upsilon\Lambda_j$ into $p_j n^2$ regions, each of area $1/ln^2$, by constructing a mesh, with sides parallel to the (u, v)-axes a distance $\alpha = 1/\sqrt{l}n$ apart, over the area $\Upsilon\Lambda_j$. Let $\Omega_{j,i}$, $i = 1, 2, \cdots, N_j$, denote the squares of the mesh which lie inside $\Upsilon\Lambda_j$ and have a distance $\geqq \alpha$ to the boundary of $\Upsilon\Lambda_j$. Let $\Pi_{j,i}$, $i = 1, 2, \cdots, M_j$, denote the remaining squares and partial squares contained in $\Upsilon\Lambda_j$. Form the unions

$$\Omega_j = \bigcup_{i=1}^{N_j} \Omega_{j,i}, \quad \Pi_j = \bigcup_{i=1}^{M_j} \Pi_{j,i}.$$

Clearly,

$$\Omega_j \cup \Pi_j = \Upsilon\Lambda_j, \quad A(\Omega_j) + A(\Pi_j) = A(\Upsilon\Lambda_j), \quad A(\Omega_{j,i}) = \alpha^2,$$
$$A(\Pi_{j,i}) \leqq \alpha^2,$$

where A is used to denote area.

Since the boundary of $\Upsilon\Lambda_j$ is rectifiable, the number M_j of squares of the mesh in Π_j, which have a distance $< \alpha$ to the boundary, is $O(n)$, $M_j = O(n)$ [5, p. 460]. The number N_j of squares in Ω_j is of order n^2, $N_j = O(n^2)$. Clearly $A(\Omega_j) = N_j\alpha^2$ and $A(\Pi_j) = A(\Upsilon\Lambda_j) - A(\Omega_j) = p_j/l - N_j\alpha^2 = (p_j n^2 - N_j)\alpha^2 = L_j\alpha^2$, where L_j is the integer $p_j \cdot n^2 - N_j$. Note that $L_j \leqq M_j$.

Decompose Π_j into $4L_j$ regions of equal area $\alpha^2/4$ and denote them by $\Pi'_{j,i}$. Since the distance between the boundary of $\Upsilon\Lambda_j$ and Ω_j is $\geqq \alpha$, a decomposition can be easily made such that the diameters of $\Pi'_{j,i}$ are $O(\alpha) = O(1/n)$, $i = 1, 2, \cdots, 4L_j$.

Summarizing, we have decomposed the plane region $\Upsilon\Lambda_j$ into N_j squares of area $1/ln^2$ and $4L_j$ regions of area $1/4ln^2$ with diameters $O(1/n)$. Also, $N_j = O(n^2)$, $L_j = O(n)$, $L_j + N_j = p_j n^2$ and $\int_{\Upsilon\Lambda_j} du\, dv = p_j/l$.

4. Numerical integration. In this section we develop, for use in § 5, sufficient numerical integration rules to approximate integrals over the square regions $\Omega_{j,i}$ and the irregular regions $\Pi'_{j,i}$.

In one dimension, a two-point Gauss numerical integration rule is exact for polynomials of degree $\leqq 3$ [1, p. 35]. For the class of twice differentiable functions with λ-Hölder continuous second derivatives, the error term

$$E(f) = \int_a^b f(x)\, dx - \frac{b - a}{2}[f(x_1) + f(x_2)]$$

for the two-point Gauss integration rule is of order $O(b-a)^{3+\lambda}$. This can be seen by expanding $f(x)$ in the Taylor expansion,

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \tfrac{1}{2}(x - x_0)^2 f''(x_0) + O(x - x_0)^{2+\lambda}.$$

The product of two 1-dimensional Gauss integration rules gives a four-point numerical integration rule which is exact for polynomials $p(x, y)$ of degree $\leqq 3$ in x and y separately. Again, an estimation of the error term, $E(f)$, for a class of functions in two variables can be carried out through a Taylor expansion of f. As a result, for the class of twice differentiable functions with λ-Hölder continuous second derivatives, $0 < \lambda < 1$,

$$\int_a^b \int_a^b f(x, y)\,dx\,dy = \frac{(b-a)^2}{4}[f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)] + E(f),$$

where $E(f) = O(b-a)^{4+\lambda}$.

Let B be a bounded region with diameter $O(\delta)$ and area δ^2. Consider the class of functions $f(x, y)$ defined in the convex hull of B which are twice continuously differentiable with uniformly bounded second partial derivatives. Expanding $f(x, y)$ in a Taylor series about (x_0, y_0) we obtain

$$\int_B f(x, y)\,dx\,dy = \delta^2 f(x_0, y_0) + O(\delta^4),$$

where

$$x_0 = \frac{\int_B x\,dx\,dy}{\int_B dx\,dy}, \qquad y_0 = \frac{\int_B y\,dx\,dy}{\int_B dx\,dy}.$$

This particular choice of x_0 and y_0 eliminates the δ^3 term.

5. Construction of an asymptotically neutral family. In this section we construct an asymptotically neutral family of points on S when $S \in L_k(B, \lambda)$, $k \geqq 3$, i.e., we shall construct a family of finite sequences $\{y_{n1}, y_{n2}, \cdots, y_{nn}\}$, $n = n_j \to \infty$, of points on S such that

$$\sum_{k=1}^{n} \frac{1}{\|x - y_{nk}\|} + C_n \to 0$$

uniformly on every compact subset $K \subset D$.

For each compact subset K, let $\mathscr{F}_K$ denote the family of functions

$$\left\{F(u, v) = \frac{1}{\|x - y(\xi(u, v), \eta(u, v))\|}, x \in K\right\}.$$

The differentiability of F as a function of u and v is determined by the differentiability of the surface S and the equilibrium distribution μ. For example, if $S \in L_k(B, \lambda)$, $F(u, v)$ is in the class $H_{k-1}(A', \lambda'')$ (see Theorem 1). Furthermore, since K is bounded away from the surface S, there exist constants A'' and λ'' such that

$$F(u, v) \in H_{k-1}(A'', \lambda'') \quad \text{for all } F \text{ in } \mathscr{F}_K.$$

We begin our construction with the normalized equilibrium distribution μ which satisfies

$$\int_S \mu(y)\, d\sigma(y) = 1, \tag{5.1}$$

and

$$\int_S \frac{\mu(y)}{\|x - y\|}\, d\sigma(y) = C \tag{5.2}$$

for all $x \in D$. Use the surface decomposition of § 3 to rewrite (5.2) as

$$\sum_{j=1}^{m} \int_{S_j} \frac{\mu(y)}{\|x - y\|}\, d\sigma(y) = C. \tag{5.3}$$

Transform each surface integral in (5.3) to a plane integral with the transformation $\Phi_j \circ \Upsilon^{-1}$ discussed in § 3. Accordingly,

$$\int_{S_j} \frac{\mu(y)}{\|x - y\|}\, d\sigma(y) = \int_{\Upsilon\Lambda_j} \frac{1}{\|x - y(\xi(u, v), \eta(u, v))\|}\, du\, dv. \tag{5.4}$$

Decompose $\Upsilon\Lambda_j$ into N_j squares of area $\alpha^2 = 1/ln^2$ and $4L_j$ bounded regions of area $\alpha^2/4$ (see § 3), where $N_j = O(n^2)$, $L_j = O(n)$. Partition the integral (5.4) and write it as

$$\int_{\Upsilon\Lambda_j} F(u, v)\, du\, dv = \sum_{i=1}^{N_j} \int_{\Omega_{j,i}} F(u, v)\, du\, dv + \sum_{i=1}^{4L_j} \int_{\Pi'_{j,i}} F(u, v)\, du\, dv. \tag{5.5}$$

To each integral $\int_{\Omega_{j,i}} F(u, v)\, du\, dv$ we apply the four-point Gauss numerical integration rule (§ 4) with $F(u, v) \in H_2(A'', \lambda'')$ to obtain

$$\begin{aligned} &\int_{\Omega_{j,i}} F(u, v)\, du\, dv = \\ &\frac{1}{4ln^2}[F(u_1, v_1) + F(u_1, v_2) + F(u_2, v_1) + F(u_2, v_2)] + O\left(\frac{1}{n^{4+\lambda}}\right). \end{aligned} \tag{5.6}$$

To each integral $\int_{\Pi'_{j,i}} F(u, v)\, du\, dv$ we apply the center of mass rule

$$\int_{\Pi'_{j,i}} F(u, v)\, du\, dv = \frac{1}{4ln^2} F(u_0, v_0) + O\left(\frac{1}{n^4}\right). \tag{5.7}$$

Note. Although it is not indicated, the four points used in (5.6) and the point in (5.7) depend on $\Omega_{j,i}$ and $\Pi'_{j,i}$ respectively. Denote the corresponding points on the surface S_j by $y_{j,i,1}, y_{j,i,2}, y_{j,i,3}, y_{j,i,4}$ and $y_{j,i,0}$.

CLAIM. *The family*

$$\bigcup_{j=1}^{m} \left\{ \bigcup_{i=1}^{N_j} \{y_{j,i,1}, \cdots, y_{j,i,4}\} \cup \bigcup_{i=1}^{4L_j} \{y_{j,i,0}\} \right\}$$

is an asymptotically neutral family for D.

Proof. Multiply both sides of (5.6) and (5.7) by $4ln^2$. The error terms are now of order $O(1/n^{2+\lambda})$ and $O(1/n^2)$, respectively.

Sum over $i = 1, 2, \cdots, N_j$ and $i = 1, 2, \cdots, 4L_j$:

$$4ln^2 \sum_{i=1}^{N_j} \int_{\Omega_{j,i}} F(u, v)\, du\, dv = \sum_{i=1}^{N_j} \frac{1}{\|x - y_{j,i,1}\|} + \frac{1}{\|x - y_{j,i,2}\|} + \frac{1}{\|x - y_{j,i,3}\|} + \frac{1}{\|x - y_{j,i,4}\|} + O(N_j/n^{2+\lambda}), \tag{5.8}$$

$$4ln^2 \sum_{i=1}^{4L_j} \int_{\Pi'_{j,i}} F(u, v)\, du\, dv = \sum_{i=1}^{4L_j} \frac{1}{\|x - y_{j,i,0}\|} + O(L_j/n^2). \tag{5.9}$$

Substitute (5.8) and (5.9) into (5.5), combine the error terms using $N_j = O(n^2)$ and $L_j = O(n)$, and sum over $j = 1, 2, \cdots, m$, to conclude

$$\sum_{j=1}^{m} \left\{ \sum_{i=1}^{N_j} \frac{1}{\|x - y_{j,i,1}\|} + \frac{1}{\|x - y_{j,i,2}\|} + \frac{1}{\|x - y_{j,i,3}\|} + \frac{1}{\|x - y_{j,i,4}\|} + \sum_{i=1}^{4L_j} \frac{1}{\|x - y_{j,i,0}\|} \right\} = 4ln^2 \sum_{j=1}^{m} \int_{\Upsilon\Lambda_j} F(u, v)\, du\, dv + O(1/n^\lambda) = 4ln^2 C + O(1/n^\lambda). \tag{5.10}$$

Note that the error term is uniform for x in K.

For integer n we abbreviate the sum in (5.10) to

$$\sum_{\nu=1}^{N_n} \frac{1}{\|x - y_\nu\|}.$$

Define $C_n = -4ln^2 C$ and let $n \to \infty$ in (5.10) to conclude

$$\sum_{\nu=1}^{N_n} \frac{1}{\|x - y_\nu\|} + C_n \to 0 \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Since the error term, $O(1/n^\lambda)$, is uniform for $x \in K$, the convergence in (5.11) is likewise uniform.

REFERENCES

[1] P. Davis and P. Rabinowitz, *Numerical Integration*, Blaisdell, Waltham, Mass., 1967.
[2] N. M. Günter, *Potential Theory and its Applications to Basic Problems of Mathematical Physics*, Frederick Ungar, New York, 1967.
[3] E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc., 3 (1952), pp. 791–793.
[4] O. D. Kellogg, *Foundations of Potential Theory*, Frederick Ungar, New York, 1929.
[5] Jacob Korevaar, *Mathematical Methods*, vol. I, Academic Press, New York, 1967.
[6] ———, *Asymptotically neutral distributions of electrons and polynomial approximation*, Ann. of Math., 80 (1964), pp. 403–410.
[7] A. Korn, *Quelques théoremes sur les potentiels des surfaces*, Ann. Sci. École Norm. Sup. Ser. 3, 24 (1907), pp. 12–30.
[8] D. T. Piele, *Approximation of harmonic functions by potentials of unit charges*, Doctoral dissertation, University of California, San Diego, 1971.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

PROPERTIES OF SOLUTIONS OF $u'' + c(t)f(u)h(u') = 0$ WITH EXPLICIT INITIAL CONDITIONS*

C. M. PETTY† AND W. E. JOHNSON‡

Abstract. Conditions are stated, determining the behavior of a solution of the nonlinear equation mentioned in the title, which are expressed wholly and explicitly in terms of the initial conditions and the given functions $h(s)$, $f(s)$ and $c(t)$. Avoiding a priori assumptions that a solution is proper or possesses some other property, the results illuminate the variety of behavior which can exist for a single equation under various initial conditions. The basic restrictions imposed here on the defining functions are that $h(s)$ and $c(t)$ have constant sign and $sf(s) > 0$ for $s \neq 0$. The method of proof for some of the results involves the introduction of two Lyapunov functions which do not require that $c(t)$ be monotone.

1. Introduction. The differential equation in the multiplicative form

$$u'' + c(t)f(u)h(u') = 0 \tag{1.1}$$

has been previously studied in the papers [2], [3], [4], [7], [9], [12], [14], [15], [17]–[20], [22]. An extensive bibliography of papers dealing with equations of a related type can be found in [21].

In part, we generalize here some of the known results. However, the major emphasis is placed on the determination of the behavior of a solution from its initial conditions. This aspect of the theory, which is important in applications, has been relatively neglected for this equation.

We assume that the functions defining the differential equation (1.1) satisfy the following conditions:

(i) $h(s)$ is positive and continuous for all real s;
(ii) $f(s)$ is continuous for all real s and $sf(s) > 0$ if $s \neq 0$;
(iii) $c(t)$ is continuous for all $t \geqq t_0$.

We shall adhere to the following terminology and notation throughout this paper. If $u(t)$ is a solution of (1.1) with initial conditions at $t_1, t_1 \geqq t_0$, then $[t_1, T)$ will denote the maximum interval to the right on which $u(t)$ is defined. The conditions (i), (ii), (iii) ensure that this interval exists. The solution is said to be proper or nonproper as $T = \infty$ or $T < \infty$, respectively. The solution is said to be oscillatory if it has an infinite number of zeros exceeding t_1, whether or not it is proper. The solution is said to be positive if it is ultimately positive, whether or not it is proper. Finally, we define the functions

$$I(t) = \int_0^t h^{-1}(\tau)\,d\tau, \qquad H(t) = \int_0^t \tau h^{-1}(\tau)\,d\tau, \qquad F(t) = \int_0^t f(\tau)\,d\tau.$$

We observe that results for negative solutions may be obtained from those for positive solutions by considering

$$u'' + c(t)f^*(u)h^*(u') = 0, \tag{1.2}$$

where $f^*(s) = -f(-s), h^*(s) = h(-s)$.

* Received by the editors October 21, 1971, and in revised form February 8, 1972.
† Department of Mathematics, University of Missouri, Columbia, Missouri 65201.
‡ Lockheed Palo Alto Research Laboratory, Palo Alto, California 94304.

As a point of reference, the reader may wish to apply some of the results proved here to the much-studied $u'' + c(t)|u|^n \operatorname{sgn} u = 0$ and, as a special case, to the Emden–Fowler equation for which $c(t) = \pm t^{\sigma}$ (see [1, Chap. 7]).

2. Positive solutions. In this section we restrict our study to solutions $u(t)$ with initial conditions

$$u(t_1) = \alpha, \quad u'(t_1) = \beta, \qquad \alpha \geqq 0, \quad \beta \geqq 0, \quad \alpha + \beta > 0, \tag{2.1}$$

for the differential equation

$$u'' - c(t)f(u)h(u') = 0, \tag{2.2}$$

where

(iv) $c(t)$ is positive and continuous for $t \geqq t_0$.

Positive solutions to (2.1) which satisfy $\alpha > 0$ and $\beta < 0$ have been studied in great detail elsewhere, for example, [16] where other references are given.

LEMMA 1. *Suppose conditions* (i), (ii), (iv) *are satisfied and* $u(t)$ *is a solution of* (2.2) *with initial conditions* (2.1). *Then* $u(t)$ *is of one of the following four types*:

Nonproper positive solutions, $T < \infty$:

Type 1. $u(t) \to k, u'(t) \to \infty$ *as* $t \to T, \alpha < k < \infty$.

Type 2. $u(t) \to \infty, u'(t) \to \infty$ *as* $t \to T$.

Proper positive solutions, $T = \infty$:

Type 3. $u(t) \to \infty, u'(t) \to \infty$ *as* $t \to \infty$.

Type 4. $u(t) \to \infty, u'(t) \to k$ *as* $t \to \infty, \beta < k < \infty$.

Proof. A proof is obtained in a straightforward way by consideration of the various possibilities under the conditions (i), (ii) and (iv). Consequently, the details are omitted.

We shall give various conditions under which each of the four types is obtained. Throughout the remainder of this section, if $H(\infty) = \infty$ we denote the inverse of $H(t), t \geqq 0$, by $G(s), s \geqq 0$.

LEMMA 2. *Suppose conditions* (i), (ii), (iv) *are satisfied and let* $u(t)$ *be a solution of* (2.2) *with initial conditions* (2.1).

(a) *If* $H(\infty) < \infty, F(\infty) = \infty$, *then* $u(t)$ *is of Type* 1, 3, *or* 4.

(b) *If* $H(\infty) = \infty$, *then* $u(t)$ *is of Type* 2, 3, *or* 4.

(c) *If* $H(\infty) = \infty$ *and*

$$\int_1^{\infty} \frac{d\tau}{G(kF(\tau))} = \infty$$

for all $k > 0$, *then* $u(t)$ *is of Type* 3 *or* 4.

Proof. Assume $u(t)$ is of Type 1 or 2 and let $0 < k_1 \leqq c(t) \leqq k_2 < \infty$ on $[t_1, T]$. Set $V_i(t) = H(u'(t)) - k_iF(u(t)), t_1 \leqq t < T$. Then $V'_1(t) \geqq 0, V'_2(t) \leqq 0$ on $[t_1, T)$ and the proofs of (a), (b) are obtained from this.

To prove (c) we may assume $\alpha > 0$. Suppose $u(t)$ is of Type 2 on $[t_1, T)$ and $c(t) \leqq k_2$ on $[t_1, T]$. Let $k \geqq \max\{k_2, H(\beta)F^{-1}(\alpha)\}$. Then, if $V(t) = H(u'(t)) - kF(u(t))$, we have $V(t) \leqq 0$ on $[t_1, T)$ and consequently

$$\int_{\alpha}^{\infty} \frac{d\tau}{G(kF(\tau))} \leqq T - t_1,$$

giving a contradiction. This completes the proof of Lemma 2.

Throughout the remainder of this paper, certain integrals need to be interpreted as Lebesgue integrals.

Let $a(t), b(t)$ be any functions which are absolutely continuous on every compact interval in $[t_1, \infty)$ and let $u(t)$ be a solution to (2.2) with initial conditions (2.1). From (2.2) we obtain for $t_1 \leqq t < T$,

$$\begin{aligned} H(u'(t)) - H(\beta) = a(t)F(u(t)) - a(t_1)F(\alpha) - \int_{t_1}^{t} a'(\tau)F(u(\tau))\, d\tau \\ + \int_{t_1}^{t} u'(\tau)f(u(\tau))[c(\tau) - a(\tau)]\, d\tau, \end{aligned} \tag{2.3}$$

$$\begin{aligned} I(u'(t)) - I(\beta) = b(t)F(u(t)) - b(t_1)F(\alpha) - \int_{t_1}^{t} b'(\tau)F(u(\tau))\, d\tau \\ + \int_{t_1}^{t} f(u(\tau))[c(\tau) - u'(\tau)b(\tau)]\, d\tau. \end{aligned} \tag{2.4}$$

We shall frequently use functions $a(t)$ such that $0 < a(t) \leqq c(t), a'(t) \leqq 0$ for $t \geqq t_1$. All such functions may be constructed as follows. Let $g(t)$ be absolutely continuous on every compact interval in $[t_1, \infty)$ and $0 < g(t) \leqq c(t)$. Define $g_1(t) = \min [0, -g'(t)/g(t)]$ and set

$$a(t) = g(t) \exp\left(\int_{t_1}^{t} g_1(\tau)\, d\tau\right), \qquad t \geqq t_1. \tag{2.5}$$

LEMMA 3. *Suppose conditions* (i), (ii), (iv) *are satisfied and let $a(t)$ be given by* (2.5). *If $u(t)$ is a solution of* (2.2) *with initial conditions* (2.1) *and $q(t)$ is absolutely continuous on every compact interval in $[t_1, T)$ with $0 \leqq q(t) \leqq u(t)$, then, for $t_1 \leqq t < T$,*

(a) $u'(t)[I(u'(t)) - I(\beta)] \geqq a(t)[F(u(t)) - F(\alpha)]$,

(b) $H(u'(t)) - H(\beta) \geqq a(t_1)[F(q(t_1)) - F(\alpha)] + \int_{t_1}^{t} q'(\tau)a(\tau)f(q(\tau))\, d\tau$.

Proof. Since $u'(t) > 0$ on (t_1, T) it is sufficient to prove the first inequality for $\beta > 0$. Set $b(t) = a(t)/u'(t)$. Then since $b'(t) \leqq 0$ and $F(s)$ is increasing for $s \geqq 0$ the first inequality follows from (2.4), where the last integral is discarded and the inequality $F(u(\tau)) \geqq F(\alpha)$ is used in the remaining integral.

The proof of the second inequality is obtained from (2.3) by omitting the last term, using the monotonicity of $F(s)$ and applying integration by parts.

THEOREM 1. *Suppose conditions* (i), (ii), (iv) *are satisfied and $u(t)$ is a solution of* (2.2) *with initial conditions* (2.1).

(a) *If for some $a(t)$ given by* (2.5),

$$H(\infty) - H(\beta) < \beta \int_{t_1}^{\infty} a(\tau)f(\beta(\tau - t_1) + \alpha)\, d\tau \leqq \infty,$$

then $u(t)$ is of Type 1 *or* 2. *If, in addition, $F(\infty) = \infty$, then $u(t)$ is of Type* 1.

(b) *If there exists $k > 0$ such that $c(t) \geqq k$ for $t \geqq t_1$ and*

$$H(\infty) - H(\beta) < k[F(\infty) - F(\alpha)] \leqq \infty,$$

then $u(t)$ is of Type 1.

(c) *If for some* $a(t)$ *given by* (2.5), $\beta I(\beta) \geqq a(t_1)F(\alpha)$ *and*

$$I(\infty) \int_{\alpha}^{\infty} \frac{ds}{F(s)} < \int_{t_1}^{\infty} a(\tau)\, d\tau \leqq \infty,$$

then $u(t)$ *is of Type* 1 *if* $H(\infty) < \infty$ *and of Type* 2 *if* $H(\infty) = \infty$.

(d) *If* $f(s)$ *is nondecreasing for* $s \geqq 0$ *and*

$$I(\infty) - I(\beta) < \int_{t_1}^{\infty} c(\tau) f(\beta(\tau - t_1) + \alpha)\, d\tau \leqq \infty,$$

then $u(t)$ *is of Type* 1 *if* $H(\infty) < \infty$ *and of Type* 2 *if* $H(\infty) = \infty$.

(e) *If there exists* $k > 0$ *such that* $c(t) \geqq k$ *for* $t \geqq t_1, H(\infty) = \infty, H(\beta) \geqq kF(\alpha)$ *and*

$$\int_{1}^{\infty} \frac{d\tau}{G(kF(\tau))} < \infty,$$

then $u(t)$ *is of Type* 2.

Proof. Since $u(t)$ is a convex function on $[t_1, T)$, $u(t) \geqq \beta(t - t_1) + \alpha$, for $t_1 \leqq t < T$. To prove (a), we apply Lemma 3 with $q(t) = \beta(t - t_1) + \alpha$ and use the fact that $H(s)$ is increasing for $s \geqq 0$. The second sentence in (a) follows from Lemma 2(a).

The proof of (b) is obtained directly from (2.3) with $a(t) = k$.

The constraints in (c) imply, by use of Lemma 3(a), that $u(t)$ is not of Type 3 or 4. The proof of (c) is then obtained from Lemma 2 since $F(\infty) = \infty$ in this case.

The proof of (d) is obtained directly from (2.4) with $b(t) \equiv 0$ and the use of Lemma 2.

To prove part (e), we may assume $\alpha > 0$. Then the result follows from Lemma 2(b) and the use of (2.3) with $a(t) = k$. This completes the proof.

The following theorem yields the solution type independent of particular initial conditions (2.1). We first list some conditions:

A. There exists $a(t)$ given by (2.5) and defined on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} a(\tau) f(k\tau)\, d\tau = \infty$$

for all $k > 0$.

B. For some $a(t)$ given by (2.5) over $[t_0, \infty)$,

$$\int_{t_0}^{\infty} a(\tau)\, d\tau = \infty, \qquad \int_{1}^{\infty} \frac{d\tau}{F(\tau)} < \infty.$$

C. The function $f(s)$ is nondecreasing for $s \geqq 0$ and

$$\int_{t_0}^{\infty} c(\tau) f(k\tau)\, d\tau = \infty$$

for all $k > 0$.

D. There exists $k_0 > 0$ such that $c(t) \geqq k_0$ for $t \geqq t_0$, $H(\infty) = \infty$ and for all $k > 0$,

$$\int_1^\infty \frac{d\tau}{G(kF(\tau))} < \infty.$$

THEOREM 2. *Suppose conditions* (i), (ii), (iv) *are satisfied and* $u(t)$ *is a solution of* (2.2) *with initial conditions* (2.1).

(a) *If* $H(\infty) < \infty$ *and at least one of the conditions* A, B, *or* C *holds, then* $u(t)$ *is of Type* 1.

(b) *If* D *holds, then* $u(t)$ *is of Type* 2. *If* B *or* C *holds and* $H(\infty) = \infty$, $I(\infty) < \infty$, *then* $u(t)$ *is of Type* 2.

(c) *If* $H(\infty) = \infty$ *and either* A *or* C *holds, then* $u(t)$ *is of Type* 2 *or* 3.

Proof. Since $a(t)$ is nonincreasing, condition A implies that $F(\infty) = \infty$. If $u(t)$ is of Type 3 or 4, then there exists $k > 0$ such that $u(t) > kt$ for all sufficiently large t. The proof of (a) and (c) for condition A now follows from Lemmas 2 and 3(b) since we may construct $q(t)$ with $q(t) = kt$ for all sufficiently large t.

To prove (a) and (b) for condition B we may assume that $\alpha > 0, \beta > 0$. Then the proof is obtained from Theorem 1(c) since $a(t)$ may be replaced by $\lambda a(t)$, $0 < \lambda \leqq 1$.

The proof of (a) and (b) from condition C is obtained from Theorem 1(d) since we may assume that $\beta > 0$. The proof of (c) from condition C is obtained from Lemma 2(b) and (2.4) with $b(t) \equiv 0$.

To prove (b) from condition D we may again assume that $\alpha > 0, \beta > 0$. Choose k, $k_0 \geqq k > 0$, such that $H(\beta) \geqq kF(\alpha)$. Then the proof is obtained from Theorem 1(e). This completes the proof.

THEOREM 3. *Suppose conditions* (i), (ii), (iv) *are satisfied,* $H(\infty) = \infty$, *and there exist positive constants* p, r *such that* $G(t) \geqq pt^r$, $t \geqq 0$, *and*

$$\int_1^\infty \frac{d\tau}{[F(\tau)]^r} < \infty.$$

Let $u(t)$ *be a solution of* (2.2) *with initial conditions* (2.1).

(a) *If for some* $a(t)$ *given by* (2.5), $H(\beta) \geqq a(t_1)F(\alpha)$ *and*

$$\int_\alpha^\infty \frac{d\tau}{[F(\tau)]^r} < p \int_{t_1}^\infty [a(\tau)]^r \, d\tau \leqq \infty, \tag{2.6}$$

then $u(t)$ *is of Type* 2.

(b) *Either* $u(t)$ *is of Type* 2 *or* $u(t)$ *is proper* (*Type* 3 *or* 4) *and for any given* $a(t)$ *expressed by* (2.5),

$$\int_{u(t)}^\infty \frac{d\tau}{[F(\tau)]^r} \geqq p \int_t^\infty [a(\tau)]^r \, d\tau$$

for all sufficiently large t.

Proof. By Lemma 2(b), $u(t)$ is of Type 2, 3, or 4. To prove (a) we shall show that $T < \infty$. A slight modification of the proof is needed if the hypothesis in (a) is valid for $\alpha = 0$. By (2.3), $H(u'(t)) \geqq a(t)F(u(t))$ for $t_1 \leqq t < T$. Consequently $u'(t)$

$\geqq G(a(t)F(u(t))) \geqq p[a(t)]^r[F(u)]^r$ and

$$\int_\alpha^\infty \frac{d\tau}{[F(\tau)]^r} \geqq p \int_{t_1}^T [a(t)]^r\, dt \tag{2.7}$$

and $T < \infty$. The inequality (2.7) gives an upper bound for T.

To prove (b) one may assume that $\alpha > 0$, $\beta > 0$ and that $u(t)$ is of Type 3 or 4. It follows from (a) that $a(t) \to 0$ as $t \to \infty$ since otherwise $a(t)$ may be replaced by $\lambda a(t), 0 < \lambda \leqq 1$, to obtain $H(\beta) \geqq a(t_1)F(\alpha)$ while (2.6) remains valid. Consequently, from (2.3), neglecting the second integral and replacing $F(u(\tau))$ by $F(\alpha)$ in the first integral, it follows that $H(u'(t)) - a(t)F(u(t)) \geqq 0$ for all sufficiently large t. The result (b) is now obtained by the procedure used in the proof of (a).

Remark. Under the hypothesis of Theorem 3, one can choose positive α, β such that not only is $u(t)$ of Type 2 but also such that the interval $[t_1, T)$ is arbitrarily small. One first chooses α sufficiently large so that both (2.6) and (2.7) are satisfied with $T - t_1$ as small as desired. Then, since $H(\infty) = \infty$, one can choose β such that $H(\beta) \geqq a(t_1)F(\alpha)$. Consequently, Theorem 3(a) generalizes Theorem 1 in [5]. Theorem 3(b) does not wholly conform to our announced intentions since the integral inequality involves the solution itself. Since this inequality places a constraint on the growth of a solution of Type 3 or 4, to conclude that a solution is actually of Type 2 we need a constraint from below. There is always available $u(t) \geqq kt$ for sufficiently large t. This is valid for some $k > 0$ for Type 4 and for all $k > 0$ for Type 3.

THEOREM 4. *Suppose* (i), (ii), (iv) *are satisfied and there exist positive constants* k_1, k_2 *such that* $k_1 \leqq h(s) \leqq k_2$ *for* $s \geqq 0$. *Then*

$$\int_1^\infty \frac{d\tau}{\sqrt{F(\tau)}} = \infty \tag{2.8}$$

is a necessary and sufficient condition that all positive solutions of (2.2) *be proper.*

Proof. The hypothesis implies that there exist positive constants p_1, p_2 such that $p_1 t^{1/2} \leqq G(t) \leqq p_2 t^{1/2}$. Since the only nonproper positive solutions must be of Type 1 or 2, the sufficiency follows from Lemma 2(c) and the necessity follows from Theorem 3(a) and the first part of the remark following this theorem. Positive decreasing solutions are not covered by our initial conditions (2.1), but such solutions are always proper.

Theorem 4 may be compared to Theorem 3 in [5], which considers a more general equation, and may be obtained from the latter if $f(s)$, $s \geqq 0$, is increasing.

COROLLARY 1. *If the hypotheses of Theorem* 2(c) *and Theorem* 4 *are valid and* (2.8) *is satisfied, then the solution* $u(t)$ *is of Type* 3.

We conclude the study in this section by giving conditions under which the solution is of Type 4.

THEOREM 5. *Suppose* (i), (ii), (iv) *are satisfied and* $u(t)$ *is a solution to* (2.2) *with initial conditions* (2.1). *If either of the following conditions* (a) *or* (b) *is satisfied, then* $u(t)$ *is of Type* 4.

(a) *The function* $c(t) \leqq k$, $t \geqq t_1$, *and* $k[F(\infty) - F(\alpha)] < H(\infty) - H(\beta) \leqq \infty$.

(b) *The function* $f(s)$ *is nondecreasing for* $s \geqq 0$ *and there exists* $K > 0$ *such that*

$$I(\beta) + \int_{t_1}^{\infty} c(\tau)f(K(\tau - t_1) + \alpha)\,d\tau < I(K).$$

Proof. To prove the result from (a), set $a(t) = k$ in (2.3). Then for $t_1 \leqq t < T$ we have $H(u'(t)) - H(\beta) \leqq k[F(u(t)) - F(\alpha)] \leqq k[F(\infty) - F(\alpha)] < H(\infty) - H(\beta)$ and it follows that $u(t)$ is of Type 4.

To prove the result from (b), define $T_1 = \sup\{t : t \geqq t_1, u'(t) < K\}$, $T_1 \leqq T$. For $t_1 \leqq t < T_1, u(t) \leqq K(t - t_1) + \alpha$ and by (2.4) with $b(t) \equiv 0$ we have

$$I(u'(t)) - I(\beta) \leqq \int_{t_1}^{t} c(\tau)f(K(\tau - t_1) + \alpha)\,d\tau.$$

Consequently, $\lim u'(t) < K$ as $t \to T_1$ and therefore $T_1 = T = \infty$ and $u(t)$ is of Type 4.

Remark. From condition (b) and (2.4), it follows that

$$I(u'(t)) = I(e) - \int_{t}^{\infty} c(\tau)f(u(\tau))\,d\tau,$$

where $e = \lim u'(t)$ as $t \to \infty, \beta < e < K$. By L'Hospital's rule we may write $u(t) = t(e + \phi(t))$ for $t > \max(0, t_1)$, where $\phi(t) \to 0$ as $t \to \infty$. For $h(s) = 1$, $s \geqq 0$, we obtain the estimate

$$|\phi(t)| \leqq t^{-1}\left[|\alpha - et_1| + \int_{t_1}^{t}\int_{s}^{\infty} c(\tau)f(K(\tau - t_1) + \alpha)\,d\tau\,ds\right],$$

where, by L'Hospital's rule, the estimate tends to zero as $t \to \infty$.

3. Oscillation and nonoscillation. In this section we shall study the behavior of the solutions to (1.1) subject to (i), (ii) and

(v) $c(t)$ is positive and absolutely continuous on every compact interval in $[t_0, \infty)$.

Under these conditions we first develop some preliminary tools. Let

$$g_1(t) = \min[0, -c'(t)/c(t)],$$
$$g_2(t) = \max[0, -c'(t)/c(t)].$$

Given $t_1 \geqq t_0$, we define $(j = 1, 2)$

$$a_j(t) = c(t)\exp\left(\int_{t_1}^{t} g_j(\tau)\,d\tau\right).$$

Then $a_1'(t) \leqq 0 \leqq a_2'(t)$ for $t \geqq t_1$. Set $(j = 1, 2)$

$$k_j = \lim_{t \to \infty} a_j(t), \quad K_j = \exp\left(\int_{t_1}^{\infty} g_j(\tau)\,d\tau\right).$$

Then $0 \leqq k_1 < \infty, 0 < k_2 \leqq \infty, 0 \leqq K_1 \leqq 1 \leqq K_2 \leqq \infty$. We observe that $k_1 \leqq k_2$ with equality if and only if $c(t)$ is a positive constant on $[t_1, \infty)$.

Let $u(t)$ be a solution to (1.1) defined on $[t_1, T)$. We define $(j = 1, 2)$

$$V_j(t) = a_j(t)F(u(t)) + H(u'(t)) \exp\left(\int_{t_1}^{t} g_j(\tau)\, d\tau\right).$$

Then $V_1'(t) \leqq 0 \leqq V_2'(t)$ on $[t_1, T)$ and $V_j(t) > 0, j = 1, 2$, unless $u(t) = u'(t) = 0$. It follows that if some $t_2, t_1 \leqq t_2 < T, u(t_2) = u'(t_2) = 0$, then $u(t) = 0$ for all $t \geqq t_1$. Consequently, the zeros of a nontrivial solution are discrete.

Similar Lyapunov functions have been applied to (1.1), e.g., [14] and [18], but, in general, their applicability is restricted to monotone $c(t)$.

LEMMA 4. *Suppose conditions* (i), (ii), (v) *are satisfied and* $u(t)$ *is a solution to* (1.1) *on* $[t_1, T)$. *Then* $u(t)$ *is of one of the following types*:

Proper solutions, $T = \infty$

A. *Nonoscillatory.*

Type 5. (a) $u(t) \to \infty, u'(t) \to b > 0, t \to \infty$.
(b) $u(t) \to -\infty, u'(t) \to b < 0, t \to \infty$.

Type 6. (a) $u(t) \to \infty, u'(t) \to 0+, t \to \infty$.
(b) $u(t) \to -\infty, u'(t) \to 0-, t \to \infty$.

Type 7. (a) $u(t) \to a > 0, u'(t) \to 0+, t \to \infty$.
(b) $u(t) \to a < 0, u'(t) \to 0-, t \to \infty$.

B. *Oscillatory.*

Type 8. (a) $u(t)$ *and* $u'(t)$ *are bounded.*
(b) $u(t)$ *or* $u'(t)$ *is unbounded.*

Nonproper solutions, $T < \infty$

Type 9. (a) $u(t) \to a > 0, u'(t) \to -\infty, t \to T$.
(b) $u(t) \to a < 0, u'(t) \to \infty, t \to T$.

Type 10. (a) $u(t) \to 0+, u'(t) \to -\infty, t \to T$.
(b) $u(t) \to 0-, u'(t) \to \infty, t \to T$.

Proof. The proof is straightforward except, possibly, for the nonexistence of oscillatory nonproper solutions. Suppose $u(t)$ is an oscillatory nonproper solution on $[t_1, T), T < \infty$. By the above remarks, $u(t)$ has a denumerable number of zeros which converge to T and on which $u'(t)$ is unbounded. For otherwise, $u'(t)$ would be bounded on $[t_1, T)$ and the interval would not be maximal. Since $V_1(t)$ is non-increasing on $[t_1, T)$ it follows from comments just stated that $\lim V_1(t) > 0$ as $t \to T$ and consequently

$$\begin{aligned} 0 < k &= \lim_{t \to T} V_1(t)/\exp\left(\int_{t_1}^{t} g_1(\tau)\, d\tau\right) \\ &= \lim_{t \to T} [c(t)F(u(t)) + H(u'(t))]. \end{aligned}$$

By considering the two subsequences of the zeros of $u(t)$ at which $u'(t)$ is positive and negative respectively, we have $k \leqq \min[H(\infty), H(-\infty)]$. Now since $c(t)$ is bounded away from zero on $[t_1, T]$, by considering a sequence of relative maxima of $u(t)$ we have $\limsup u(t) \geqq a > 0$ as $t \to T$. Let $\{t_i\}$ be an increasing sequence such that $t_i \to T$ and $u(t_i) = a/2$. Then $\lim H(u'(t_i)) = k - c(T)F(a/2) < \min[H(\infty),$

$H(-\infty)]$. Therefore, the sequence $\{|u'(t_i)|\}$ is bounded. But this is inconsistent with $\limsup u(t) \geqq a$. This completes the proof.

With initial conditions $u(t_1) = \alpha, u'(t_1) = \beta$ we set

$$V = c(t_1)F(\alpha) + H(\beta), \qquad V = V_1(t_1) = V_2(t_1).$$

LEMMA 5. *Suppose conditions* (i), (ii), (v) *are satisfied and* $u(t)$ *is a solution to* (1.1) *with initial conditions at* $t_1, t_1 \geqq t_0$. *If either* $H(-\infty) = \infty$ *or* $K_1 > 0$ *and* $V < K_1 H(-\infty)$, *then* $u(t)$ *is not of Type* 9(a) *or* 10(a). *If either* $H(\infty) = \infty$ *or* $K_1 > 0$ *and* $V < K_1 H(\infty)$, *then* $u(t)$ *is not of Type* 9(b) *or* 10(b).

Proof. Suppose $u(t)$ is a nonproper solution on $[t_1, T)$. By considering $V_1(t)$, $t_1 \leqq t < T$, we obtain

$$V \geqq H(u'(t)) \exp\left(\int_{t_1}^{T} g_1(\tau)\, d\tau\right) \geqq H(u'(t))K_1$$

and the proof follows from this inequality.

The importance of the condition $H(\infty) = H(-\infty) = \infty$ in connection with the continuation of all solutions has been emphasized in [4]. This is also seen in Theorem 6 below.

The initial conditions $u(t_1) = \alpha, u'(t_1) = \beta$ for a nontrivial solution $u(t)$ may be partitioned into 4 cases, viz.: *Case* 1. $\alpha \geqq 0, \beta > 0$. *Case* 2. $\alpha > 0, B \leqq 0$. *Case* 3. $\alpha \leqq 0, \beta < 0$. *Case* 4. $\alpha < 0, \beta \geqq 0$. For $t, t_1 \leqq t < T$, it will be convenient also to speak of the pair $u(t), u'(t)$ as belonging to one of these four cases.

We note, for $t_1 \leqq t < T$, the relation

$$I(\beta) - I(u'(t)) = \int_{t_1}^{t} c(\tau)f(u(\tau))\, d\tau. \tag{3.1}$$

THEOREM 6. *Suppose conditions* (i), (ii), (v) *are satisfied. Let* $u(t)$ *be a nontrivial solution of* (1.1) *with initial conditions at* $t_1, t_1 \geqq t_0$, *and suppose* $0 < k_1 \leqq k_2 < \infty$, *where* k_1, k_2, K_1, K_2 *are as previously defined.*

(a) *If the initial conditions correspond to case j (as mentioned preceding the theorem), then a test for the type of* $u(t)$ *is obtained by starting with statement (j) (as numbered below), and continuing cyclically, when permitted, until the first valid hypothesis is encountered.*

(1) *If* $V > k_2 F(\infty)$, *then* $u(t)$ *is of Type* 5(a). *If* $V = k_2 F(\infty)$, *then* $u(t)$ *is of Type* 5(a) *or* 6(a) *as* $k_1 < k_2$ *or* $k_1 = k_2$ *respectively. If* $V < k_1 F(\infty)$ *continue to* (2).

(2) *If* $V > K_2 H(-\infty)$, *then* $u(t)$ *is of Type* 9(a). *If* $V = K_2 H(-\infty)$, *then* $u(t)$ *is of Type* 9(a) *or* 10(a). *In particular, if* $V = K_2 H(-\infty)$, $u(t)$ *is of Type* 9(a) *unless* $K_2 = 1$ *and* $c(t)$ *is constant on an interval which includes* $[t_1, T)$, *in which case* $u(t)$ *is of Type* 10(a). *If* $V < K_1 H(-\infty)$ *continue to* (3).

(3) *If* $V > k_2 F(-\infty)$, *then* $u(t)$ *is of Type* 5(b). *If* $V = k_2 F(-\infty)$, *then* $u(t)$ *is of Type* 5(b) *or* 6(b) *as* $k_1 < k_2$ *or* $k_1 = k_2$ *respectively. If* $V < k_1 F(-\infty)$ *continue to* (4).

(4) *If* $V > K_2 H(\infty)$, *then* $u(t)$ *is of Type* 9(b). *If* $V = K_2 H(\infty)$, *then* $u(t)$ *is of Type* 9(b) *or* 10(b). *In particular, if* $V = K_2 H(\infty)$, $u(t)$ *is of Type* 9(b) *unless* $K_2 = 1$ *and* $c(t)$ *is constant on an interval which includes* $[t_1, T)$, *in which case* $u(t)$ *is of Type* 10(b). *If* $V < K_1 H(\infty)$ *continue to* (1).

(b) *If* $V < \min[k_1F(\infty), K_1H(-\infty), k_1F(-\infty), K_1H(\infty)] \leqq \infty$, *then* $u(t)$ *is of Type* 8(a).

Proof. The condition $0 < k_1 \leqq k_2 < \infty$ implies that $c(t)$ tends to a positive constant as $t \to \infty$ and $0 < K_1 \leqq 1 \leqq K_2 < \infty$. Consequently, by (3.1), there is no solution of Type 7(a) or 7(b).

Suppose that for some $t, t \geqq t_1$, the pair $u(t), u'(t)$ corresponds to Case 1. If $V \geqq k_2F(\infty)$, then by studying $V_2(t)$ one concludes that the solution remains in Case 1 for all larger values of t. Consequently, the solution is of Type 5(a) or 6(a). Again, if $V \geqq k_2F(\infty)$ and if $u(t)$ is of Type 6(a), then it follows from $V_2'(t) \geqq 0$ and

$$V = V_2(t_1) \leqq V_2(t) \leqq V_2(\infty) = k_2F(\infty) \leqq V$$

that $V_2(t) = V = k_2F(\infty)$ for all $t \geqq t_1$. This implies, by the expressions for the derivatives of $a_2(t)$ and $V_2(t)$, that $c(t)$ is constant on $[t_1, \infty)$ and therefore $k_1 = k_2$. Also, if $k_1 = k_2$ and $V = k_2F(\infty)$, then $K_1 = K_2 = 1$ and $V = V_1(t) = V_2(t)$ for all $t \geqq t_1$. Consequently $u(t)$ is of Type 6(a). On the other hand, if $V < k_1F(\infty)$, then by studying $V_1(t)$ one concludes that the solution must pass to Case 2, i.e., there exists a larger value of t such that $u(t) > 0, u'(t) \leqq 0$.

Now, suppose that for some $t, t \geqq t_1$, the pair $u(t), u'(t)$ corresponds to Case 2. If $V \geqq K_2H(-\infty)$, then by studying $V_2(t)$ one concludes that the solution remains in Case 2 for all larger values of t on $[t_1, T)$. Consequently, the solution is of Type 9(a) or 10(a). Again if $V \geqq K_2H(-\infty)$ and $u(t)$ is of Type 10(a), then it follows as before that $c(t)$ is constant on $[t_1, T)$ and since $\int_T^\infty g_2(\tau)\,d\tau = 0$, it follows that $c(t)$ is nondecreasing on $[T, \infty)$. Consequently, $K_2 = 1$, $V = H(-\infty)$. Also, if $K_2 = 1$, $V = H(-\infty)$ and $c(t)$ is constant on $[t_1, T)$, then $u(t)$ is of Type 10(a). On the other hand, if $V < K_1H(-\infty)$, then, by Lemma 5, one concludes that the solution must pass to Case 3. The proof of part (a) is now completed by similar arguments.

If the hypothesis in part (b) is valid, then, by the above argument, the solution passes cyclically from case to case and is therefore oscillatory. Since $V_1(t) \leqq V$ for $t \geqq t_1$, it follows that both $u(t)$ and $u'(t)$ are bounded and consequently the solution is of Type 8(a).

Remark. If $c(t)$ is a positive constant on $[t_1, \infty)$, the above result gives a definitive statement. However, in part, the theorem falls short of our announced intentions due to the question of the decidability of the statement "$c(t)$ is constant on an interval which includes $[t_1, T)$." In principle, however, this question may be reduced to that of quadratures. For example, suppose $K_2 = 1, k_1F(\infty) > V = H(-\infty)$ and the initial conditions correspond to Case 1. We would then conclude that $u(t)$ is of Type 9(a) or 10(a). If $u(t)$ is of Type 10(a), then $c(t_1)F(u) + H(u') = H(-\infty)$ for $t_1 \leqq t < T$ and one may calculate $[t_1, T)$ by two quadratures over known intervals in the variable u. One may then compare $[t_1, T)$ to intervals over which $c(t)$ is constant.

A number of papers [4], [7], [9], [14], [17]–[20], [22] have dealt with boundedness of solutions to (1.1). Using Lemmas 4 and 5 and $V_1(t)$ as in the proof of Theorem 6(b), we obtain the following corollary which generalizes, in part, some of the boundedness results in the above papers. In particular, it generalizes Theorems 1 and 2 in [18].

COROLLARY 2. *Suppose conditions* (i), (ii), (v) *are satisfied and let* $u(t)$ *be a solution of* (1.1) *with initial conditions at* $t_1, t_1 \geqq t_0$. *If* $k_1 > 0$ *and* $V < \min(k_1F(\infty),$

$k_1F(-\infty))$, *then* $u(t)$ *is bounded. If* $K_1 > 0$ *and* $V < \min(K_1H(\infty), K_1H(-\infty))$, *then* $u'(t)$ *is bounded. If* $k_1 > 0, K_1 > 0$ *and* $V < \min[k_1F(\infty), k_1F(-\infty), K_1H(\infty), K_1H(-\infty)]$, *then* $u(t)$ *is Type* 8(a).

COROLLARY 3. *Suppose conditions* (i), (ii), (v) *are satisfied and* $k_1 > 0, K_1 > 0$. *Then the zero solution of the system*

$$x' = y,$$
$$y' = -c(t)f(x)h(y)$$

is Lyapunov stable.

Proof. The result follows in the usual way by considering

$$V(x, y) = k_1F(x) + K_1H(y),$$
$$V_1(x, y, t) = a_1(t)F(x) + H(y)\exp\left(\int_{t_1}^{t} g_1(\tau)\,d\tau\right).$$

Similar stability theorems are given in [4], [14], [18]. In particular, Corollary 3 generalizes Theorem 3 in [18], whereas [4] has essentially the same restrictions as those imposed here.

LEMMA 6. *Suppose conditions* (i), (ii) *are satisfied and* $c(t)$ *is nonnegative and continuous on* $[t_0, \infty)$. *Let* $u(t)$ *be a solution to* (1.1) *with initial conditions corresponding to Case* 1 (*as defined preceding Theorem* 6). *If either of the following conditions* (a) *or* (b) *is satisfied, then* $u(t)$ *is of Type* 5(a).

(a) *There exists* $g(s)$, *continuous and nondecreasing for* $s \geqq 0$, *such that* $f(s) \leqq g(s), s \geqq 0$, *and*

$$I(\beta) > \int_{t_1}^{\infty} c(\tau)g(\alpha + \beta(\tau - t_1))\,d\tau.$$

(b) *There exists* $g(s)$, *continuous and nonincreasing for* $s \geqq 0$, *such that* $f(s) \leqq g(s), s \geqq 0$, *and for some* $k > 0$,

$$I(\beta) > I(k) + \int_{t_1}^{\infty} c(\tau)g(\alpha + k(\tau - t_1))\,d\tau.$$

Proof. To prove the lemma from (a), let (t_1, t_2) be the maximum open interval to the right on which $u(t)$ is positive. We shall show that $t_2 = \infty$. For $t \in (t_1, t_2)$ we have $0 < u(t) \leqq \alpha + \beta(t - t_1)$. From (3.1) and the hypothesis we obtain

$$I(u'(t)) \geqq I(\beta) - \int_{t_1}^{\infty} c(\tau)g(\alpha + \beta(\tau - t_1))\,d\tau > 0$$

for all $t \in (t_1, t_2)$. Consequently $t_2 = \infty$ and since $u'(t)$ is nonincreasing, $u(t)$ is of Type 5(a).

To prove the lemma from (b), let (t_1, t_2) be the maximum open interval to the right on which $\beta \geqq u'(t) > k$. We shall show that $t_2 = \infty$. For $t \in (t_1, t_2)$ we have $u(t) \geqq \alpha + k(t - t_1)$ and by (3.1) and the hypothesis we obtain

$$I(u'(t)) \geqq I(\beta) - \int_{t_1}^{\infty} c(\tau)g(\alpha + k(\tau - t_1))\,d\tau > I(k)$$

for all $t \in (t_1, t_2)$. It follows that $t_2 = \infty$ and $u(t)$ is of Type 5(a). This completes the proof.

Under certain conditions, the finiteness of the integrals in Lemma 6 are also necessary as is shown by Theorems 1 and 2 in [6].

For the following nonoscillation theorem we introduce an additional condition:

(vi) There exists $g(s)$, continuous and nondecreasing for $s \geqq 0$, such that $f(s) \leqq g(s), s \geqq 0$, and

$$\int_{\max[0,t_0]}^{\infty} c(\tau)g(k\tau)\,d\tau < \infty$$

for all $k \geqq 0$.

Suppose conditions (i), (ii), (v), (vi) are satisfied and set

$$F(k,t) = \int_t^{\infty} c(\tau)g(k(\tau - t))\,d\tau.$$

Then $F(k, t)$ is defined for $k \geqq 0, t \geqq t_0$, and is nondecreasing in k. For a fixed $k > 0$, $F(k, t)$ is decreasing in t and $F(k, t) \to 0$ as $t \to \infty$. We define

$$\phi(k) = \inf\{t : t \geqq t_0, I(k) \geqq F(k,t)\}.$$

Then $\phi(k)$ is defined and bounded on every compact subset of the positive real numbers. For suppose $0 < k' \leqq k \leqq k''$ and for some $t^*, t^* \geqq t_0, I(k') \geqq F(k'', t^*)$. Then $I(k) \geqq I(k') \geqq F(k'', t^*) \geqq F(k, t^*)$ and therefore $\phi(k) \leqq t^*$.

THEOREM 7. *Suppose conditions* (i), (ii), (v), (vi) *are satisfied. Let $u(t)$ be a nontrivial solution of* (1.1) *with initial conditions at $t_1, t_1 \geqq t_0$.*

(a) *If $\phi(k), k > 0$, is bounded, then $u(t)$ is nonoscillatory.*

(b) *If $K_1 > 0, V < K_1 \min(H(\infty), H(-\infty))$ and* $\limsup \phi(k) < \infty$ *as $k \to 0$, then $u(t)$ is proper and nonoscillatory.*

Proof. To prove (a), let $\lambda > \phi(k)$ for all $k > 0$. Assume that $u(t)$ is oscillatory. Then, since $u(t)$ is nontrivial and proper, there exists $t^* > \lambda$ such that $u(t^*) = 0$, $u'(t^*) = \beta^* > 0$. Consequently, $I(\beta^*) > F(\beta^*, t^*)$ and by Lemma 6, $u(t)$ is of Type 5(a) contrary to assumption.

By the hypothesis of (b), one concludes by studying $V_1(t)$ that $u(t)$ is proper and $u'(t)$ is bounded. Let $\mu > |u'(t)|, t \geqq t_1$, and $\lambda > \phi(k)$ for $0 < k \leqq \mu$. Suppose $u(t)$ is oscillatory. Then, there exists $t^* > \lambda$ such that $u(t^*) = 0, u'(t^*) = \beta^* > 0$. Consequently, $I(\beta^*) > F(\beta^*, t^*)$ and by Lemma 6 we obtain a contradiction. This completes the proof.

Theorem 7(b), above, generalizes Theorem 3 in [19]. A nonoscillation theorem involving solutions of Type 7 can be found in [8].

We conclude with an oscillation theorem. Some recent papers on oscillation which are applicable to (1.1) are [10], [11], [13].

We may establish an oscillation theorem from a result which implies the solution is not positive; for the corresponding theorem for (1.2) would then yield conditions under which the solution is not negative.

If $I(\infty) = \infty$, we denote the inverse of $I(t), t \geqq 0$, by $J(s), s \geqq 0$.

THEOREM 8. *Suppose conditions* (i), (ii), (v) *are satisfied. Let $u(t)$ be a solution to* (1.1) *with initial conditions at $t_1, t_1 \geqq t_0$, and suppose either $H(-\infty) = \infty$ or*

$K_1 > 0$ *and* $V < K_1 H(-\infty)$. *If, in addition, at least one of the following conditions is satisfied, then* $u(t)$ *is not a positive solution:*

(1) $k_1 > 0$, $V < k_1 F(\infty)$.

(2) $\int_{t_0}^{\infty} c(\tau)\,d\tau = \infty$, $\liminf f(s) > 0$ *as* $s \to \infty$.

(3) $I(\infty) = \infty$ *and there exist positive constants* p, r $(r \geqq 1)$ *such that* $J(t) \geqq pt^r$, $t \geqq 0$. *There exists* $g(s)$, *continuous and nondecreasing on* $[0, \infty)$, *such that* $0 < g(s) \leqq f(s)$, $s > 0$, *and*

$$\int_{t_0}^{\infty} \left(\int_t^{\infty} c(\tau)\,d\tau \right)^r dt = \infty, \quad \int_1^{\infty} g^{-r}(s)\,ds < \infty.$$

(4) $I(\infty) = \infty$ *and* $J(t) \geqq pt$, $t \geqq 0$, *for some positive constant p. There exists* $g(s)$, *continuous and nondecreasing on* $[0, \infty)$, *such that* $0 < g(s) \leqq f(s)$, $s > 0$. *If*

$$Q_\alpha(t) = \int_\alpha^t g^{-1}(s)\,ds, \qquad t \geqq \alpha > 0,$$

then $Q_\alpha(\infty) = \infty$ *and*

$$\int_t^{\infty} c(\tau) g\left(R_\alpha\left(p \int_t^{\tau} \int_s^{\infty} c(x)\,dx\,ds \right) \right) d\tau = \infty$$

for $t \geqq t_0$, $\alpha > 0$ *and where* $R_\alpha(s)$, $s \geqq 0$, *is the inverse of* $Q_\alpha(t)$, $t \geqq \alpha$.

Proof. It follows from the hypothesis and Lemma 5 that the only possible positive solutions are of Types 5(a), 6(a), or 7(a). The conclusion of the theorem follows readily from conditions (1) or (2) by studying $V_1(t)$ and (3.1).

Now suppose that $u(t)$ is of Type 5(a), 6(a), or 7(a) and condition (3) or (4) is satisfied. For sufficiently large t, say $t \geqq \bar{t}$, $u(t)$ is positive and from (3.1) we obtain, after some manipulations,

$$I(u'(t)) \geqq \int_t^{\infty} c(\tau) g(u(\tau))\,d\tau, \qquad t \geqq \bar{t}. \tag{3.2}$$

If the remaining conditions in (3) hold, then

$$u'(t) \geqq p\left(\int_t^{\infty} c(\tau)\,d\tau \right)^r g^r(u(t)), \qquad t \geqq \bar{t},$$

and consequently, for $\alpha = u(\bar{t}) > 0$,

$$\int_\alpha^{u(t)} g^{-r}(s)\,ds \geqq p \int_{\bar{t}}^{t} \left(\int_s^{\infty} c(x)\,dx \right)^r ds, \qquad t \geqq \bar{t},$$

which yields a contradiction.

If the remaining conditions in (4) hold, we obtain the estimate

$$u(t) \geqq R_\alpha\left(p \int_{\bar{t}}^{t} \int_s^{\infty} c(x)\,dx\,ds \right), \qquad t \geqq \bar{t},$$

and by use of (3.2) we obtain the desired contradiction.

Remark. For the linear equation $u'' + c(t)u = 0$, the condition (4) reduces to

$$\int_t^\infty c(\tau)\exp\left(\int_t^\tau \int_s^\infty c(x)\,dx\,ds\right)d\tau = \infty.$$

For $c(t) = kt^{-2}$ we obtain, according to the above theorem, oscillation for $k \geqq 1$ which is not far removed from the best condition $k > 1/4$.

REFERENCES

[1] R. BELLMAN, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

[2] I. BIHARI, *Oscillation and monotonicity theorems concerning nonlinear differential equations of the second order*, Acta Math. Acad. Sci. Hungar., 9 (1958), pp. 83–104.

[3] ———, *On periodic solutions of certain second order ordinary differential equations with periodic coefficients*, Ibid., 12 (1961), pp. 11–16.

[4] T. A. BURTON AND R. C. GRIMMER, *Stability properties of* $(r(t)u')' + a(t)f(u)g(u') = 0$, Monatsh. Math., 74 (1970), pp. 211–222.

[5] ———, *On continuability of solutions of second order differential equations*, Proc. Amer. Math. Soc., 29 (1971), pp. 277–283.

[6] ———, *On the asymptotic behavior of solutions of* $x'' + a(t)f(x) = 0$, Proc. Cambridge Philos. Soc., 70 (1971), pp. 77–88.

[7] S. H. CHANG, *Boundedness theorems for certain second order nonlinear differential equations*, J. Math. Anal. Appl., 31 (1970), pp. 509–516.

[8] L. H. ERBE, *Nonoscillatory solutions of second order nonlinear differential equations*, Pacific J. Math., 28 (1969), pp. 77–85.

[9] I. V. KAMENEV, *Oscillatoriness and boundedness of solutions of a nonlinear second order equation with a multiplicatively separated right hand side*, Differencial'nye Uravnenija, 4 (1968), pp. 845–849 (Russian): MR 37 # 3105.

[10] A. KARTSATOS, *Properties of bounded solutions of nonlinear equations of second order*, Proc. Amer. Math. Soc., 19 (1968), pp. 1057–1059.

[11] ———, *On oscillation and boundedness of solutions of second order nonlinear equations*, Boll. Un. Mat. Ital. (4), 1 (1968), pp. 357–361.

[12] ———, *A stability theorem for the nonlinear differential equation* $x'' + p(t)g(x)h(x') = 0$, J. Austral. Math. Soc., 10 (1969), pp. 169–172.

[13] ———, *Some properties of solutions of* $\ddot{x} + p(t)g(x, \dot{x}) = 0$, Math. Nachr., 40 (1959), pp. 299–304.

[14] B. S. LALLI, *On boundedness of solutions of certain second order differential equations*, J. Math. Anal. Appl., 25 (1969), pp. 182–188.

[15] LIANG CHUNG-CH'AO, *Asymptotic character of the solutions of a class of second order nonlinear differential equations*, Shuxue Jinzhan, 9 (1966), pp. 251–264 (Chinese); MR 38 #4767.

[16] W. E. SHREVE, *Terminal value problems for second order nonlinear differential equations*, SIAM J. Appl. Math., 18 (1970), pp. 783–791.

[17] J. S. W. WONG, *A note on boundedness theorems of certain second order differential equations*, SIAM Rev., 6 (1964), pp. 175–176.

[18] J. S. W. WONG AND T. A. BURTON, *Some properties of solutions of* $u''(t) + a(t)f(u)g(u') = 0$, *(II)*, Monatsh. Math., 69 (1965), pp. 368–374.

[19] ———, *Some properties of solutions of* $u''(t) + a(t)f(u)g(u') = 0$, *(III)*, SIAM J. Appl. Math., 14 (1966), pp. 209–214.

[20] ———, *Boundedness theorems for solutions of* $u''(t) + a(t)f(u)g(u') = 0$, *(IV)*, Enseignement Math., (2) 13 (1967), pp. 157–165.

[21] ———, *On second order nonlinear oscillation*, Funkcial. Ekvac., 11 (1968), pp. 207–234.

[22] M. S. ZARGHAMEC AND B. MEHRI, *A note on boundedness of solutions of certain second order differential equations*, J. Math. Anal. Appl., 31 (1970), pp. 504–508.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

SINGULAR PERTURBATIONS FOR A NONLINEAR DIFFERENTIAL EQUATION WITH A SMALL PARAMETER*

GEORGE C. HSIAO†

Abstract. This paper concerns singular perturbation problems such as those of slow viscous flow past a cylinder. A nonlinear second order differential equation with a small parameter is used as a model to discuss the validity of the method of inner and outer expansions (MIO) for treating such problems. Based on a regular perturbation procedure by Finn and Smith, it is shown that the formal asymptotic expansions constructed by MIO are indeed in some sense the asymptotic expansions for the exact solution of the problem.

1. Introduction. It is the purpose of this paper to point out certain intimate connections between the regular perturbation procedure developed by Finn and Smith [6] for existence proofs in the theory of two-dimensional viscous flow problems and the method of *inner and outer expansions* for treating such problems [9]. As a model, we consider here the boundary value problem (P_ε) defined by

$$L[y] = y'' + \frac{1}{x}y' = \varepsilon yy', \qquad 1 \leqq x < \infty, \tag{1.1}$$

$$y = 0 \quad \text{at} \quad x = 1 \tag{1.2}$$

and

$$y \to -a \quad \text{as} \quad x \to \infty, \tag{1.3}$$

where ε is a small positive parameter and a is a positive constant. For this simple model we can give a complete discussion which illustrates the ideas. These ideas are extended to the viscous flow problem in [8] but the results there are less complete.

The problem (P_ε) is *singular* in the sense that the linearized problem (P_0),

$$L[y] = 0 \tag{1.4}$$

together with (1.2) and (1.3) has no solution. This is the analogue of the Stokes paradox in fluid flow [1]. On the other hand, if we let $y = -a + v$, then (1.1)–(1.3) read

$$\mathscr{L}[v] = v'' + \left(\frac{1}{x} + a\varepsilon\right)v' = \varepsilon avv', \qquad 1 \leqq x < \infty, \tag{1.1'}$$

$$v = -a \quad \text{at} \quad x = 1, \tag{1.2'}$$

$$v \to 0 \quad \text{as} \quad x \to \infty, \tag{1.3'}$$

and it is easy to see that the corresponding linearized problem,

$$\mathscr{L}[v] = 0 \tag{1.4'}$$

* Received by the editors July 29, 1971, and in revised form April 27, 1972.

† Department of Mathematics, University of Delaware, Newark, Delaware 19711. The preparation of this paper was supported by the University of Delaware Research Foundation under Research Grant UDRF-2-97-414.

together with (1.2′) and (1.3′), does possess a solution. In fluid flow (1.4′) will represent the Oseen equation [5]. That there exists a solution of the Oseen problem, (1.4′) together with (1.2′) and (1.3′), is the basis for the procedure of Finn and Smith. The solution $y(x;\varepsilon)$ of (P_ε), which we show exists (see Theorem 1), is then sought as a regular perturbation of the solution v of the Oseen problem.

The singular nature of the linearized problem, such as (P_0), has the effect of giving the solution $y(x;\varepsilon)$ a nonuniform asymptotic structure for small ε. This leads to the construction of the *inner* and *outer* expansions for the problem (P_ε). Although the method of *inner* and *outer* expansions has been used successfully in a wide variety of problems (cf. [3] and [4]), the rigorous justification of this formal procedure seems still in its infancy[1] and further clarification and development are needed. In the present work, we first define what we mean by the *inner* and *outer* expansions for the problem (P_ε) (Definitions 1 and 2, § 4), then we state, in a precise way, the *matching principle* (§ 5). Finally, we justify the procedure by showing that the formal inner and outer expansions obtained from the matching principle are indeed the inner and outer expansions for the solution $y(x,\varepsilon)$. This last task is really the major goal of this paper. It should be mentioned that in [4] some partial justification of the method has been given for a problem similar to (P_ε) (see (5.5)), but our results are much more complete than those given there. From our results, the model (P_ε) may serve as another example in the class of singular perturbation problems (such as those of slow viscous flow past a cylinder) for demonstrating the validity of the *asymptotic matching principle* in [7]; there a nonlinear fourth order ordinary differential equation was used.

The main results can now be summarized in the following three theorems.

THEOREM 1. *There exists a solution $y(x;\varepsilon)$ of the problem (P_ε) defined by* (1.1)–(1.3) *for ε sufficiently small.*

Comment. A kind of uniqueness theorem is indicated in § 3.

In order to state the next theorem we need some notation. Notice that, if the condition at infinity, (1.3), is relaxed the problem $L[y] = 0$, together with (1.2), has a one-parameter family of solutions,

$$y = A\log x, \tag{1.5}$$

which is uniquely determined by the parameter A. We denote it by $u_A(x)$. Then, we have the following theorem.

THEOREM 2 (Inner expansion). *There exists a sequence of functions $\{y_n(x)\}$, independent of ε and defined for $x \geqq 1$, such that for any positive integer N, the relation*

$$(1.6)^2 \qquad y(x;\varepsilon) - \sum_{n=1}^{N} \frac{y_n(x)}{(\log\varepsilon)^n} = o\left(\frac{1}{(\log\varepsilon)^N}\right) \quad as \quad \varepsilon \to 0^+$$

holds uniformly in x on any compact set in $[1,\infty)$. Moreover, $y_n(x) = u_{a_n}(x)$ for all $n \geqq 1$, where the constants a_n can be obtained by the matching principle.

[1] With the understanding that boundary layer problems are excluded. In this case, a great deal of work has been done in recent years; one is particularly referred to the book by Wasow [12] and the article by O'Malley [11], where other references can be found.

[2] Without loss of generality, here we may assume $0 < \varepsilon < 1$, and hence $(\log\varepsilon)^n = (-|\log\varepsilon|)^n$, although the sign of $\log\varepsilon$ is immaterial, since ε is a positive parameter.

Some further notation is needed. We define the function $E_0(x;\alpha)$ by

$$E_0(x;\alpha) = \int_x^\infty \frac{e^{-\alpha t}}{t}\,dt, \qquad \alpha = \varepsilon a, \tag{1.7}$$

and let

$$\bar{V}(x;\alpha;f) = \int_x^\infty \frac{e^{-\alpha t}}{t}\,dt \int_t^\infty s\,e^{\alpha s} f(s)\,ds$$

denote the particular solution of the problem,

$$\begin{gathered} \mathscr{L}[v] = f(x), \qquad 1 \leqq x < \infty, \\ v \to 0 \quad \text{as} \quad x \to \infty. \end{gathered} \tag{1.8}$$

Here $\mathscr{L}$ is the operator defined in (1.1′) and $f(\cdot)$ is some given function.

Remark. The function E_0 in (1.7) is a solution of (1.8) for $f(x) = 0$.

THEOREM 3 (Outer expansion). *There exists a sequence of functions* $\{Y_n(\tilde{x})\}$, *independent of* ε *and defined for all* $\tilde{x} > 0$, *such that for any positive integer* N, *the relation*

$$y\left(\frac{\tilde{x}}{\varepsilon};\varepsilon\right) - \left\{-a + \sum_{n=1}^{N} \frac{Y_n(\tilde{x})}{(\log \varepsilon)^n}\right\} = o\left[\frac{1}{(\log \varepsilon)^N}\right] \quad as \quad \varepsilon \to 0^+ \tag{1.9}$$

holds uniformly in $\tilde{x}$ *on any interval* $I_\delta : \tilde{x} \geqq \delta > \varepsilon$. *Moreover,*

$$\begin{gathered} Y_1(\tilde{x}) = d_1 E_0\left(\frac{\tilde{x}}{\varepsilon};\alpha\right), \\ Y_n(\tilde{x}) = d_n E_0\left(\frac{\tilde{x}}{\varepsilon};\alpha\right) + \bar{V}\left(\frac{\tilde{x}}{\varepsilon};\alpha;f_n\right) \quad for \quad n \geqq 2, \end{gathered} \tag{1.10}$$

where $f_n(\tilde{x}/\varepsilon) = \alpha\varepsilon \sum_{k=1}^{n-1} Y_k(\tilde{x})\,dY_{n-k}(\tilde{x})/d\tilde{x}$, *and the constants* d_n *can be obtained by the matching principle.*

The proof of Theorem 1 is given in § 3 and uses estimates for solutions of the linearized problem. These latter are obtained in § 2. Section 4 contains a constructive scheme for obtaining the asymptotic expansion of the solution $y(x;\varepsilon)$ in Theorem 1. Based on this scheme, Theorems 2 (1.6) and 3 (1.9) are established. In § 5, we establish the remaining statements of Theorems 2 and 3.

2. A priori estimates for the linear equation. We consider the problem:

$$\begin{gathered} \mathscr{L}[w] = \alpha\phi\psi, \qquad 1 \leqq x < \infty, \\ w = 0 \quad \text{at} \quad x = 1, \\ w \to 0 \quad \text{as} \quad x \to \infty, \end{gathered} \tag{Q}$$

where $\mathscr{L}$ is defined in (1.1′) and $\alpha = \varepsilon a$. Here it is assumed that

$$\phi \in C^2(1,\infty), \qquad |\phi| \leqq A E_0(x;\alpha), \tag{2.1}$$

and

$$\psi \in C^1(1, \infty), \qquad |\psi| \leqq B\frac{e^{-\alpha x}}{x}, \tag{2.2}$$

where A, B are constants and E_0 is defined in (1.7). Then the theorem is as follows.

THEOREM 4. *Let $w = w(x;\alpha;\alpha\phi\psi)$ be a solution of the problem* (Q). *Then there exists a constant H, independent of $\alpha = \varepsilon a$, such that*

$$\begin{aligned} &|w| \leqq ABHE_0(x;\alpha), \quad |w'| \leqq ABH\frac{e^{-\alpha x}}{x}, \\ &|w''| \leqq AB\frac{e^{-\alpha x}}{x}\{(1+\alpha)H + \alpha E_0(x;\alpha)\}. \end{aligned} \tag{2.3}$$

The following result will be needed to prove Theorem 4 and is easily verified by direct computation.

LEMMA 2.1. *There exists a constant s independent of α such that*

$$\int_x^\infty E_0(t;\alpha)\,dt \leqq \frac{s}{\alpha} \quad \textit{for all} \quad 0 \leqq x. \tag{2.4}$$

The solution $w(x;\alpha;\alpha\phi\psi)$ of (Q) can be written as

$$w(x;\alpha;\alpha\phi\psi) = M(x;\alpha;\alpha\phi\psi) + N(x;\alpha;\alpha\phi\psi), \tag{2.5}$$

where

$$M(x;\alpha;\alpha\phi\psi) = \int_x^\infty \frac{e^{-\alpha t}}{t}\,dt \int_t^\infty \alpha\phi\psi\, e^{\alpha\tau}\tau\, d\tau$$

and

$$N(x;\alpha;\alpha\phi\psi) = -M(1;\alpha;\alpha\phi\psi)\frac{E_0(x;\alpha)}{E_0(1;\alpha)}.$$

The proof of Theorem 4 follows easily from (2.4) and (2.5).

3. Existence theorem. In this section we shall establish the existence of a solution of the problem (P_ε) defined by (1.1)–(1.3) for ε sufficiently small. Our method follows that in [6].

We consider the family of problems,

$$\begin{aligned} &\mathscr{L}[v] = \alpha\tau v v', \qquad 1 \leqq x < \infty, \qquad \alpha = \varepsilon a, \\ &v = 1 \quad \text{at} \quad x = 1, \\ &v \to 0 \quad \text{as} \quad x \to \infty, \end{aligned} \tag{R_τ}$$

for $0 \leqq \tau \leqq 1$. Let us denote the solution of (R_τ) by $v(x;\alpha;\tau)$ (if it exists). Let

$$y(x;\alpha;\tau) = -a + a\tau v(x;\alpha;\tau). \tag{3.1}$$

Then $y(x;\alpha;1)$ is a solution of (P_ε).

We can now state the main results of this section.

THEOREM 5. *For α sufficiently small there exists a solution $v(x;\alpha;\tau)$ of* (R_τ) *for* $0 \leqq \tau \leqq 1$. *The solution can be represented by a convergent expansion*

$$v(x;\alpha;\tau) = \sum_{n=0}^{\infty} v_n(x;\alpha)\tau^n, \tag{3.2}$$

where $v_n(x;\alpha)$ is the solution of the following problem:

$$\mathscr{L}[v_n] = \begin{cases} 0, & n = 0, \\ \alpha \sum_{k=0}^{n-1} v_k v'_{n-k}, & n \geqq 1, \end{cases} \qquad 1 \leqq x < \infty,$$

$$v_n(1;\alpha) = \begin{cases} 1, & n = 0, \\ 0, & n \geqq 1, \end{cases} \tag{3.3}$$

$$v_n(x;\alpha) \to 0 \quad \textit{as} \quad x \to \infty \quad \textit{for all} \quad n \geqq 0.$$

COROLLARY 1. *For ε sufficiently small, there exists a solution $y(x;\varepsilon)$ of* (P_ε), *and the solution can be represented as*

$$y(x;\varepsilon) = -a + a \sum_{n=0}^{\infty} v_n(x;\alpha). \tag{3.4}$$

THEOREM 6. *There exists a constant Γ such that for sufficiently small ε, the solution $y(x;\varepsilon)$ of Corollary* 1 *satisfies*

$$|y(x;\varepsilon) + a| \leqq \Gamma \frac{E_0(x:\alpha)}{E_0(1;\alpha)}, \tag{3.5}$$

where $E_0(x;\alpha)$ is defined in (1.7).

Our first task in proving Theorem 5 is to obtain some estimates for the v_n's in (3.2). We see that,

$$v_0(x;\alpha) = \frac{E_0(x;\alpha)}{E_0(1;\alpha)}. \tag{3.6}$$

Hence if $C_0 = 1/E_0(1;\alpha)$, we have

$$|v_0(x;\alpha)| = C_0 E_0(x;\alpha), \qquad |v'_0(x;\alpha)| \leqq C_0 \frac{e^{-\alpha x}}{x},$$
$$|v''_0(x;\alpha)| \leqq C_0(1+\alpha)\frac{e^{-\alpha x}}{x}, \tag{3.7}$$

uniformly for $x \geqq 1$.

An immediate consequence of Theorem 4 and equations (3.6) and (3.7) is the following.

LEMMA 3.1. *Let the sequence $\{C_n\}$ of constants be defined by*

$$C_{n+1} = H \sum_{k=0}^{n} C_k C_{n-k}, \qquad C_0 = \frac{1}{E_0(1;\alpha)}. \tag{3.8}$$

Then

$$|v_n(x;\alpha)| \leqq C_n E_0(x;\alpha), \qquad |v_n'(x;\alpha)| \leqq C_n \frac{e^{-\alpha x}}{x},$$

and

$$|v_n''(x;\alpha)| \leqq C_n \frac{e^{-\alpha x}}{x}\left\{(1+\alpha) + \frac{\alpha}{H}E_0(x;\alpha)\right\},$$

uniformly for $x \geqq 1$.

It follows from Lemma 3.1 that the series

$$C(\tau;\alpha) = \left(\sum_{k=0}^{\infty} C_k \tau^k\right) E_0(1;\alpha) \tag{3.9}$$

will dominate (3.2). Our next result concerns the convergence of the series in (3.9).

LEMMA 3.2. *The series in* (3.9) *can be written as*

$$C(\tau;\alpha) = \sum_{n=0}^{\infty} \gamma_{n+1} \frac{(2HC_0)^n}{(n+1)!}\tau^n, \tag{3.10}$$

where

$$\gamma_{n+1} = \begin{cases} 1, & n \leqq 1, \\ \prod_{k=2}^{n+1} (2k-3), & n \geqq 1. \end{cases}$$

Consequently (3.9) *will converge for*

$$\tau < (4HC_0)^{-1}. \tag{3.11}$$

Proof. It is easy to see that the convergence of the series (3.9) implies that $C = C(\tau;\alpha)$ satisfies the equation

$$E_0 C = C_0 E_0^2 + H\tau C^2. \tag{3.12}$$

The solution of (3.12) has a branch which is analytic in τ in a circle about the origin. Then (3.10) follows from Taylor's theorem, and (3.11) follows from (3.10) by the ratio test.

Remarks. 1. From (3.10) we obtain

$$|C_n| \leqq \frac{\gamma_{n+1}(2HC_0)^n}{(n+1)!} C_0, \tag{3.13}$$

where γ_{n+1} is defined in (3.10).

2. Since $C_0 = 1/E_0(1;\alpha)$ and $E_0(1;\alpha) \to \infty$ as $\varepsilon \to 0^+$, the inequality (3.11) holds for all $\tau \in [0, 1]$ for ε sufficiently small. Consequently the series (3.2) converges uniformly and absolutely.

We now return to the proofs of Theorems 5 and 6. Observe that the series (3.2) is formally a solution of (R_τ) and thus it remains only to check the convergence.

Completion of the proof of Theorem 5. If we collect all the results (3.9)–(3.11), we find that for ε sufficiently small and for all $\tau \in [0, 1]$ the series (3.2) converges uniformly and absolutely. Similarly it can be shown the series may be twice termwise differentiated with respect to x. Hence the sum $v(x;\alpha;\tau)$ defined in (3.2) is a solution of (R_τ) for $\tau \in [0, 1]$. This completes the proof of Theorem 5.

Proof of Theorem 6. From Corollary 1 of Theorem 5 and Lemma 3.1, we obtain

$$|y(x;\varepsilon) + a| = a\left|\sum_{n=0}^{\infty} v_n(x;\alpha)\right| \leqq a\left(\sum_{n=0}^{\infty} C_n\right)E_0(x;\alpha).$$

Then it follows from (3.13) that

$$|y(x;\varepsilon) + a| \leqq a \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(2HC_0)^n}{(n+1)!}\,\frac{E_0(x;\alpha)}{E_0(1;\alpha)}.$$

It follows from Lemma 3.2 that the series $a\sum_{n=0}^{\infty}(\gamma_{n+1}(2HC_0)^n/(n+1)!)$ converges and we set Γ equal to this series. This completes the proof of Theorem 6.

The techniques of [6] can also be used to establish the following uniqueness result. We omit the proof.

THEOREM 7. *Let S_v denote the class of functions $y(x;\varepsilon)$ such that*

$$|y(x;\varepsilon) + a| < avE_0(x;\alpha), \quad \alpha = \varepsilon a,$$

uniformly in $1 \leqq x < \infty$. Let v be any given number $0 < v < (2H)^{-1}$, H as in Theorem 4. Then there exists at most one solution y of (P_ε) such that $y \in S_v$.

Remark. The solution $y(x;\varepsilon)$ of Theorem 6 satisfies

$$|y(x;\varepsilon) + a| \leqq \Gamma\frac{E_0(x;\alpha)}{E_0(1;\alpha)}.$$

Since $\Gamma/E_0(1;\alpha)$ can be made arbitrarily small by choosing ε small, we see that the solution of Theorem 6 is in S_v for ε small.

4. Asymptotic structure. In this section we shall investigate the asymptotic behavior of solutions of (P_ε) for small ε. Our aim is to develop asymptotic expansions of the exact solution $y(x;\varepsilon)$ in (3.4), which we shall define to be the *inner* and *outer* expansions of the solution.

We introduce the *outer variable* $\tilde{x} = \varepsilon x$, and set

$$(4.1)\qquad Y(\tilde{x};\varepsilon) = y\left(\frac{\tilde{x}}{\varepsilon};\varepsilon\right), \qquad V(\tilde{x};\varepsilon) = v\left(\frac{\tilde{x}}{\varepsilon};\alpha;1\right) \quad\text{and}\quad V_n(\tilde{x};\varepsilon) = v_n\left(\frac{\tilde{x}}{\varepsilon};\alpha\right),$$

where v and v_n are the functions appearing in (3.2) and $\alpha = \varepsilon a$. We use dots to indicate differentiations with respect to $\tilde{x}$ and set

$$(4.2)\qquad \tilde{L} = \frac{1}{\varepsilon^2}L \quad\text{and}\quad \tilde{\mathscr{L}} = \frac{1}{\varepsilon^2}\mathscr{L}.$$

Then the development can be stated in the form of the following theorems.

THEOREM 8. *There exist two sequences* $\{Y_k(\tilde{x})\}$, *independent of* ε, *defined for all* $\tilde{x} > 0$, *and* $\{\bar{V}_m(\tilde{x};\varepsilon)\}$ *defined for* $\tilde{x} \geqq \varepsilon$, *and having the following properties:*

$$(4.3)\qquad \begin{aligned} &\text{(a)}\ Y_k(\tilde{x}) = A_k \log \tilde{x} + B_k + O(\tilde{x}\log\tilde{x}) \text{ as } \tilde{x} \to 0^+,\\ &\text{(b)}\ Y_k(\tilde{x}) \to 0 \text{ as } \tilde{x} \to \infty,\\ &\text{(c)}\ \bar{V}_m(\tilde{x};\varepsilon) = O(1) \text{ as } \varepsilon \to 0^+ \text{ uniformly on } \tilde{x} \geqq \delta > 0,\\ &\text{(d)}\ \bar{V}_m(\tilde{x};\varepsilon) = O(\log\varepsilon) \text{ as } \varepsilon \to 0^+ \text{ uniformly on } \tilde{x} \geqq \varepsilon, \end{aligned}$$

and such that if $y(x;\varepsilon)$ *is the solution of* (P_ε) *defined by* (3.4) *and* m *is any positive integer, then*

$$(4.4)\qquad y(x;\varepsilon) = -a + \sum_{k=1}^{m} \frac{Y_k(\varepsilon x)}{(\log\varepsilon)^k} + \frac{\bar{V}_m(\varepsilon x;\varepsilon)}{(\log\varepsilon)^{m+1}}.$$

Remark. This theorem needs some explanation. It yields a kind of asymptotic expansion for the solution but reflects the nonuniformity of this expansion. Observe that by (4.3) (c) of the theorem we have

$$(4.5)\qquad \left| y(x;\varepsilon) - \left\{ -a + \sum_{k=1}^{m} \frac{Y_k(\varepsilon x)}{(\log\varepsilon)^k} \right\} \right| \leqq \frac{\Gamma_m}{(\log\varepsilon)^{m+1}} \quad \text{as} \quad \varepsilon \to 0^+,$$

if $x \geqq \delta/\varepsilon$ for any $\delta > 0$, where $\Gamma_m < \infty$ is a constant. Thus the expansion can be used in a straightforward way for large x. However, suppose one wishes to calculate y for small values of x. Then (4.3) (d) yields a result of the following type:

$$(4.6)\qquad \left| y(x;\varepsilon) - \left\{ -a + \sum_{k=1}^{m} \frac{Y_k(\varepsilon x)}{(\log\varepsilon)^k} \right\} \right| \leqq \frac{\bar{\Gamma}_m}{(\log\varepsilon)^{m}} \quad \text{as} \quad x \to 0^+.$$

Thus in order to have accuracy up to a given power of $1/\log\varepsilon$, one must keep one extra term.

As a consequence of Theorem 8 and the fact that $y(1;\varepsilon) = 0$ we have the following corollary.

COROLLARY 2. *There exists a constant* $M < \infty$ *such that*

$$(4.7)\qquad \left| a - \sum_{k=1}^{m} Y_k(\varepsilon)(\log\varepsilon)^{-k} \right| |(\log\varepsilon)^m| \leqq M \quad \text{as} \quad \varepsilon \to 0^+.$$

The next theorem concerns the differentiability of the expansion in (4.4).

THEOREM 9. *The expansion in* (4.4) *can be differentiated termwise. Moreover, we have*

$$(4.8)\qquad \begin{aligned} &\text{(a)}\ |e^{a\tilde{x}}\tilde{x}\dot{Y}_k(\tilde{x})| \leqq \bar{A}_k < \infty,\ \bar{A}_k \text{ a constant independent of } \tilde{x};\\ &\text{(b)}\ \dot{Y}_k(\tilde{x}) = A_k/\tilde{x} + O(\log\tilde{x}) \text{ as } \tilde{x} \to 0, \text{ where the } A_k\text{'s are the constants in (4.3)(a)};\\ &\text{(c)}\ e^{a\tilde{x}}\tilde{x}\dot{\bar{V}}_m(\tilde{x};\varepsilon) = O(1) \text{ as } \varepsilon \to 0, \text{ uniformly on } \tilde{x} \geqq \varepsilon. \end{aligned}$$

Remark. Observe that these estimates would enable us to calculate an approximation to $y'(x;\varepsilon)$ for x near 1, in fact, even at $x = 1$. We have

$$\left| y'(1;\varepsilon) - \sum_{k=1}^{m} \frac{A_k}{(\log \varepsilon)^k} \right| \leqq \frac{D}{(\log \varepsilon)^{m+1}}, \tag{4.9}$$

as $\varepsilon \to 0$, where $D < \infty$ is a constant.

The proofs of the above results require that we obtain more detailed information about the function $V_n(\tilde{x};\varepsilon)$ which we derive from the $v_n(x;\alpha)$ of § 3 by substituting $\tilde{x}/\varepsilon$ for x (see also (4.1)). *The essential fact is that the V_k's are of increasing order in* $1/\log \varepsilon$. More precisely we have the following results.

LEMMA 4.1. *Let $V(\tilde{x};\varepsilon)$ and $V_n(\tilde{x};\varepsilon)$ be functions defined in* (4.1). *Then, for any integer $m \geqq 0$, the relation*

$$V(\tilde{x};\varepsilon) - \sum_{n=0}^{m} V_n(\tilde{x};\varepsilon) = O\left(\frac{1}{(\log \varepsilon)^{m+1}}\right) \quad as \quad \varepsilon \to 0^+ \tag{4.10}$$

holds uniformly for $\tilde{x} \geqq \varepsilon$.

Proof. By Lemma 3.1 and (3.13), we obtain

$$|v_n(x;\alpha)| \leqq \tilde{A}_n(E_0(1;\alpha))^{-(n+1)}E_0(x;\alpha), \qquad x \geqq 1, \tag{4.11}$$

where

$$\tilde{A}_n = \begin{cases} \prod_{k=2}^{n+1} (2k-3)(2H)^k/(k+1)!, & n \geqq 1, \\ (2H)^n/(n+1)!, \quad 0 \leqq n \leqq 1. \end{cases}$$

H is the constant in Theorem 4, and $E_0(x;\alpha)$ is defined in (1.7). Since $E_0(x;\alpha) = E_0(\varepsilon x; a)$, we obtain, by (4.1) and (4.11),

$$|V_n(\tilde{x};\varepsilon)| = \left| v_n\left(\frac{\tilde{x}}{\varepsilon}; \varepsilon a\right)\right| \leqq \tilde{A}_n(E_0(\varepsilon;a))^{-(n+1)}E_0(\tilde{x};a), \qquad \tilde{x} \geqq \varepsilon. \tag{4.12}$$

Hence we have

$$\begin{aligned} \left| V(\tilde{x};\varepsilon) - \sum_{n=0}^{m} V_n(\tilde{x};\varepsilon)\right| &= \left| \sum_{n=m+1}^{\infty} V_n(\tilde{x};\varepsilon)\right| \\ &\leqq \sum_{n=m+1}^{\infty} \tilde{A}_n(E_0(\varepsilon;a))^{-(n+1)}E_0(\tilde{x};a) \\ &\leqq \sum_{n=0}^{\infty} \tilde{A}_{n+m+1}(E_0(\varepsilon;a))^{-n}(E_0(\varepsilon;a))^{-(m+2)}E_0(\tilde{x};a) \\ &\leqq (E_0(\varepsilon;a))^{-(m+1)} \sum_{n=0}^{\infty} \tilde{A}_{n+m+1}(E_0(\varepsilon;a))^{-n}. \end{aligned} \tag{4.13}$$

Since $E_0(\varepsilon;a)$ is dominated by $|\log \varepsilon|$ for small ε, the series on the right-hand side of (4.13) will converge for ε sufficiently small and thus Lemma 4.1 follows.

Remark. From (4.13) it follows that for ε small there exists a constant $\tilde{M}$ such that

$$\left|V(\tilde{x};\varepsilon) - \sum_{n=0}^{m} V_n(\tilde{x};\varepsilon)\right| \leqq \frac{\tilde{M}E_0(\tilde{x};a)}{|(\log\varepsilon)^{m+2}|}. \tag{4.14}$$

Also this inequality may be differentiated with respect to $\tilde{x}$. This follows from the uniform convergence of the derivatives of the infinite series (3.4).

LEMMA 4.2. *For each fixed $j \geqq 0$, there exist two sequences $\{V_{jk}(\tilde{x})\}$, independent of ε and defined for all $\tilde{x} > 0$, and $\{\bar{V}_{jm}(\tilde{x};\varepsilon)\}$, defined for $\tilde{x} \geqq \varepsilon$, having the following properties:*

(4.15)

(a) $V_{jk}(\tilde{x}) = a_{jk}\log\tilde{x} + b_{jk} + O(\tilde{x}\log\tilde{x})$ *as* $\tilde{x} \to 0^+$, *where* a_{jk} *and* b_{jk} *are constants,*

(b) $V_{jk}(\tilde{x}) \to 0$ *as* $\tilde{x} \to \infty$,

(c) $\int_{\tilde{x}}^{\infty} |V_{jk}(t)|\,dt \leqq A_{jk}$, $A_{jk} < \infty$, *a constant,*

(d) $\bar{V}_{jm}(\tilde{x};\varepsilon) = O(1)$ *as* $\varepsilon \to 0^+$ *uniformly on* $\tilde{x} \geqq \delta > 0$, *for any* $\delta > 0$,

(e) $\bar{V}_{jm}(\tilde{x};\varepsilon) = O(\log\varepsilon)$ *as* $\varepsilon \to 0^+$ *uniformly on* $\tilde{x} \geqq \varepsilon$,

(f) $\int_{\tilde{x}}^{\infty} |\bar{V}_{jm}(t;\varepsilon)|\,dt \leqq \bar{A}_{jm}$ *as* $\varepsilon \to 0^+$ *uniformly on* $\tilde{x} \geqq \varepsilon$,

and such that if $V_j(\tilde{x};\varepsilon)$ is the function defined in (4.1) *and m is any integer, then*

$$V_j(\tilde{x};\varepsilon) = \sum_{k=j+1}^{m+1} \frac{V_{jk}(\tilde{x})}{(\log\varepsilon)^k} + \frac{\bar{V}_{jm}(\tilde{x};\varepsilon)}{(\log\varepsilon)^{m+2}}. \tag{4.16}$$

Moreover, the expansion (4.16) *can be differentiated termwise and we have:*

(4.17)

(a) $|\dot{V}_{jk}(\tilde{x})\, e^{a\tilde{x}}\tilde{x}| \leqq c_{jk} < \infty$, *a constant independent of* $\tilde{x}$,

(b) $\dot{V}_{jk}(\tilde{x}) = a_{jk}/\tilde{x} + O(\log\tilde{x})$ *as* $\tilde{x} \to 0$, *where the* a_{jk}*'s are the same constants as in* (4.15) (a), *and*

(c) $|\dot{\bar{V}}_{jm}(\tilde{x};\varepsilon)\, e^{a\tilde{x}}\tilde{x}| \leqq \bar{c}_{jm} < \infty$ *as* $\varepsilon \to 0$, *where* $\bar{c}_{jm}$ *is a constant independent of* ε *and* $\tilde{x}$.

Remark. This lemma is the key to Theorems 8 and 9 and again it requires some explanation. Observe that the lemma *does not* say that the difference between the V_j's and the finite sums on the right of (4.16) are *uniformly* $O((\log\varepsilon)^{-(m+2)})$. The reason is that the error terms $\bar{V}_{jm}$, like the $V_{jk}(\tilde{x})$'s, contain expressions which become arbitrarily large as $\tilde{x}$ becomes small. Thus, (4.15) (e) is the best that can be said about the $\bar{V}_{jm}$'s. On the other hand, the singularities which appear for small $\tilde{x}$ are integrable, both for the V_{jk}'s and the $\bar{V}_{jm}$'s, and hence one obtains the uniform estimates (4.15) (c) and (f) for the integrals.

The proof of the lemma, an induction argument, is tedious. Instead of presenting all details we indicate the idea in the Appendix by examining the first two terms in the series (3.4).

Theorems 8 and 9 follow readily from Lemmas 4.1 and 4.2, and a rearrangement of terms. The Y_k's are defined by

$$Y_k(\tilde{x}) \equiv a \sum_{j=0}^{k-1} V_{jk}(\tilde{x}), \qquad k \geqq 1, \tag{4.18}$$

and from (3.6) it is not difficult to verify that

$$Y_1(\tilde{x}) = aV_{01}(\tilde{x}) = -aE_0(\tilde{x}; a). \tag{4.19}$$

We can now give the definitions of the *inner* and *outer* expansions of $y(x;\varepsilon)$.

DEFINITION 1. Let $\{y_n(x)\}$ be a sequence of functions such that for any N, the relations

$$y(x;\varepsilon) - \sum_{n=0}^{N} \frac{y_n(x)}{(\log \varepsilon)^n} = o\left(\frac{1}{(\log \varepsilon)^N}\right) \quad \text{as} \quad \varepsilon \to 0^+ \tag{4.20}$$

hold uniformly in x on any compact subset of $[1, \infty)$. Then the expansion in (4.20) is called the *inner expansion* of $y(x;\varepsilon)$.

DEFINITION 2. Let $\{Y_n(\tilde{x})\}$ be a sequence of functions such that for any N, the relation

$$y\left(\frac{\tilde{x}}{\varepsilon};\varepsilon\right) - \sum_{n=0}^{N} \frac{Y_n(\tilde{x})}{(\log \varepsilon)^n} = o\left(\frac{1}{(\log \varepsilon)^N}\right) \quad \text{as} \quad \varepsilon \to 0^+ \tag{4.21}$$

holds uniformly in $\tilde{x}$ on any set $\tilde{x} \geqq \delta > 0$. Then the expansion in (4.21) is called the *outer expansion* of $y(x;\varepsilon)$.

If we compare (4.21) with (4.4) and use (4.3) (c) we see immediately that the Y_k's of (4.4) yield the outer expansion for the solution y. We can also produce the inner expansion. Observe that (4.18) and (4.15) (a) yield

$$Y_k(\varepsilon x) \doteq a \sum_{j=0}^{k-1} \{a_{jk}(\log x + \log \varepsilon) + b_{jk}\} + O(\varepsilon \log \varepsilon) \tag{4.22}$$

uniformly on any compact subset of $[1, \infty)$. Thus we have

$$Y_k(\varepsilon x) = \phi_k(x) \log \varepsilon + \psi_k(x) + O(\varepsilon \log \varepsilon), \tag{4.23}$$

where

$$\phi_k(x) = a \sum_{j=0}^{k-1} a_{jk}, \qquad \psi_k(x) = a \sum_{j=0}^{k-1} (a_{jk} \log x + b_{jk}).$$

We substitute (4.23) into (4.4) with $m = N + 1$ and obtain, from (4.3) (d),

$$\begin{aligned} y(x,\varepsilon) &= -a + \sum_{k=1}^{N+1} \frac{\phi_k(x)\log\varepsilon + \psi_k(x)}{(\log \varepsilon)^k} + O\left(\frac{1}{(\log \varepsilon)^{N+1}}\right) \\ &= \sum_{n=0}^{N} \frac{y_n(x)}{(\log \varepsilon)^n} + o\left(\frac{1}{(\log \varepsilon)^N}\right), \end{aligned} \tag{4.24}$$

where

$$y_0(x) = -a + \phi_1(x), \qquad y_n(x) = \psi_n(x) + \phi_{n+1}(x), \qquad n = 1, 2, 3, \cdots, N.$$

Comparison with (4.20) shows that the y_n's yield the inner expansion.

Remark. One may question the desirability for replacing the convergent series (3.4) with the two asymptotic series (4.20) and (4.21) since the terms in the latter surely decrease very slowly unless ε is very small. The reasons for this replacement are related to the fact that our procedure here is supposed to serve as a model for an analogous one in fluid flow. In the hydrodynamic situation of [6] the calculation of the v_n's in (3.4) requires the solution of inhomogenous Oseen problems in the exterior of an obstacle and this is very difficult. On the other hand the quantities corresponding to the y_n's and Y_n's can be calculated from Stokes flows and special solutions of Oseen's equations without boundary conditions as indicated in [9]. These calculations are much simpler but they are based on an analogue of the matching principle of the next section. The point of our model is that it lends credence to these calculations by showing that in the present simplified context they yield the correct asymptotic terms for the exact solution.

5. Matching principle. According to the method of inner and outer expansions [9], [10], a formal matching procedure can be established to obtain two asymptotic expansions of (P_ε) similar to those defined in (4.20) and (4.21). The basis for this procedure is the so-called *matching principle* which will be stated later. We refer to these expansions as the formal inner and outer expansions respectively. However, we shall show later these are indeed the inner and outer expansions of the actual solution $y(x;\varepsilon)$. In this section we shall describe this formal procedure (matching principle) by computing the first few terms of the formal inner and outer expansions. The results here will be needed in the proof of Theorems 2 and 3.

We begin with the formal inner expansion. This has the form

$$\sum_{k=1}^{\infty} \frac{u_{a_k}(x)}{(\log \varepsilon)^k}, \tag{5.1}$$

where the functions u_{a_k}, $k \geqq 1$, are solutions of the problems

$$\begin{aligned} &L[u_{a_k}] = 0 \quad \text{for} \quad x > 1, \\ &u_{a_k} = 0 \quad \text{at} \quad x = 1, \quad u_{a_k}(x) - a_k \log x = O(1) \quad \text{as} \quad x \to \infty, \end{aligned} \tag{5.2}$$

where the a_k, $k \geqq 1$, are constants to be determined by the matching principle.

Recall that the problem defined by (5.2) has a unique solution, namely,

$$u_{a_k}(x) = a_k \log x \tag{5.3}$$

for every fixed constant a_k.

Now we formally substitute (5.3) into the expansion (5.1) and rewrite the functions $a_k \log x$ in terms of outer variable $\tilde{x} = \varepsilon x$. This yields the series

$$-a_1 + \sum_{k=1}^{\infty} \frac{a_k \log \tilde{x} - a_{k+1}}{(\log \varepsilon)^k}. \tag{5.4}$$

Next we construct the outer expansion. Observe first that in the notation of (4.1) equation (1.1) becomes

$$\tilde{L}[Y] = \ddot{Y} + \frac{1}{\tilde{x}}\dot{Y} = Y\dot{Y}, \qquad \varepsilon < \tilde{x} < \infty, \tag{5.5}$$

(5.5)[3]

and conditions (1.2) and (1.3) become, respectively,

$$Y = 0 \quad \text{at} \quad \tilde{x} = \varepsilon \tag{5.6}$$

and

$$Y \to -a \quad \text{as} \quad \tilde{x} \to \infty. \tag{5.7}$$

The outer expansion is of the form

$$-a + \sum_{k=1}^{\infty} \frac{U_k(\tilde{x})}{(\log \varepsilon)^k}. \tag{5.8}$$

This expansion is required to satisfy the equation (5.5) and the condition (5.7), *but not* (5.6). Formally substituting (5.8) into (5.5), (5.7) and equating coefficients of like powers of $(\log \varepsilon)^{-1}$, one obtains the conditions for the functions $U_k(\tilde{x})$; that is,

$$\begin{aligned} &\tilde{\mathscr{L}}[U_k] = \ddot{U}_k + \left(\frac{1}{\tilde{x}} + a\right)\dot{U}_k = \begin{cases} 0, & k = 1, \\ a \sum_{\nu=1}^{k-1} U_\nu \dot{U}_{k-\nu}, & k \geqq 2, \end{cases} \\ &U_k \to 0^+ \quad \text{as} \quad \tilde{x} \to \infty \quad \text{for all} \quad k \geqq 1. \end{aligned} \tag{5.9}$$

The general solutions of (5.9) have the following forms:

$$\begin{aligned} &U_1(\tilde{x}) = d_1 E_0(\tilde{x}; a), \\ &U_k(\tilde{x}) = d_k E_0(\tilde{x}; a) + a \sum_{\nu=1}^{k-1} \int_{\tilde{x}}^{\infty} \frac{e^{-at}}{t}\, dt \int_t^{\infty} e^{a\tau} \tau U_\nu \dot{U}_{k-\nu}\, d\tau, \qquad k \geqq 2, \end{aligned} \tag{5.10}$$

where the d_k, $k \geqq 1$, are constants to be determined by the matching principle.

In order to formulate the matching principle we need the following lemma which yields information about the U_k's as $\tilde{x}$ tends to zero.

LEMMA 5.1. *Let* $\phi(\tilde{x})$ *and* $\psi(\tilde{x})$ *be functions defined for* $\tilde{x} > 0$ *and such that*

$$|\phi(\tilde{x})| \leqq A E_0(\tilde{x}; a), \qquad |\psi(\tilde{x})| \leqq B \tilde{x}^{-1} e^{-a\tilde{x}} \tag{5.11}$$

for some constants A and B. Define $\chi(\tilde{x})$ *by*

$$\chi(\tilde{x}) = \int_{\tilde{x}}^{\infty} \frac{e^{-at}}{t}\, dt \int_t^{\infty} e^{a\tau} \tau \phi(\tau) \psi(\tau)\, d\tau. \tag{5.12}$$

[3] This is a variant of the Lagerstrom model for the incompressible low Reynolds number flow [2], [3], [10]; the original form can be obtained with Y replaced by $-Y$. A calculation of the first three terms in the formal inner and outer expansions for this Lagerstrom model is given in [2], [3]. The calculation in [3] is carried out by introducing an intermediate limiting procedure; while in [2], the method of limit process expansions is used with different inner and outer variables than those used here.

Then we have, for some constant H and for all $\tilde{x} > 0$,

$$|\chi(\tilde{x})| \leqq HABE_0(\tilde{x}; a), \tag{5.13}$$

$$|\dot{\chi}(\tilde{x})| \leqq HAB\tilde{x}^{-1} e^{-a\tilde{x}}. \tag{5.14}$$

Moreover there exist constants M and N such that

$$\chi(\tilde{x}) = M \log \tilde{x} + N + o(1) \quad as \quad \tilde{x} \to 0^+. \tag{5.15}$$

The proof is a straightforward computation.

Now (1.7) yields

$$E_0(\tilde{x}; a) = -\log \tilde{x} + \tilde{\gamma} + O(\tilde{x} \log \tilde{x}) \quad \text{as} \quad \tilde{x} \to 0^+, \tag{5.16}$$

where $\tilde{\gamma} \neq 0$ is a constant. Then, (5.15) and (5.16) show that the functions U_k of (5.10) satisfy estimates of the form

$$U_k(\tilde{x}) = \hat{a}_k \log \tilde{x} + \hat{b}_k + O(\tilde{x} \log \tilde{x}) \quad \text{as} \quad \tilde{x} \to 0^+, \tag{5.17}$$

where

$$\hat{a}_k = -d_k + M_k(d_1, \cdots, d_{k-1}), \qquad \hat{b}_k = d_k\tilde{\gamma} + N_k(d_1, \cdots, d_{k-1});$$

M_k and N_k are certain functions with $M_1 = N_1 = 0$.

MATCHING PRINCIPLE.[4] Determine the constants a_k of (5.1) and d_k of (5.10) so that the coefficients of $\log \tilde{x}$ and the constant terms for corresponding powers of $(\log \varepsilon)^{-1}$ are equal. That is, $a_1 = a$ and

$$\begin{aligned} a_k &= -d_k + M_k(d_1, \cdots, d_{k-1}) = \hat{a}_k, \\ -a_{k+1} &= d_k\tilde{\gamma} + N_k(d_1, \cdots, d_{k-1}) = \hat{b}_k, \qquad k \geqq 2. \end{aligned} \tag{5.18}$$

Equations (5.18) and $a_1 = a$ clearly determine the a_k and d_k recursively in the order $a_1, d_1, a_2, d_2, a_3, d_3, \cdots$. It follows that the functions u_{a_k} of (5.1) and U_k of (5.8) are well-determined. The final assertion of Theorem 2 is that the u_{a_k}'s are identical with the functions y_k in the inner expansion (4.20) of the exact solution. Similarly the final assertion of Theorem 3 is that the U_k's are identical with the Y_k's of the outer expansion (4.21). We saw in the sequence of formulas (4.22)–(4.24) that the inner expansion can be obtained from the outer expansion so that if we prove $U_k = Y_k$ it will follow easily that $y_k = u_{a_k}$. Thus we need only prove $U_k = Y_k$ and this we do now.

The proof is by induction. For $k = 1$ we have, by (5.18), $d_1 = -a$ and hence,

$$U_1(\tilde{x}) = -aE_0(\tilde{x}; a), \tag{5.19}$$

and by (4.19) this is the same as $Y_1(\tilde{x})$. Next we need the following lemma.

LEMMA 5.2. *The functions* U_m *constructed by the matching principle satisfy the recursive relations:*

$$U_m(\tilde{x}) = \left\{a - \sum_{k=1}^{m-1} \frac{U_k(\tilde{x})}{(\log \varepsilon)^k}\right\}(\log \tilde{x})^m + O(1) \quad as \quad \tilde{x} \to 0^+, \tag{5.20}$$

for any $m \geqq 1$.

[4] See comment at the end of this section.

Proof. By (5.17) and the matching principle (5.18), we have

$$
\begin{aligned}
U_k(\tilde{x}) &= \hat{a}_k \log \tilde{x} + \hat{b}_k + O(\tilde{x} \log \tilde{x}) \\
&= a_k \log \tilde{x} - a_{k+1} + O(\tilde{x} \log \tilde{x}) \quad \text{as} \quad \tilde{x} \to 0^+.
\end{aligned} \tag{5.21}
$$

Equation (5.21) yields

$$a_{k+1} = a_k \log \tilde{x} - U_k(\tilde{x}) + O(\tilde{x} \log \tilde{x}) \quad \text{as} \quad \tilde{x} \to 0^+$$

and

$$U_m(\tilde{x}) = a_{m-1}(\log \tilde{x})^2 - U_{m-1}(\tilde{x})(\log \tilde{x}) - a_{m+1} + O(\tilde{x} \log \tilde{x}^2). \tag{5.22}$$

Repeated applications of these formulas yield

$$
\begin{aligned}
U_m(\tilde{x}) = a_{m-k}(\log \tilde{x})^{k+1} &- \sum_{l=1}^{k} U_{m-l}(\tilde{x})(\log \tilde{x})^l \\
&- a_{m+1} + O(\tilde{x}(\log \tilde{x})^{k+1}) \quad \text{as} \quad \tilde{x} \to 0^+.
\end{aligned} \tag{5.23}
$$

Then the result (5.20) follows from (5.23) with $k = m - 1$, since by the matching principle we have $a_1 = a$. This completes the proof of Lemma 5.2.

Assume now that

$$U_k(\tilde{x}) \equiv Y_k(\tilde{x}), \qquad 1 \leqq k \leqq m - 1. \tag{5.24}$$

If we let $W(\tilde{x}) = U_m(\tilde{x}) - Y_m(\tilde{x})$, then $W(\tilde{x})$ is a solution of the problem,

$$\tilde{\mathscr{L}}[W] = 0, \qquad \varepsilon < \tilde{x} < \infty; \qquad W \to 0 \quad \text{as} \quad \tilde{x} \to \infty.$$

Then, it follows that

$$W(\tilde{x}) = dE_0(\tilde{x}; a), \tag{5.25}$$

where d is some constant.

Assume $d \neq 0$. Then we obtain, by (5.25), $W(\varepsilon) = U_m(\varepsilon) - Y_m(\varepsilon) = dE_0(\varepsilon; a) = O(\log \varepsilon)$ as $\varepsilon \to 0^+$. But Corollary 2 of Theorem 8 and Lemma 5.2 with $\tilde{x} = \varepsilon$ imply that

$$U_m(\varepsilon) - Y_m(\varepsilon) = O(1) \quad \text{as} \quad \varepsilon \to 0^+,$$

which is a contradiction. This concludes the proof of Theorem 3 and thus of Theorem 2.

Remark. The considerations above show that for the problem studied in this paper there occurs important simplification. This is as follows. Not only can the y_n's in the expansion (1.6) of the exact solution be obtained from the matching procedure, but they are in fact *precisely equal* to the leading terms in U_k of the outer expansion for small $\tilde{x}$. This simplification will not be present in general [8].

Comment. The matching principle (cf. (5.18)) presented in this section may be considered, at least for the model (P_ε), as a simplified version of what is called the *asymptotic matching principle* in [7]. With the notation there, as will be seen, (5.18) can be rewritten in the form:

$$H_q E_p y = E_p H_q y \tag{5.26}$$

for any integer $p \geqq 1$ and $q = p - 1$. Here E_p and H_q denote, respectively, the outer and inner expansion operators.[5] In view of the definitions of E_p and H_q, we obtain from (5.8) and (5.17),

$$
\begin{aligned}
H_q E_p y &= H_{p-1}\left\{-a + \sum_{k=1}^{p} [\hat{a}_k(\log \varepsilon + \log x) + \hat{b}_k + O(\varepsilon x \log \varepsilon x)]/(\log \varepsilon)^k\right\} \\
&= (-a + \hat{a}_1) + \sum_{k=1}^{p-1} (\hat{a}_{k+1} + \hat{a}_k \log x + \hat{b}_k)/(\log \varepsilon)^k.
\end{aligned}
\tag{5.27}
$$

Similarly, we obtain from (5.1) and (5.3),[6]

$$
\begin{aligned}
E_p H_q y &= E_p\left\{\sum_{k=1}^{p-1} [a_k(-\log \varepsilon + \log \tilde{x})]/(\log \varepsilon)^k\right\} \\
&= \sum_{k=1}^{p-1} (a_k \log x)/(\log \varepsilon)^k.
\end{aligned}
\tag{5.28}
$$

Thus, a comparison of (5.27) and (5.28) shows that the result (5.26) follows indeed from the matching principle (5.18).

Appendix. Proof of Lemma 4.2. We consider first the leading term in the series (3.4), that is,

$$v_0(x; \varepsilon a) = \frac{E_0(x; \varepsilon a)}{E_0(1; \varepsilon a)},$$

by (3.6). By definition (1.7),

$$E_0(x; \varepsilon a) = \int_x^\infty \frac{e^{-\varepsilon a t}}{t}\, dt = \int_{\varepsilon x a}^\infty \frac{e^{-t}}{t}\, dt.$$

For simplicity, we set

$$E_0(a\varepsilon x) = \int_{a\varepsilon x}^\infty \frac{e^{-t}}{t}\, dt = E_0(x; \varepsilon a). \tag{A.1}$$

Then

$$V_0(\tilde{x}; \varepsilon) \equiv v_0\left(\frac{\tilde{x}}{\varepsilon}; \varepsilon a\right) = \frac{E_0(a\tilde{x})}{E_0(a\varepsilon)}. \tag{A.2}$$

---

[5] For convenience, the definitions of the operators E_p and H_q in [7] are cited here. Given asymptotic sequences $\{\alpha_m(\varepsilon)\}$, $m = 0, \cdots, p$, and $\{\beta_m(\varepsilon)\}$, $m = 0, \cdots, q$, the outer expansion operator, E_p, on a given function $f(x; \varepsilon)$ is defined by the relations

$$E_0 f = \alpha_0 \lim_{\substack{\tilde{x}\,\text{fixed} \\ \varepsilon \to 0}} \frac{f}{\alpha_0}, \quad E_m f = E_{m-1} f + \alpha_m \lim_{\substack{\tilde{x}\,\text{fixed} \\ \varepsilon \to 0^+}} \frac{f - E_{m-1} f}{\alpha_m},$$

$$m = 1, \cdots, p,$$

and the inner expansion operator H_q is defined similarly with the outer variable $\tilde{x}$ replaced by the inner variable x and $\alpha_m(\varepsilon)$ by $\beta_m(\varepsilon)$.

[6] Strictly speaking, it is used here in the asymptotic form $\lim_{x \uparrow \infty} u_{a_k}(x)$ which turns out to be the same as $u_{a_k}(x)$.

A simple calculation yields

$$E_0(a\varepsilon) = -\log\varepsilon + \Gamma_0 + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0^+, \tag{A.3}$$

where Γ_0 is a constant. Hence,

$$\frac{1}{E_0(a\varepsilon)} = -\sum_{k=1}^{m+1} \frac{\Gamma_0^{k-1}}{(\log\varepsilon)^k} + O\left(\frac{1}{(\log\varepsilon)^{m+2}}\right) \quad \text{as} \quad \varepsilon \to 0^+. \tag{A.4}$$

Therefore, we see that

$$V_0(\tilde{x};\varepsilon) = \sum_{k=1}^{n+1} \frac{V_{0k}(\tilde{x})}{(\log\varepsilon)^k} + \frac{\bar{V}_{0m}(\tilde{x};\varepsilon)}{(\log\varepsilon)^{m+2}}, \tag{A.5}$$

where

$$\begin{aligned} V_{0k} &= -\Gamma_0^{k-1}E_0(a\tilde{x}), \qquad k = 1,2,\cdots,m+1,\\ \bar{V}_{0m}(\tilde{x};\varepsilon) &= \left\{\Gamma_0^{m+1} + O\left(\frac{1}{\log\varepsilon}\right)\right\}E_0(a\tilde{x}). \end{aligned} \tag{A.6}$$

It remains to check that the V_{0k}'s and $\bar{V}_{0m}$ defined in (A.6) satisfy the properties in (4.15). This is a straightforward calculation except perhaps (4.15) (c) and (d). But these follow from Lemma 2.1 with $x = \tilde{x}/\varepsilon$ in (2.4). Hence we have established the lemma for $j = 0$ (in sharper form).

Now we investigate the term $v_1(x;\varepsilon a)$ in the series (3.4). The analysis proceeds in the same manner, in general. By the definition of $v_1(x;\varepsilon)$ in (3.3) and formulas (2.5), it is not difficult to see that

$$V_1(\tilde{x};\varepsilon) \equiv v_1\left(\frac{\tilde{x}}{\varepsilon};\varepsilon a\right) = c_1E_0(a\tilde{x}) + a\int_{\tilde{x}}^{\infty}\frac{e^{-at}}{t}\,dt\int_t^{\infty} e^{as}sV_0(s;\varepsilon)\dot{V}_0(s;\varepsilon)\,ds, \tag{A.7}$$

where

$$c_1 = -\frac{a}{E_0(a\varepsilon)}\int_{\varepsilon}^{\infty}\frac{e^{-at}}{t}\,dt\int_t^{\infty} e^{as}sV_0\dot{V}_0\,ds.$$

Hence if we substitute the function V_0 in the form defined in (A.5) and the corresponding $\dot{V}_0$ into (A.7), we shall obtain terms of the form:

$$\begin{aligned} I_1 &= \frac{1}{(\log\varepsilon)^{l+k}}a\int_{\tilde{x}}^{\infty}\frac{e^{-at}}{t}\,dt\int_t^{\infty} e^{as}sV_{0l}\dot{V}_{0k}\,ds,\\ I_2 &= \left(\frac{1}{(\log\varepsilon)^{l+k}}a\int_{\varepsilon}^{\infty}\frac{e^{-at}}{t}\,dt\int_t^{\infty} e^{as}sV_{0l}\dot{V}_{0k}\,ds\right)\frac{E_0(a\tilde{x})}{E_0(a\varepsilon)}, \end{aligned} \tag{A.8}$$

and similar terms involving $\dot{\bar{V}}_{0m}$ and $\bar{V}_{0m}$. Observe that I_1 is a function of $\tilde{x}$ only times a power of $(\log\varepsilon)$. Thus, terms of this form can be collected and rearranged to yield contributions to V_{1m} if $l + k \leqq m + 1$ or $\bar{V}_{1m}$ if $k + l > m + 1$. One need only check that these contributions satisfy the various estimates in Lemma 4.2 and this is an easy computation which we omit.

The term I_2 is more complicated since ε appears in a nontrivial way. Consider the terms,

$$I_2' = \int_\varepsilon^\infty \frac{e^{-at}}{t}\,dt \int_t^\infty e^{as} s \dot{V}_{0k} V_{0l}\,ds. \tag{A.9}$$

We have

$$I_2' = -\int_\varepsilon^\infty \frac{d}{dt}E_0(at)\left(\int_t^\infty e^{as} s\dot{V}_{0k}V_{0l}\,ds\right) dt.$$

An integration by parts yields

$$\begin{aligned} I_2' &= E_0(a\varepsilon)\int_\varepsilon^\infty e^{as}s\dot{V}_{0k}V_{0l}\,ds - \int_\varepsilon^\infty E_0(at)\,e^{at}t\dot{V}_{0k}V_{0l}\,dt \\ &= M_1E_0(a\varepsilon) + M_2 + \mathscr{R}, \end{aligned} \tag{A.10}$$

where

$$M_1 = \int_0^\infty e^{as}s\dot{V}_{0k}V_{0l}\,ds,$$

$$M_2 = -\int_0^\infty E_0(at)\,e^{at}t\dot{V}_{0k}V_{0l}\,dt$$

and

$$\mathscr{R} = -\left\{E_0(a\varepsilon)\int_0^\varepsilon e^{as}s\dot{V}_{0k}V_{0l}\,ds + \int_0^\varepsilon E_0(at)e^{at}t\dot{V}_{0k}V_{0l}\,dt\right\}.$$

By the properties in (4.15) and (4.17), we can show that both M_1 and M_2 exist and we note that they do not depend on ε. On the other hand, we have

$$\mathscr{R} = O(\varepsilon(\log\varepsilon)^2) \quad \text{as} \quad \varepsilon \to 0^+.$$

Consequently we write, by (A.8) and (A.10),

$$I_2 = \frac{aM_1}{(\log\varepsilon)^{l+k}}E_0(a\tilde{x}) + \frac{aM_2}{(\log\varepsilon)^{l+k}}\frac{E_0(a\tilde{x})}{E_0(a\varepsilon)} + \frac{a\mathscr{R}}{(\log\varepsilon)^{l+k}}\frac{E_0(a\tilde{x})}{E_0(a\varepsilon)}. \tag{A.11}$$

The first terms can be summed and rearranged again to contribute to V_{1m} or $\bar{V}_{1m}$. Moreover, since all these terms are dominated by a constant times $E_0(a\tilde{x})$, it is clear that the estimates in Lemma 4.2 are satisfied. Finally we observe that all the terms involving $\bar{V}_{0m}$ and $\dot{\bar{V}}_{0m}$ will only contribute to $\bar{V}_{1m}$ since all of these involve powers of $(\log\varepsilon)^{-\rho}$ for $\rho > m + 1$. Calculations similar to those just given show that these contributions all satisfy the estimates of Lemma 4.2. This concludes the proof of Lemma 4.2.

Acknowledgments. The author would like to express his gratitude to Professor Richard C. MacCamy for suggesting the topic and his guidance throughout the development of the work.

REFERENCES

[1] G. BIRKHOFF, *Hydrodynamics: A Study in Logic, Fact and Similitude*, 1st ed., Princeton University Press, Princeton, 1950.

[2] W. B. BUSH, *On the Lagerstrom mathematical model for flow at low Reynolds number*, SIAM J. Appl. Math., 20 (1971), pp. 279–287.

[3] J. D. COLE, *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, Mass., 1968.

[4] M. VAN DYKE, *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.

[5] R. FINN AND D. R. SMITH, *On the linearized hydrodynamical equations in two dimensions*, Arch. Rational Mech. Anal., 25 (1967), pp. 1–23.

[6] ———, *On the stationary solutions of the Navier–Stokes equations in two dimensions*, Ibid., 25 (1967), pp. 26–39.

[7] L. E. FRAENKEL, *On the method of matched asymptotic expansions. Part I: A matching principle; Part II: Some applications of the composite series; Part III: Two boundary-value problems*, Proc. Cambridge Philos. Soc., 65 (1969), pp. 209–284.

[8] G. C. HSIAO, *Singular perturbations of boundary value problems for a class of non-linear differential equations with a small parameter*, Doctoral dissertation, Carnegie-Mellon University, Pittsburgh, Pa., 1969.

[9] S. KAPLUN, *Low Reynolds number flow past a circular cylinder*, J. Math. Mech., 6 (1957), pp. 595–603.

[10] ———, *Fluid mechanics and singular perturbations*, P. A. Lagerstrom, L. N. Howard and C. S. Liu, eds., Academic Press, New York, 1967.

[11] R. E. O'MALLEY, JR., *Topics in singular perturbations*, Advances in Mathematics, vol. 2, Academic Press, New York, 1968, pp. 365–470.

[12] W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York, 1965.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

LIPSCHITZ BEHAVIOR AND CHARACTERISTIC FUNCTIONS*

K. SONI AND R. P. SONI†

Abstract. Let F be a distribution function. Its characteristic function belongs to Lip α, $0 < \alpha < 1$, if and only if $F(-x)$ and $1 - F(x)$ are $O(x^{-\alpha})$ as $x \to \infty$ (see Boas [1]). The n-dimensional Fourier transform of a radial function reduces to the Hankel transform of a function in one variable. Results similar to those given by Boas are obtained for this transform. The problem, however, is discussed in a rather general form. The class of functions Φ, $\Phi(x) = \int_0^\infty k(xt)\,dF(t)$, is considered. It is assumed that k is essentially bounded and has a nonzero Peano derivative of some definite order at zero, whereas $F(t)$ is nonincreasing but not necessarily bounded.

1. Introduction. It is well known that if $f(x)$ and $xf(x)$ are absolutely integrable in $(-\infty, \infty)$, then $\phi(x)$, the Fourier transform of $f(x)$, has uniformly continuous derivative. The converse is not true. If, however, $\phi(x)$ is a characteristic function corresponding to some distribution function $F(x)$,

$$\phi(x) = \int_{-\infty}^{\infty} e^{ixt}\,dF(t), \tag{1.1}$$

there exists a definite relationship between the Lipschitz behavior of $\phi(x)$ and the asymptotic behavior of $F(x)$ near $\pm\infty$. This relation was given explicitly by Boas [1] as follows.

THEOREM A. *If* $0 < \gamma < 1$, *then* $\phi \in \mathrm{Lip}\,\gamma$ *if and only if*

$$F(x) - F(\pm\infty) = O(|x|^{-\gamma}), \qquad |x| \to \infty. \tag{1.2}$$

The condition (1.2) is to be read as $F(x) = O(|x|^{-\gamma})$ as $x \to -\infty$ and $1 - F(x) = O(x^{-\gamma})$ as $x \to \infty$. If $\gamma = 1$, the theorem fails. This problem is related to the existence of $\phi'(x)$ at $x = 0$. Zygmund [12] proved that if

$$\lim_{T\to\infty} \int_{-T}^{T} t\,dF(t) \tag{1.3}$$

exists, then $\phi'(0)$ exists if and only if $\phi(h) + \phi(-h) - 2\phi(0) = o(h)$ as $h \to 0$. Later, Pitman [8] showed that if the limit in (1.3) exists, then $\phi'(0)$ exists if and only if $F(x) - F(\pm\infty) = o(1/x)$ as $x \to \infty$. In this connection, a result of Boas can be stated as follows.

THEOREM B. $F(x) - F(\pm\infty) = O(1/|x|)$ *or* $o(1/|x|)$, $|x| \to \infty$, *if and only if*

$$\phi(x + h) + \phi(x - h) - 2\phi(x) = O(h) \quad or \quad o(h) \tag{1.4}$$

uniformly in x as $h \to 0$.

Thus the conditions given by Pitman and Zygmund regarding the existence of $\phi'(0)$ are equivalent.

We consider the general class of transforms defined by

$$\Phi(x) = \int_0^\infty k(xt)\,dF(t), \tag{1.5}$$

* Received by the editors March 20, 1972.

† Mathematics Department, The University of Tennessee, Knoxville, Tennessee 37916.

where $k(t)$ is uniformly bounded and $F(t) \downarrow 0$ as $t \to \infty$. Many integral transforms used in applications have this form. A characteristic function (1.1) can also be studied with the help of such integrals [9], [10]. A particularly interesting special case of (1.5) is the Hankel transform because the Fourier transform of radial functions in several variables becomes the Hankel transform of a function in one variable [3, p. 69]. Some Abelian-type results pertaining to this transform are given in [7].

Our object in this paper is to determine a set of necessary and sufficient conditions so that the transform $\Phi(x)$ may have properties similar to those given in Theorems A and B. We note that the kernel e^{ix} is bounded and satisfies Lipschitz condition of order one uniformly in $-\infty < x < \infty$, yet the significance of these properties is not obvious from the results. We prove that in general the Lipschitz behavior of $\Phi(x)$ depends not only on the asymptotic behavior of $F(x)$ as $x \to \infty$ but also on the Lipschitz behavior of the kernel $k(x)$. A theorem of the type A fails when γ equals the order of the Lipschitz condition satisfied by the kernel. Whenever the kernel has suitable behavior at $x = 0$, this can be avoided by considering the symmetric difference of $\phi(x)$ as in (1.4) since the effect is the same as that of replacing the kernel $k(x)$ by the kernel $k^*(x) = k(x) + k(-x)$.

2. Main results. Let $\Phi(x)$ be the transform of the function $F(t)$ defined by

$$\Phi(x) = \int_0^\infty k(xt)\,dF(t), \qquad x \geqq 0, \tag{2.1}$$

where $F(t)$ and $k(t)$ satisfy the following assumptions:

$$|k(t)| \leqq M, \qquad 0 \leqq t < \infty, \tag{2.2}$$

$$k(t) = k(0) + Bt^\beta + o(t^\beta), \qquad t \to 0, B \neq 0, \beta > 0, \tag{2.3}$$

(2.4) $F(t)$ is nonincreasing in $0 < t < \infty$ and $F(t) \to 0$ as $t \to \infty$.

The function $F(t)$ is not necessarily bounded. In what follows, it is understood that these assumptions are satisfied.

THEOREM 2.1. *If*

$$\left|\int_0^1 k(t)\,dF(t)\right| < \infty, \tag{2.5}$$

then $\Phi(x)$ exists for all $x \geqq 0$. Furthermore,

(a) $\alpha > 0, \alpha \neq \beta, F(t) = O(t^{-\alpha}), t \to \infty$, *implies that* $\Phi(x) - \Phi(0) = O(x^{\min(\alpha,\beta)})$, $x \to 0$;

(b) $0 < \alpha < \beta$, $F(t) = o(t^{-\alpha})$, $t \to \infty$, *implies that* $\Phi(x) - \Phi(0) = o(x^\alpha)$, $x \to 0$;

(c) $\alpha > 0$, $k(t) \in \operatorname{Lip}(\gamma)$ *uniformly in* $0 \leqq t < \infty$, $\alpha \neq \gamma$ *and in case* $k(0) = 0$,

$$\left|\int_0^1 t^\gamma\,dF(t)\right| < \infty.$$

Under these conditions $F(t) = O(t^{-\alpha})$, $t \to \infty$, *implies that* $\Phi(x) \in \operatorname{Lip}(\min(\alpha, \gamma))$ *uniformly in* $0 \leqq x < \infty$.

THEOREM 2.2. *Let* $0 < \alpha < \beta$. *If*

$$\Phi(x) - \Phi(0) = O(x^\alpha), \quad o(x^\alpha), \qquad x \to 0, \tag{2.6}$$

then

$$F(t) = O(t^{-\alpha}), \quad o(t^{-\alpha}), \qquad t \to \infty, \tag{2.7}$$

provided that the kernel $k(t)$ *satisfies either one of the following conditions*:

(a) $k(t)$ *assumes its absolute maximum or minimum at* $t = 0$;

(b) *there exists a nontrivial function* $\omega(x)$ *such that*

(2.8)
(i) $\omega(x) \geqq 0, \quad \omega(x) \in L(0, 1)$,

(ii) $k^*(x) = \int_0^1 \omega(t)k(xt)\,dt$ *assumes its absolute maximum or minimum at* $x = 0$.

We shall need the following results which we state as lemmas. The first one is due to Sz.-Nagy [11].

LEMMA 1. *Let* $\phi(x)$ *and* $\psi(x)$ *be two monotone functions* ($\phi\uparrow$, $\psi\downarrow$) *defined in* $0 < x \leqq a$ *such that* $\phi(0+) = 0$. *If either one of the two integrals* $\int_0^a \phi(x)\,d\psi(x)$ *or* $\int_0^a \psi(x)\,d\phi(x)$ *exists, then both integrals exist and* $\lim_{x \to 0+} \phi(x)\psi(x) = 0$.

LEMMA 2. *Let* $F(t)$ *be nonincreasing in* $0 < t < \infty$, $F(t) \to 0$ *as* $t \to \infty$; *and let* $t^{\beta-1}F(t)$ *be integrable into* $t = 0$. *If* $0 < \alpha \leqq \beta$, *then*

$$\int_0^y t^\beta\,dF(t) = O(y^{\beta-\alpha}), \qquad y \to \infty, \quad \textit{implies} \quad F(t) = O(t^{-\alpha}), \qquad t \to \infty. \tag{2.9}$$

In (2.9), O can be replaced by o provided that $0 < \alpha < \beta$.

Proof of Lemma 2. Let

$$A(y) = \int_0^y t^\beta\,dF(t), \qquad y > 0.$$

By Lemma 1, $A(y)$ exists for all $y > 0$. Also $|A(y)|$ is a nondecreasing function of y. Since

$$F(x) = -\int_x^\infty t^{-\beta}t^\beta\,dF(t)$$
$$= x^{-\beta}A(x) - \beta\int_x^\infty t^{-\beta-1}A(t)\,dt,$$

the conclusion is obvious.

Proof of Theorem 2.1. By (2.3) we can choose $\delta > 0$ such that $|k(t) - k(0)| \leqq 2|B|t^\beta$ for $0 \leqq t \leqq \delta$. If $k(0) \neq 0$, $F(0)$ is finite by (2.5) so that $\Phi(x)$ exists for all $x \geqq 0$. If $k(0) = 0$, $\Phi(0) = 0$ and by (2.5) again,

$$\left|\int_0^1 t^\beta\,dF(t)\right| < \infty. \tag{2.10}$$

For $x > 0$,

$$\left|\int_0^\infty k(xt)\,dF(t)\right| \leqq \left|\int_0^{\delta/x} + \int_{\delta/x}^\infty k(xt)\,dF(t)\right|$$
$$\leqq 2|B|x^\beta\left|\int_0^{\delta/x} t^\beta\,dF(t)\right| + MF(\delta/x). \tag{2.11}$$

Hence $\Phi(x)$ exists for all $x \geqq 0$.

To prove (a), let $x > 0$. By similar reasoning as above,

$$
\begin{aligned}
|\Phi(x) - \Phi(0)| &= \left|\int_0^\infty [k(xt) - k(0)]\, dF(t)\right| \\
&\leqq 2|B|x^\beta \left|\int_0^{\delta/x} t^\beta\, dF(t)\right| + 2MF(\delta/x).
\end{aligned}
\tag{2.12}
$$

$F(\delta/x) = O(x^\alpha)$ as $x \to 0$. By (2.10) and Lemma 1, $t^\beta F(t) \to 0$ as $t \to 0$. Integrating by parts,

$$
\begin{aligned}
\left|\int_0^{\delta/x} t^\beta\, dF(t)\right| &= \left|(\delta/x)^\beta F(\delta/x) - \beta \int_0^{\delta/x} t^{\beta-1} F(t)\, dt\right| \\
&\leqq O(x^{\alpha-\beta}) + \beta \left|\int_0^1 + \int_1^{\delta/x} t^{\beta-1} F(t)\, dt\right| \\
&= O(x^{\alpha-\beta}) + O(1) + O\left(\int_1^{\delta/x} t^{\beta-1-\alpha}\, dt\right) \\
&= O(1) + O(x^{\alpha-\beta}), \qquad x \to 0.
\end{aligned}
\tag{2.13}
$$

Hence by (2.12) and (2.13),

$$|\Phi(x) - \Phi(0)| = O(x^{\min(\alpha,\beta)}), \qquad x \to 0.$$

This completes the proof of (a). The proof of (b) is similar. In (c), we consider $|\Phi(x \pm h) - \Phi(x)|$, $h > 0$, $x - h > 0$ and again complete the proof as in (a).

Proof of Theorem 2.2. First we note that $\Phi(x)$ is defined in some interval $0 \leqq x \leqq \delta_1$. This implies that $\Phi(x)$ is defined in $0 \leqq x < \infty$. (If $\Phi(0) \neq 0$, then $k(0) \neq 0$ and so $F(0)$ is finite. If $\Phi(0) = 0$, then $k(0) = 0$ unless $F(t) \equiv 0$ and the argument can be completed in the same manner as in the proof of Theorem 2.1.) Without loss of generality we may assume that $B > 0$ in (2.3). Determine $\delta > 0$ such that

$$k(t) - k(0) \geqq (B/2)t^\beta, \qquad 0 \leqq t \leqq \delta. \tag{2.14}$$

Let $k(t)$ satisfy condition (a). By (2.14), $k(t) - k(0) \geqq 0$ for all $t > 0$. Since

$$
\begin{aligned}
\Phi(x) - \Phi(0) &= \int_0^\infty [k(xt) - k(0)]\, dF(t) \\
&= O(x^\alpha), \qquad x \to 0,
\end{aligned}
\tag{2.15}
$$

it follows that

$$\int_0^{\delta/x} [k(xt) - k(0)]\, dF(t) = O(x^\alpha), \qquad x \to 0.$$

By (2.14),

$$\int_0^{\delta/x} (xt)^\beta\, dF(t) = O(x^\alpha), \qquad x \to 0,$$

or

$$\int_0^x t^\beta \, dF(t) = O(x^{\beta-\alpha}), \qquad x \to \infty.$$

The proof is now complete by Lemma 2. If $k(t)$ satisfies condition (b), then by (2.14),

$$(2.16) \qquad \begin{aligned} k^*(t) - k^*(0) &= \int_0^1 \omega(x)[k(xt) - k(0)]\, dx \\ &\geqq \frac{B}{2}\int_0^1 \omega(x)(xt)^\beta \, dx \qquad (0 \leqq t \leqq \delta) \\ &\geqq ct^\beta \qquad (0 \leqq t \leqq \delta) \end{aligned}$$

for some constant $c > 0$. Hence $k^*(t) - k^*(0) \geqq 0$ for all $t > 0$. From (2.15) we obtain

$$\int_0^\infty [k(xyt) - k(0)]\, dF(t) = O((xy)^\alpha), \qquad xy \to 0,$$

and in particular for $0 \leqq x \leqq 1$, $y \to 0$. Hence,

$$\int_0^1 \omega(x)\, dx \int_0^\infty [k(xyt) - k(0)]\, dF(t) = O(y^\alpha)\int_0^1 \omega(x)x^\alpha \, dx.$$

The interchange of the order of integration is easily justified and we obtain

$$\int_0^\infty [k^*(yt) - k^*(0)]\, dF(t) = O(y^\alpha), \qquad y \to 0.$$

With the help of (2.16), the proof can now be completed as before.

The proof is essentially the same when O is replaced by o.

3. Remarks. Theorem 2.2 gives the Tauberian counterpart of Theorem 2.1. The function $\omega(x)$ in (2.8) is such that for some constant c, $cT^{-1}\omega(x/T)$ is a regular summability kernel [6, p. 50]. Obviously the kernels e^{-x} and $x^{-\nu}J_\nu(x)$, $\nu \geqq -1/2$, satisfy the condition (a) of Theorem 2.2. The special case $\nu = -1/2$ when $x^{-\nu}J_\nu(x) = (2/\pi)^{1/2}\cos x$ is essentially contained in Boas's results [1]. A kernel which satisfies condition (b) but not (a) is $k(x) = \sqrt{x}J_\nu(x)$, $\nu > -1/2$. Let

$$\omega(t) = \begin{cases} t^{-1/2}(1 - t^2)^{-1/2}, & 0 < t < 1, \\ 0, & t > 1. \end{cases}$$

By [5, p. 24, (22)],

$$\begin{aligned} k^*(x) - k^*(0) &= \int_0^1 \omega(t)(xt)^{1/2}J_\nu(xt)\, dt \\ &= (\pi/2)x^{1/2}[J_{\nu/2}(x/2)]^2 \geqq 0. \end{aligned}$$

If $\nu = 1/2$, $k(x) = (2/\pi)^{1/2} \sin x$. In this case, we can consider the following simpler expression for $\omega(t)$,

$$\omega(t) = \begin{cases} 1, & 0 \leqq t \leqq 1, \\ 0, & t > 1, \end{cases}$$

so that $k^*(x) - k^*(0) = (2/\pi)^{1/2} x^{-1}(1 - \cos x) \geqq 0$, $x > 0$.

For $\nu > -1/2$, the kernel $x^{1/2} J_\nu(x)$ illustrates nicely how the behavior of a kernel at zero influences the behavior of the transform Φ. We note that $x^{1/2} J_\nu(x) \in \operatorname{Lip}(\nu + 1/2)$ at $x = 0$. The two theorems together imply that if $0 < \alpha < \nu + 1/2$, $\Phi(x) = O(x^\alpha)$, $o(x^\alpha)$, $x \to 0$ if and only if $F(t) = O(t^{-\alpha})$, $o(t^{-\alpha})$, $t \to \infty$. It is immaterial whether $\nu + 1/2$ is greater than, equal to or less than 1. On the other hand $x^{1/2} J_\nu(x) \in \operatorname{Lip} \gamma$, $\gamma = \min(\nu + 1/2, 1)$ uniformly in $0 \leqq x < \infty$. Hence if $0 < \alpha < \gamma$, $\Phi(x) \in \operatorname{Lip} \alpha$ uniformly in $0 \leqq x < \infty$ if and only if $F(t) = O(t^{-\alpha})$, $t \to \infty$. It is not difficult to find examples to show that the behavior of $\Phi(x)$ may be significantly better locally or uniformly than that indicated in Theorem 2.1. For example, let

$$k(t) = \begin{cases} 0, & 0 \leqq t \leqq 1, \\ 1, & t > 1, \end{cases}$$

and

$$F(t) = \begin{cases} 1, & 0 \leqq t \leqq 1, \\ 1/t, & t > 1. \end{cases}$$

Then

$$\Phi(x) = \begin{cases} 0, & x = 0, \\ -x, & 0 < x < 1, \\ -1, & x \geqq 1. \end{cases}$$

$\Phi(x) \in \operatorname{Lip} 1$ uniformly but $k(t)$ is discontinuous. The following examples, however, show that Theorem 2.1 (a) and (c) are the best possible.

1. Let

$$\Phi(x) = \int_0^\infty \sin xt \, d(t^{-1/2}) = -(\pi x/2)^{1/2}.$$

Here $\alpha = 1/2$, $\gamma = \beta = 1$, $\Phi(x) \in \operatorname{Lip}(1/2)$ at $x = 0$.

2. Let $k(t) = t^{-1/2} J_1(t)$, and

$$F(t) = \begin{cases} (2/3)(1 - t^{3/2}), & 0 \leqq t \leqq 1, \\ 0, & t > 1. \end{cases}$$

By [5, p. 18, (1)],

$$\Phi(x) = -x^{-3/2}[1 - J_0(x)].$$

Here $\gamma = \beta = 1/2$, α can be taken arbitrarily large but $\Phi(x) \in \operatorname{Lip}(1/2)$ at $x = 0$.

Finally, since $\Phi(x)$ is defined only for $x \geqq 0$, it would be meaningless to consider the symmetric differences [12] of $\Phi(x)$ at $x = 0$. However, we can still consider the behavior of Φ in an analogous manner. For example, if $k(t) = \sin t$,

$$\Phi(2x) - 2\Phi(x) = \int_0^\infty (\sin 2xt - 2 \sin xt)\, dF(t).$$

The kernel $\sin 2x - 2 \sin x$ is uniformly bounded and belongs to Lip (3) at $x = 0$. It can easily be verified that

$$k_1^*(x) - k_1^*(0) = \int_0^1 (\sin 2xt - 2 \sin xt)\, dt \leqq 0 \quad \text{for all} \quad x > 0.$$

Hence, if $0 < \alpha < 3$, $\Phi(2x) - 2\Phi(x) \in \mathrm{Lip}\,(\alpha)$ at $x = 0$ if and only if $F(t) = O(t^{-\alpha})$, $t \to \infty$. Results like this can take the place of the one in (1.4).

REFERENCES

[1] R. P. Boas, Jr., *Lipschitz behavior and integrability of characteristic functions*, Ann. Math. Statist., 38 (1967), pp. 32–36.

[2] ———, *Integrability Theorems for Trigonometric Transforms*, Springer-Verlag, New York, 1967.

[3] S. Bochner and K. Chandrasekharan, *Fourier Transforms*, Princeton University Press, Princeton, 1949.

[4] Z. Ditzian, *Laplace transform of functions satisfying the Lipschitz condition*, Compositio Math., 22 (1970), pp. 29–38.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, vol. 2, McGraw-Hill, New York, 1954.

[6] G. H. Hardy, *Divergent Series*, Oxford University Press, London, 1956.

[7] G. E. McKelvy, *Hankel transforms: Asymptotic behavior and Lipschitz conditions*, Thesis, The University of Tennessee, 1971.

[8] E. J. G. Pitman, *On the derivatives of a characteristic function at the origin*, Ann. Math. Statist., 27 (1956), pp. 1156–1160.

[9] ———, *Some theorems on characteristic functions of probability distributions*, Proc. 4th. Berkeley Sympos. Math. Statist. and Prob., vol. II, 1961, pp. 393–402.

[10] ———, *On the behavior of the characteristic function of a probability distribution in the neighborhood of the origin*, J. Austral. Math. Soc., 8 (1968), pp. 423–443.

[11] B. Sz.-Nagy, *Séries et intégrales de Fourier des fonctions monotones non bornées*, Acta. Sci. Math. (Szeged), 13 (1949), pp. 118–135.

[12] A. Zygmund, *A remark on characteristic functions*, Ann. Math. Statist., 18 (1947), pp. 272–276.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

A COMPLETE SET OF ORTHONORMAL HARMONIC FUNCTIONS*

A. S. FAROOQUI†

Abstract. A complete sequence of orthogonal harmonic functions on a domain is constructed. The boundary values of these harmonic functions are found to be the eigenfunctions of a certain integral operator.

Introduction. It is a familiar fact that the set of functions $1; \cos\theta, \sin\theta; \cos 2\theta, \sin 2\theta; \cdots$ is orthogonal as well as complete in the space of all square integrable functions defined over the interval $(0, 2\pi)$. On the other hand, the set of harmonic polynomials $1; r\cos\theta, r\sin\theta; r^2\cos 2\theta, r^2\sin 2\theta, \cdots$ (defined over the unit circle with center at the origin) is orthogonal in the usual sense of the inner product for a domain as well as complete in the space of all harmonic functions defined on the unit circle with continuous boundary values. Moreover, the former set is clearly the boundary values of the latter. An analogous situation is found to hold for an arbitrary simply connected domain bounded by a simple smooth closed contour.

For expedience, an inner product for functions defined and continuous on the boundary is introduced. Two functions orthogonal in this inner product space are then termed orthogonal on the boundary. Incidentally, for the unit circle, this inner product coincides with the usual inner product for the interval $(0, 2\pi)$. It is established that there is a sequence of functions, harmonic in the interior and continuous on the closure of the domain, which is orthogonal as well as complete in the space of all harmonic functions with continuous boundary values. Moreover, the boundary values of this sequence form a complete orthonormal set on the boundary.

1. Notation and terminology. Let D be the interior of some simply connected domain bounded by a simple closed contour ∂D. The positive direction of the contour is taken as counterclockwise. The symbol $\nu(Q)$, $Q \in \partial D$, is designated for the unit normal at Q directed into the interior of D. We assume, of course, that ∂D is smooth enough to have a meaningful normal.

The line integral $\oint_{\partial D} f(Q)\, dl_Q$ is defined in the usual way. By the area integral $\iint_D f(P)\, dP$, we mean $\iint_D f(x, y)\, dx\, dy$. In terms of these integrals the following inner products are introduced:

$$\langle f, g\rangle = \oint_D f(Q)g(Q)\, dl_Q \tag{1.1}$$

and

$$[f, g] = \iint_D f(P)g(P)\, dP. \tag{1.2}$$

* Received by the editors October 28, 1971, and in revised form May 22, 1972.

† Department of Mathematics, Simon Fraser University, Burnaby 2, British Columbia, Canada.
This work was supported by the National Research Council of Canada.

The inner product defined by (1.1) is referred to as the inner product on the boundary. Two functions are said to be orthogonal on the boundary if their inner product vanishes. Similar remarks apply for the inner product defined on the domain by means of (1.2).

We say that the system of functions Ψ_i defined and continuous on the boundary is orthonormal on the boundary if

$$\langle \Psi_i, \Psi_j \rangle = \delta_{i,j}, \qquad i,j = 1,2,3,\cdots, \tag{1.3}$$

where $\delta_{i,j}$ is the Kronecker delta. The system is said to be complete if, for any function f defined and continuous on the boundary, the relation $\langle f, \Psi_i \rangle = 0$ for all Ψ_i implies that $f \equiv 0$. Similar remarks apply for the system of functions defined on a domain, of course, using the appropriate inner product defined on the same domain for such functions.

The symbol $G(P, R)$, P, $R \in \bar{D}$, is reserved for the Dirichlet-type harmonic Green's function.

2. Generalized Poisson's kernel. Associated with the domain, there is a function which plays an important role in this investigation. It is termed as the generalized Poisson's kernel. More precisely, we have the following definition.

DEFINITION 2.1. The function $G_v(P, Q), P \in D, \ Q \in \partial D$ is defined as the *generalized Poisson's kernel for D.*

Incidentally, when D is the unit circle, $G_v(P, Q)$ reduces to the well-known Poisson's kernel. Several properties of the Poisson's kernel are also valid for the generalized Poisson's kernel. For example,

$$G_v(P, Q) > 0, \qquad P \in D, \tag{2.1}$$

$$\oint_{\partial D} G_v(P, Q)\, dl_Q = 1, \qquad P \in D, \tag{2.2}$$

or, in the language of inner product:

$$\langle 1, G_v(P, Q) \rangle = 1, \qquad P \in D. \tag{2.3}$$

Functions defined and continuous on the boundary may be extended to the closure of the domain in several meaningful ways. One extension, however, has proved quite useful. It is termed the Dirichlet extension and is prescribed by the following.

DEFINITION 2.2. Let f be a continuous function defined on the boundary. The *Dirichlet extension of f on* $\bar{D}$ is given by

$$\begin{aligned} f(P) &= \oint_{\partial D} f(Q) G_v(P, Q)\, dl_Q \\ &= \langle f(Q), G_v(P, Q) \rangle. \end{aligned} \tag{2.4}$$

The extended function is harmonic in D and of class C on $\bar{D}$.

3. Poisson's kernel of the second kind. Another function, this time defined on $\partial D \times \partial D$, is also found quite useful in the sequel. For expedience, we introduce it now.

DEFINITION 3.1. The *Poisson's kernel of the second kind* is defined by means of the formula

$$K(Q, S) = \int\int_D G_v(P, Q)G_v(P, S)\, dP = [G_v(P, Q), G_v(P, S)], \qquad Q, S \in \partial D. \tag{3.1}$$

Many properties of this kernel are quite interesting and can be easily established. For the purpose of this investigation, the following are listed:

(i) The kernel is symmetric and square integrable. The symmetry is, of course, obvious from the definition. Upon a closer examination, one notices that $K(Q, S)$ is continuous on $\partial D \times \partial D$ except when Q coincides with S where it has a weak (logarithmic) singularity.[1] Consequently, $K(Q, S) \in L_2(\partial D \times \partial D)$.

(ii) The kernel is positive definite.

In order to prove the assertion, we let $f(Q)$, $Q \in \partial D$, be any function of class C. Then, if $f(P)$ is its Dirichlet extension, it follows that

$$\oint_{\partial D}\oint_{\partial D} K(Q, S)f(Q)f(S)\, dl_Q\, dl_S = \int\int_D \{f(P)\}^2\, dP.$$

The left-hand side vanishes only if $f = 0$, which proves the proposition.

Let $\{\psi_i\}$ be the normalized system of eigenfunctions and $\{\lambda_i\}$ be the corresponding eigenvalues for the kernel $K(Q, S)$ so that

$$\psi_i(Q) = \lambda_i \oint_{\partial D} K(Q, S)\psi_i(S)\, dl_S \tag{3.2}$$

and

$$\langle \psi_i, \psi_j \rangle = \delta_{ij}. \tag{3.3}$$

Since the kernel is positive definite, it follows that the set $\{\psi_i\}$ forms a complete orthonormal system in the space of all continuous functions defined on the boundary. This completeness may be extended to all L_2-functions on the boundary by means of the triangular inequality in the Hilbert space generated by this inner product. Furthermore, all λ_i are positive.

With each eigenfunction $\psi_i(Q)$ defined on ∂D, we consider its Dirichlet extension $\psi_i(P)$ defined on $\bar{D}$. Clearly, $\psi_i(P)$ is harmonic in D and coincides with $\psi_i(Q)$ on ∂D.

We are now ready to state and prove the principal result of this investigation in the following form.

THEOREM 3.1. *The sequence $\{\psi_i(P)\}$ forms a complete orthogonal set of harmonic functions in the space of all harmonic functions defined on D with continuous boundary values.*

[1] For details, see Appendix.

Fortunately, the theorem is quite easy to establish. Let us consider

$$\begin{aligned}\delta_{ij} &= \langle \psi_i, \psi_j \rangle \\ &= \lambda_j \int_{\partial D} \int_{\partial D} \psi_i(S) K(Q, S) \psi_j(Q)\, dl_Q\, dl_S,\end{aligned}$$

by (3.2). Using (3.1), we have

$$\begin{aligned}\delta_{ij} &= \lambda_j \oint_{\partial D} \oint_{\partial D} \int \int_D G_v(P, Q) G_v(P, S) \psi_i(S) \psi_j(Q)\, dl_Q\, dl_S\, dP \\ &= \lambda_j \int \int_D \psi_i(P) \psi_j(P)\, dP,\end{aligned}$$

which proves the orthogonality. Moreover, the system $\{\sqrt{\lambda_i}\psi_i(P)\}$ forms an orthogonal set with respect to the inner product defined for D. To prove completeness in the space of all harmonic functions on D, with continuous boundary values, we consider any function $f(P)$ which is harmonic in D and of class C on $\bar{D}$. Then by (2.4), (3.1) and (3.2),

$$\lambda_i \int \int_D f(P) \psi_i(P)\, dP = \int_{\partial D} f(S) \psi_i(S)\, dl_S. \tag{3.4}$$

Hence, $[f, \psi_i] = 0$ implies that $\langle f, \psi_i \rangle = 0$ for all ψ_i. The completeness of ψ_i on ∂D implies that $f \equiv 0$ on ∂D. Hence its Dirichlet extension $f(P)$ vanishes on $\bar{D}$ identically, which proves the theorem.

In order to illustrate the theory, we take the simplest possible example, namely, when D is the unit circle $r = 1$. The generalized Poisson's kernel is then the ordinary Poisson's kernel:

$$\begin{aligned}G_v(r, \theta; 1, \alpha) &= \frac{1 - r^2}{2\pi\{1 - 2r\cos(\theta - \alpha) + r^2\}} \\ &= \frac{1}{2\pi} \sum_0^\infty \varepsilon_n r^n \cos n(\theta - \alpha),\end{aligned} \tag{3.5}$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$, $n = 1, 2, 3, \cdots$. The kernel of the second kind is therefore given by

$$\begin{aligned}K(\alpha, \beta) &= \int_0^1 \int_0^{2\pi} G_v(r, \theta; 1, \alpha) G_v(r, \theta; 1, \beta) r\, dr\, d\theta \\ &= \frac{1}{4\pi} \sum_0^\infty \frac{\varepsilon_n}{n + 1} \cos n(\beta - \alpha),\end{aligned} \tag{3.6}$$

giving the expansion of $K(\alpha, \beta)$ in terms of its eigenfunctions. The series may be summed into a closed form if desired. Comparing (3.6) with the usual bilinear expansion for $K(\alpha, \beta)$, namely,

$$K(\alpha, \beta) = \sum_{i=1}^\infty \frac{\psi_i(\alpha)\psi_i(\beta)}{\lambda_i}, \tag{3.7}$$

one finds that the complete set of orthonormal eigenfunctions for K is

$$\left\{\sqrt{\frac{\varepsilon_n}{2\pi}}\cos n\alpha, \sqrt{\frac{\varepsilon_n}{2\pi}}\sin n\alpha\right\}$$

with eigenvalue $\lambda_n = 2(n+1)$. The corresponding complete set of orthogonal harmonic functions obtained by means of Dirichlet extension (using (2.4) and (3.5)) is

$$\left\{\sqrt{\frac{\varepsilon_n}{2\pi}}r^n\cos n\alpha, \sqrt{\frac{\varepsilon_n}{2\pi}}r^n\sin n\alpha\right\}.$$

Theorem 3.1 can now be verified immediately. Incidently, one notices the multiplicity two of each eigenvalue after the first one because there are two eigenfunctions $\cos n\alpha$, $\sin n\alpha$ corresponding to each eigenvalue λ_n, $n > 1$.

Appendix. Proof of the fact that $K \in L_2(\partial D \times \partial D)$**.** We first show that the integral (3.1) representing the Poisson's kernel of the second kind is convergent whenever Q does not coincide with S. For any point $R \in \partial D$, let the symbol $D(R, \delta)$ denote the common intersection of the domain D and the neighborhood of R with radius δ. For a small $\delta > 0$, this domain is approximately a semicircular region with its diameter along ∂D. For Q not coinciding with S, we can choose δ so small that $D(Q, \delta)$ and $D(S, \delta)$ are disjoint. For expedience, the position of a point $P \in D(Q, \delta)$ is represented by the polar coordinates (r, θ) with pole at Q and the initial line being the tangent line at Q. It is now easily seen that

$$G_v(P, Q) = \frac{\sin\theta}{\pi|PQ|} + O(1), \qquad P \in D(Q, \delta).$$

The dominant singular part of $K(Q, S)$ over $D(Q, \delta)$ in absolute value can therefore be majorized by

$$\text{(A.1)} \qquad \int\int_{D(Q,\delta)} \frac{\sin\theta}{\pi^2|PS|}\,d\theta\,dr,$$

where a factor similar to $\sin\theta$ has been dropped in the numerator. The expression (A.1) in turn is dominated by $\delta/(\pi(h-\delta))$, where $h = |QS|$. The same result is also valid for the integral over $D(S, \delta)$. Consequently, the integral in (3.1) converges absolutely. Moreover, the convergence is uniform for $h \geqq k > 0$.

Next we consider the behavior of $K(Q, S)$ as $h \to 0$. The point S may now be considered on the diameter of $D(Q, \delta)$ at a distance h from Q where $h < \delta$. The dominant singular term of the integral over $D(Q, \delta)$ corresponding to (A.1) is

$$\text{(A.2)} \qquad \frac{1}{\pi^2}\int_0^\delta\int_0^\pi \frac{\sin\theta\,d\theta\,dr}{\sqrt{\{h^2 + r^2 - 2hr\cos\theta\}}} = \frac{1}{\pi^2}\left(1 + \log\frac{\delta}{h}\right).$$

Thus the kernel becomes logarithmically singular as $h \to 0$ which proves the assertion (i) in § 3. Moreover, the eigenfunctions are continuous on the boundary.

Acknowledgment. The author is grateful to Professor R. W. Lardner for many valuable discussions.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

SYMMETRIES OF DIFFERENTIAL EQUATIONS. THE HYPERGEOMETRIC AND EULER–DARBOUX EQUATIONS*

WILLARD MILLER, JR.†

Abstract. A general technique is introduced which uses the symmetry group of a linear homogeneous partial differential equation to obtain solutions of the equation and transformation properties of these solutions. As an application it is shown that the Euler–Poisson–Darboux equation $u_{xx} - u_{yy} - (k/y)u_y = 0$ admits the symmetry group $SL(2, \mathfrak{C})$ and, if k-changing operators are admitted, the group $SO(5, \mathfrak{C})$. Certain quadratic transformation formulas for hypergeometric functions are related to the $SO(5, \mathfrak{C})$ symmetry. Similarly it is shown that the Euler–Darboux equation $u_{\xi\eta} + (\xi - \eta)^{-1}(\alpha u_\eta - \beta u_\xi) = 0$ admits the symmetry group $SL(2, \mathfrak{C})$ and, if (α, β)-changing operators are admitted, the group $SL(4, \mathfrak{C})$. The transformation formulas for the hypergeometric functions and the 24 solutions of Kummer are related to the $SL(4, \mathfrak{C})$ symmetry.

Introduction. The notion of the symmetry group of a partial differential equation has proved useful for the construction of symmetry adapted solutions of the differential equation. This is particularly true for the nonlinear equations of hydrodynamics (see [1]).

A number of papers have appeared recently which show how one can compute the symmetry group G of a given equation and then use various one-parameter subgroups K of G to find solutions of the equation which are invariant under K (see [2]–[5]). We make particular mention of [4] in which the authors show that the heat equation in two variables admits a six-parameter Lie symmetry group. These papers follow the geometric approach of Lie himself [6] and contribute to a *static* theory of symmetry in the sense that they are concerned primarily with solutions invariant under one-parameter transformation groups.

In this paper we exploit the elementary fact that the solutions of a linear homogeneous partial differential equation form a vector space. Thus, the action of the symmetry group G on the solution space defines a representation of G and we can use representation theory to study the transformation properties of solutions under the action of the full group rather than limit ourselves to one-parameter subgroups. In this sense the theory presented here is *dynamic.*

As an application of the method we show that if parameter-changing operators are allowed, the Euler–Poisson–Darboux (EPD) equation $u_{xx} - u_{yy} - (k/y)u_y = 0$ and the Euler–Darboux equation $u_{\xi\eta} + (1/(\xi - \eta))(\alpha u_\eta - \beta u_\xi) = 0$ admit the Lie symmetry groups $SO(5, \mathfrak{C})$ and $SL(4, \mathfrak{C})$, respectively. Certain of these symmetries have already been exploited to solve boundary value and initial value problems for the above equations [10], [13, Chap. 1], but it appears that the full symmetry groups have not been computed until now.

We show also in this paper that $SO(5, \mathfrak{C})$ and $SL(4, \mathfrak{C})$ are intimately related to the hypergeometric functions ${}_2F_1(a, b, c, z)$. (Indeed, one can consider $SL(4, \mathfrak{C})$ as the dynamical symmetry group of the ${}_2F_1$.) A detailed study of this relationship will be undertaken in another publication.

A dynamic treatment of the heat equation can be derived from [12] in which Weisner treats an equivalent equation.

* Received by the editors June 28, 1971, and in revised form April 3, 1972.

† School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455. This work was supported in part by the National Science Foundation under Contract GP 29321.

1. The general method. Let $Qu = 0$ be a linear homogeneous partial differential equation where $u = u(x, t)$ and

$$Q = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial t} + C\frac{\partial^2}{\partial t^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial t} + F. \tag{1.1}$$

Here A, B, C, D, E, F are analytic functions of (x, t) in some common domain $\mathscr{D}$. We are interested in obtaining families of solutions u of $Qu = 0$ defined in some common subdomain of $\mathscr{D}$. (To be explicit we have chosen Q to be a second order differential operator in two independent variables. Actually the order of the operator and the number of variables are immaterial.)

Consider the set $\mathscr{G}$ of all linear differential operators

$$L = X(x, t)\frac{\partial}{\partial x} + T(x, t)\frac{\partial}{\partial t} + U(x, t) \tag{1.2}$$

with analytic coefficients such that $QLu = 0$ whenever $Qu = 0$. Thus, $\mathscr{G}$ consists of all operators L which map the solution space $\mathfrak{F}$ of Q into itself. Clearly, $L \in \mathscr{G}$ if and only if

$$[L, Q]u = LQu - QLu = 0 \tag{1.3}$$

for all $u \in \mathfrak{F}$, where $[L, Q] = LQ - QL$ is the commutator of L and Q. It follows from (1.3) that $L \in \mathscr{G}$ if and only if

$$[L, Q] = R(x, t)Q, \tag{1.4}$$

where the analytic function R depends on L.

It is easy to check that $\mathscr{G}$ is a (possibly infinite-dimensional) Lie algebra. That is, if $L_1, L_2 \in \mathscr{G}$, then

$$a_1L_1 + a_2L_2 \in \mathscr{G}, \qquad [L_1, L_2] \in \mathscr{G}, \tag{1.5}$$

for all constants a_1, a_2 (see [7]). We can associate with $\mathscr{G}$ the local Lie group G consisting of all finite products $\exp(\alpha_1 L_1) \cdots \exp(\alpha_n L_n)$ of operators

$$\exp \alpha L = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} L^n, \qquad L \in \mathscr{G}, \tag{1.6}$$

defined for the constants α_j sufficiently close to 0. The operators $\exp \alpha L$ can be explicitly computed and take the form

$$[\exp(\alpha L)f](x, t) = v(x, t, \alpha)f(x(\alpha), t(\alpha)), \tag{1.7}$$

where f is any analytic function and $x(\alpha)$, $t(\alpha)$, $v(x, t, \alpha)$ are uniquely determined by the equations

$$\frac{dx(\alpha)}{d\alpha} = X(x(\alpha), t(\alpha)), \quad \frac{dt(\alpha)}{d\alpha} = T(x(\alpha), t(\alpha)), \tag{1.8}$$
$$\frac{d}{d\alpha} v(x, t, \alpha) = v(x, t, \alpha)U(x(\alpha), t(\alpha)),$$
$$x(0) = x, \qquad t(0) = t, \qquad v(x, t, 0) = 1$$

(see [8]). Here L is given by (1.2).

Since L_1 and L_2 in $\mathcal{G}$ each leave $\mathfrak{F}$ invariant, so does their product L_1L_2. Similarly L^n leaves $\mathfrak{F}$ invariant for $L \in \mathcal{G}$ and $n = 0, 1, 2, \cdots$. Finally $\exp \alpha L$ leaves $\mathfrak{F}$ invariant for $L \in \mathcal{G}$ and α sufficiently small so any element of G maps $\mathfrak{F}$ into itself. Indeed $\mathfrak{F}$ is the basis space for a representation of the Lie group G and the Lie algebra $\mathcal{G}$. Thus, we can use the techniques of representation theory to study $\mathfrak{F}$. With this in mind we designate G as the symmetry group of Q.

We first describe a method for computing special solutions with a minimum of effort. Let $L_1, \cdots, L_k$ be a linearly independent set of pairwise commuting operators in $\mathcal{G}$ which is maximal with respect to these properties. Since these operators are commuting and leave $\mathfrak{F}$ invariant, it is possible that they have a simultaneous eigenvector in $\mathfrak{F}$, that is, a nonzero u such that

$$L_j u = \lambda_j u, \qquad 1 \leqq j \leqq k, \qquad Qu = 0, \qquad \lambda_j \in \mathfrak{C}. \tag{1.9}$$

If such a u exists it can often be computed rather easily from (1.9). Indeed the extra information $L_j u = \lambda_j u$ frequently reduces the problem to one of solving a series of ordinary differential equations rather than a partial differential equation [1]–[5]. (See § 2 for some examples.) These remarks relate our approach to the static theory mentioned above. Note that the eigenfunction u satisfies $(L_j - \lambda_j)u = 0$, where $L_j' = L_j - \lambda_j \in \mathcal{G}$. Thus $\exp(\alpha L_j')u = u$ for all α and u is invariant under the one-parameter subgroup of G generated by L_j'.

We remark that G may be a trivial one-dimensional Lie group in which case our method yields no information about the solutions of $Qu = 0$.

Ovsjannikov [16] has constructed a general theory of symmetries of (nonlinear) partial differential equations which essentially includes the above as a special case. However, for computational purposes the author's formulation is superior.

2. The Euler–Poisson–Darboux equation. We apply the method of § 1 to the EPD equation

$$u_{xx} - u_{yy} - \frac{k}{y} u_y = 0$$

by computing all linear differential operators

$$L = X(x, y)\partial_x + Y(x, y)\partial_y + U(x, y) \tag{2.1}$$

such that $[L, Q] = R(x, y)Q$, where

$$Q = \partial_{xx} - \partial_{yy} - \frac{k}{y}\partial_y \tag{2.2}$$

and $R(x, y)$ is a function depending on L. The results are

$$X = a(x^2 + y^2) + bx + c, \qquad Y = 2axy + by, \qquad U = axk + b\frac{k}{2} + d, \qquad a, b, c, d \in \mathfrak{C}. \tag{2.3}$$

Thus the EPD equation admits a four-dimensional symmetry algebra with basis

$$L_1 = -(x^2 + y^2)\partial_x - 2xy\partial_y - kx, \qquad L_2 = x\partial_x + y\partial_y + \frac{k}{2} \tag{2.4}$$

$$L_3 = \partial_x, \qquad E = 1.$$

Here

$$[L_2, L_1] = L_1, \qquad [L_2, L_3] = -L_3, \qquad [L_1, L_3] = 2L_2 \tag{2.5}$$

so the symmetry algebra $\mathscr{G}$ is isomorphic to $sl(2, \mathfrak{C}) \oplus \{E\}$. (In the exceptional cases, $k = 0, 2$, the symmetry algebra is infinite-dimensional and contains (2.4) as a subalgebra. For these values of k the EPD equation is equivalent to the wave equation.) Neglecting the trivial symmetry group generated by $\{E\}$, we can consider $SL(2, \mathfrak{C})$ as the symmetry group of the EPD equation. The group action of $SL(2, \mathfrak{C})$ is given in terms of the Lie algebra action by

$$T(g) = \exp\left(-\frac{b}{d}L_1\right)\exp(-cdL_3)\exp(\tau L_2), \qquad e^{\tau/2} = d^{-1}, \tag{2.6}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathfrak{C}), \qquad ad - bc = 1 \tag{2.7}$$

(see [8, p. 21]). A straightforward computation yields

$$\begin{aligned} [T(g)f](x, y) &= [(d - bx)^2 - b^2y^2]^{-k/2} \\ &\cdot f\left[\frac{ab(y^2 - x^2) + x(1 + 2bc) - cd}{(d - bx)^2 - b^2y^2}, \frac{y}{(d - bx)^2 - b^2y^2}\right], \qquad ad - bc = 1. \end{aligned} \tag{2.8}$$

Thus, for any solution f of $Qf = 0$ and any $g \in SL(2, \mathfrak{C})$ we have $Q(T(g)f) = 0$ whenever expression (2.8) makes sense. In the special case

$$e_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.9}$$

we find

$$[T(e_0)f](x, y) = [x^2 - y^2]^{-k/2} f\left[\frac{-x}{x^2 - y^2}, \frac{y}{x^2 - y^2}\right] \tag{2.10}$$

is a solution of the EPD equation whenever $f(x, y)$ is a solution.

We can obtain special solutions of the EPD equation by requiring that these solutions be invariant under one-parameter subgroups of $SL(2, \mathfrak{C})$. For example, it is easy to show that the space of solutions of the simultaneous equations

$$Qf = 0, \qquad L_2 f = (\mu + k/2)f \tag{2.11}$$

is spanned by

$$f_1(x, y) = (x - y)^{\mu}{}_2F_1(-\mu, k/2; 1 - \mu - k/2; (x + y)/(x - y)),$$

(2.12) $$f_2(x, y) = (x - y)^{-k/2}(x + y)^{\mu+k/2}{}_2F_1(k + \mu, k/2; 1 + \mu + k/2; (x + y)/(x - y)),$$

where ${}_2F_1(a, b; c; z)$ is a hypergeometric function [9]. Furthermore, the space of solutions of

(2.13) $$Qf = 0, \qquad L_3 f = \lambda f$$

is spanned by

(2.14) $$f(x, y) = y^{(1-k)/2}\, e^{\lambda x} J_{\pm(k-1)/2}(\lambda y),$$

where $J_\nu(z)$ is a Bessel function. We could use the $SL(2, \mathfrak{C})$ symmetry to derive identities for hypergeometric and Bessel functions, and transformation formulas for solutions of the EPD equation. However, very similar derivations are given in [8] and [14] so we shall not reproduce them here.

Next we look for transformations which map solutions of the EPD equation for one value of the parameter k into solutions for another value of k. In particular we study the operator

(2.15) $$Q^{(1)} = \partial_{xx} - \partial_{yy} - \frac{t}{y}\partial_{yt}.$$

The solutions $f_k(x, y)$ of $Qf_k = 0$ correspond to solutions $f(x, y, t)$ of $Q^{(1)}f = 0$ such that $f(x, y, t) = f_k(x, y)t^k$.

In analogy with our previous problem we compute all linear differential operators

(2.16) $$L = X(x, y, t)\partial_x + Y(x, y, t)\partial_y + T(x, y, t)\partial_t + U(x, y, t)$$

such that $[L, Q^{(1)}] = R(x, y, t)Q^{(1)}$. A tedious computation yields an eleven-dimensional symmetry algebra $\mathcal{G}$ with basis

(2.17) $$\begin{aligned}
L_1 &= -(x^2 + y^2)\partial_x - 2xy\partial_y - xt\partial_t,\\
L_2 &= x\partial_x + y\partial_y + \tfrac{1}{2}t\partial_t, \qquad L_3 = \partial_x,\\
L_4 &= \frac{t^2}{y}\partial_y, \qquad L_5 = t^2\partial_x + \frac{xt^2}{y}\partial_y,\\
L_6 &= \frac{y^2}{t^2}\partial_x + \frac{xy}{t^2}\partial_y + \frac{x}{t}\partial_t - \frac{x}{t^2},\\
L_7 &= \frac{y}{t^2}\partial_y + \frac{1}{t}\partial_t - \frac{1}{t^2}, \qquad L_8 = t\partial_t - 1,\\
L_9 &= 2t^2x\partial_x + \frac{(x^2 + y^2)}{y}t^2\partial_y + t^3\partial_t,\\
L_{10} &= \frac{2xy^2}{t^2}\partial_x + y\left(\frac{x^2 + y^2}{t^2}\right)\partial_y + \frac{x^2}{t}\partial_t + \frac{y^2 - x^2}{t^2}, \qquad E = 1.
\end{aligned}$$

Here $\mathscr{G} \cong \mathscr{G}' \oplus \{E\}$, where $\mathscr{G}'$ is a ten-dimensional Lie algebra with basis $L_1, \cdots, L_{10}$. Explicitly computing the commutation relations of the L_j one can verify that $\mathscr{G}'$ is a simple Lie algebra. Thus $\mathscr{G}' \cong so(5, \mathfrak{C})$ since $so(5, \mathfrak{C})$ is the only simple algebra of dimension ten [15]. The generators $\{L_2, L_8\}$ form a basis for a Cartan subalgebra of $\mathscr{G}'$.

The generators L_j of $so(5, \mathfrak{C})$ map solutions of the EPD equation into solutions, either fixing the parameter k or changing it by ± 2. Indeed, denoting a general solution of the EPD equation by $f_k(x, y)$ or $h_k(x, y)$ we see from (2.17) and the remarks following (2.15) that

$$\begin{aligned}
&\frac{1}{y}\partial_y f_k = f_{k+2}, \qquad y\partial_y f_k + (k-1)f_k = f_{k-2},\\
&2x\partial_x f_k + \frac{(x^2+y^2)}{y}\partial_y f_k + kf_k = f_{k+2},\\
&2xy^2\partial_x f_k + y(x^2+y^2)\partial_y f_k + (kx^2+y^2-x^2)f_k = f_{k-2},\\
&-(x^2+y^2)\partial_x f_k - 2xy\partial_y f_k - kxf_k = h_k, \qquad \partial_x f_k = h_k,\\
&\partial_x f_k + \frac{x}{y}\partial_y f_k = f_{k+2}, \qquad y^2\partial_x f_k + xy\partial_y f_k + (xk - x)f_k = f_{k-2}.
\end{aligned} \tag{2.18}$$

(Each of these eight relations is independent of the remaining ones.) In his study of boundary value problems for the EPD equation, Weinstein [10] made use of the first recurrence relation.

To determine the group action of $SO(5, \mathfrak{C})$ we note that each of the triplets

$$\begin{aligned}
\{J^+, J^-, J^3\} \equiv \{L_1, L_3, L_2\}, \qquad \{L_5, L_6, \tfrac{1}{2}L_8\},\\
\{\tfrac{1}{2}L_9, -\tfrac{1}{2}L_7, \tfrac{1}{2}L_2 + \tfrac{1}{4}L_8\}, \qquad \{\tfrac{1}{2}L_{10}, -\tfrac{1}{2}L_4, \tfrac{1}{2}L_2 - \tfrac{1}{4}L_8\}
\end{aligned} \tag{2.19}$$

satisfies the commutation relations

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^3$$

and forms a basis for a subalgebra of $so(5, \mathfrak{C})$ isomorphic to $sl(2, \mathfrak{C})$. It is easy to show that each triplet generates a Lie subgroup of $SO(5, \mathfrak{C})$ isomorphic to $SL(2, \mathfrak{C})$ and that the four subgroups so obtained suffice to generate the full group $SO(5, \mathfrak{C})$.

A straightforward computation [8] shows that $\{L_1, L_3, L_2\}$ generates the group action

$$\begin{aligned}
&[T_1(g)f](x, y, t)\\
&= f\left[\frac{ab(y^2-x^2)+x(1+2bc)-cd}{(d-bx)^2-b^2y^2}, \frac{y}{(d-bx)^2-b^2y^2}, \frac{t}{[(d-bx)^2-b^2y^2]^{1/2}}\right],\\
&\qquad g \in SL(2, \mathfrak{C}).
\end{aligned} \tag{2.20}$$

(For $f(x, y, t) = f_k(x, y)t^k$, (2.8) and (2.20) agree.) The triplet $\{L_5, L_6, \frac{1}{2}L_8\}$

generates the group action

$$(2.21)\quad \begin{aligned}[T_2(g)f](x,y,t) &= \left[a^2 - \frac{2xac}{t^2} + \frac{c^2y^2}{t^4}\right]^{-1/2} f\left[x(1+2bc) - t^2ab - cd\frac{y^2}{t^2},\right.\\ &\left. y\left(d^2 - \frac{2t^2}{y^2}xbd + \frac{t^4b^2}{y^2}\right)^{1/2}\left(a^2 - \frac{2xac}{t^2} + \frac{c^2y^2}{t^4}\right)^{1/2}, t\left(a^2 - \frac{2xac}{t^2} + \frac{c^2y^2}{t^4}\right)^{1/2}\right],\\ & g \in SL(2,\mathfrak{C}).\end{aligned}$$

If $g = e_0$ (see (2.9)) and $f(x,y,t) = f_k(x,y)t^k$, then (2.21) simplifies to

$$(2.22)\quad [T_2(e_0)f](x,y,t) = y^{k-1}f_k(-x,y)t^{2-k}$$

or $y^{k-1}f_k(-x,y) = f_{2-k}(x,y)$. The triplet $\{\frac{1}{2}L_9, -\frac{1}{2}L_7, \frac{1}{2}L_2 + \frac{1}{4}L_8\}$ generates the group action

$$(2.23)\quad \begin{aligned}[T_3(g)f](x,y,t) &= [a + c/t^2]^{-1/2}\\ &\cdot f\left[\frac{x}{d+t^2b}, \frac{y(a+c/t^2)^{1/2}}{d+bt^2}(d + bt^2(1 - x^2/y^2))^{1/2}, \frac{t(a+c/t^2)^{1/2}}{(d+t^2b)^{1/2}}\right],\end{aligned}$$

and the triplet $\{\frac{1}{2}L_{10}, -\frac{1}{2}L_4, \frac{1}{2}L_2 - \frac{1}{4}L_8\}$ generates the action

$$(2.24)\quad \begin{aligned}[T_4(g)f](x,y,t) &= [d + b(y^2 - x^2)/t^2]^{-1/2}\\ &\cdot f\left[\frac{x}{d + by^2/t^2}, \frac{y(d + b(y^2-x^2)/t^2)^{1/2}}{d + by^2/t^2}(a + ct^2/y^2)^{1/2},\right.\\ &\left. t\frac{(d + b(y^2-x^2)/t^2)^{1/2}}{(d + by^2/t^2)^{1/2}}\right].\end{aligned}$$

The operators (2.20), (2.21), (2.23), (2.24) determine the action of $SO(5,\mathfrak{C})$. In addition the operator $Q^{(1)}$ admits certain non-Lie symmetries. The most important are

$$(2.25)\quad S_1f(x,y,t) = f(-x,y,t), \qquad S_2f(x,y,t) = f(x,-y,t).$$

(In fact, $SO(5,\mathfrak{C})$ and the reflection S_2 generate a symmetry group isomorphic to $O(5,\mathfrak{C})$ and $S_1 \in O(5,\mathfrak{C})$.) The transformation $S_1T_2(e_0)$ or $y^{k-1}f_k(x,y) = f_{2-k}(x,y)$ was used by Weinstein in [10].

As with the ordinary EPD equation, we can obtain special solutions of $Q^{(1)}f = 0$ by requiring that these solutions be invariant under one-parameter subgroups of $SO(5,\mathfrak{C})$. In particular, the space of solutions of the simultaneous equations

$$(2.26)\quad Q^{(1)}f = 0, \qquad L_8f = (k-1)f, \qquad L_2f = (\mu + k/2)f$$

is spanned by

$$(2.27)\quad \begin{aligned} f_1(x,y,t) &= (x-y)^\mu t^k\, {}_2F_1(-\mu, k/2; 1-\mu-k/2; (x+y)/(x-y)),\\ f_2(x,y,t) &= (x-y)^{-k/2}(x+y)^{\mu+k/2}t^k\\ &\quad {}_2F_1(k+\mu, k/2; 1+\mu+k/2; (x+y)/(x-y))\end{aligned}$$

(for $\mu + k/2$ not an integer). Another basis is

$$f_3(x, y, t) = (x - y)^{\mu} t^k \, {}_2F_1(-\mu, k/2; k; -2y/(x - y)),$$

$$f_4(x, y, t) = (x - y)^{\mu+k-1}(-2y)^{1-k} t^k \cdot {}_2F_1(1 - \mu - k, 1 - k/2; 2 - k; -2y/(x - y)) \tag{2.28}$$

(for k not an integer). By applying Weisner's method [11], [14], [8], we can use the $SO(5, \mathfrak{C})$ symmetry to derive a variety of generating functions for the ${}_2F_1$. This will be carried out in another publication. Here, we merely show the intimate relationship between $SO(5, \mathfrak{C})$ symmetry and the quadratic transformation formulas for hypergeometric functions. From (2.23) and (2.28),

$$T_3(e_0) f_3(x, y, t) = h(w, z, t)$$

$$= i^{k-1} 2^{-\mu} w^{\mu} (1 - \sqrt{z})^{2\mu} t^{1-k-2\mu} \, {}_2F_1\left(-\mu, \frac{k}{2}; k; \frac{-4\sqrt{z}}{(1 - \sqrt{z})^2}\right), \tag{2.29}$$

$$w = x - y, \qquad z = (x + y)/(x - y).$$

Now h is a solution of

$$Q^{(1)} h = 0, \qquad L_8 h = (-k - 2\mu) h, \qquad L_2 h = (-\mu/2 - k/2 + \tfrac{1}{2}) h,$$

which is bounded for general μ, k at $z = 0$. Hence h is a constant multiple of $f_1(x, y, t)$ (for $k = 1 - k - 2\mu$, $\mu = \mu$):

$$(1 - \sqrt{z})^{2\mu} \, {}_2F_1\left(-\mu, \frac{k}{2}; k; \frac{-4\sqrt{z}}{(1 - \sqrt{z})^2}\right) = c_0 \, {}_2F_1\left(-\mu, \frac{1}{2} - \frac{k}{2} - \mu; \frac{1}{2} + \frac{k}{2}; z\right). \tag{2.30}$$

Letting $z = 0$ we find $c_0 = 1$ and (2.30) yields a quadratic transformation formula for the ${}_2F_1$. Similarly, $T_3(e_0) f_1(x, y, t)$ leads to the identity

$$\left(\frac{1 - \sqrt{1 - v}}{2}\right)^{2\mu} {}_2F_1\left(-\mu, \frac{k}{2}; 1 - \mu - \frac{k}{2}; \left(\frac{1 + \sqrt{1 - v}}{1 - \sqrt{1 - v}}\right)^2\right) = {}_2F_1\left(-\mu, \frac{1}{2} - \mu - \frac{k}{2}; 1 - 2\mu - k; v\right), \qquad v = -2y/(x - y), \tag{2.31}$$

and $T_4(e_0) f_3(x, y, t)$ leads to

$$(1 - \sqrt{z})^{2\mu} (1 - z)^{-k-2\mu} \, {}_2F_1\left(-\mu, \frac{k}{2}; k; \frac{-4\sqrt{z}}{(1 - \sqrt{z})^2}\right) = {}_2F_1\left(\mu + \frac{k}{2} + \frac{1}{2}, \mu + k; \frac{k}{2} + \frac{1}{2}; z\right). \tag{2.32}$$

Additional formulas of this type can be derived from (2.23) and (2.24) by making use of the transformation formulas for the ${}_2F_1$.

Note added later. Just as in the remark preceding (3.23) in the next section one can easily show that the equation $Q^{(1)} f = 0$ is equivalent to the partial differential equation obtained from (3.23) by setting $c = a - b + 1$ and replacing a and b by differential operators. In this sense $O(5,\mathfrak{C})$ is the natural symmetry group of the ultraspherical functions. Past studies of these functions have used only $SL(2,\mathfrak{C})$ symmetry [8], [17].

3. The Euler–Darboux equation. The Euler–Darboux equation

$$u_{\xi\eta} + \frac{1}{\xi - \eta}(\alpha u_\eta - \beta u_\xi) = 0 \tag{3.1}$$

is a generalization of the EPD equation. In fact, if the constants α, β are chosen so that $\alpha = \beta = k/2$ and new variables x, y are introduced so that $\xi = y - x$, $\eta = -y - x$, then (3.1) becomes the EPD equation. To find the symmetry algebra of (3.1) we look for all linear differential operators

$$L = X(\xi,\eta)\partial_\xi + Y(\xi,\eta)\partial_\eta + V(\xi,\eta)$$

such that $[L, P] = R(\xi,\eta)P$, where

$$P = \partial_{\xi\eta} + \frac{\alpha}{\xi - \eta}\partial_\eta - \frac{\beta}{\xi - \eta}\partial_\xi. \tag{3.2}$$

The results are

$$\begin{gathered} X = a\xi + b\xi^2 - c, \qquad Y = a\eta + b\eta^2 - c, \\ V = \frac{a}{2}(\alpha + \beta) + b(\xi\alpha + \eta\beta) + d, \qquad a,b,c,d \in \mathfrak{C}. \end{gathered} \tag{3.3}$$

Clearly the Euler–Darboux equation admits a four-dimensional symmetry algebra with basis

$$\begin{gathered} L_1 = \xi^2\partial_\xi + \eta^2\partial_\eta + \alpha\xi + \beta\eta, \qquad L_2 = \xi\partial_\xi + \eta\partial_\eta + (\alpha + \beta)/2, \\ L_3 = -\partial_\xi - \partial_\eta, \qquad E = 1. \end{gathered} \tag{3.4}$$

The commutation relations of the L_j are

$$[L_2, L_1] = L_1, \qquad [L_2, L_3] = -L_3, \qquad [L_1, L_3] = 2L_2 \tag{3.5}$$

so the symmetry algebra is again isomorphic to $sl(2,\mathfrak{C}) \oplus \{E\}$. The group action is given in terms of the Lie algebra action by (2.6) and (2.7). A standard computation yields

$$[T(g)f](\xi,\eta) = (d + b\xi)^{-\alpha}(d + b\eta)^{-\beta} f\left[\frac{a\xi + c}{d + b\xi}, \frac{a\eta + c}{d + b\eta}\right], \qquad ad - bc = 1. \tag{3.6}$$

Thus, if $Pf = 0$, then $P(T(g)f) = 0$ for any $g \in SL(2,\mathfrak{C})$ such that (3.6) makes sense. An interesting special case is obtained for $g = e_0$: If $f(\xi,\eta)$ is a solution of the Euler–Darboux equation, then so is $\xi^{-\alpha}\eta^{-\beta}f(-\xi^{-1}, -\eta^{-1})$.

It is easy to show that the space of solutions of the simultaneous equations

$$Pf = 0, \qquad L_2 f = \left(\mu + \frac{\alpha}{2} + \frac{\beta}{2}\right)f, \qquad \mu \in \mathfrak{C}, \tag{3.7}$$

is spanned by

$$f_1(\xi,\eta) = \xi^{\mu}\,{}_2F_1(-\mu,\beta;1-\mu-\alpha;\eta/\xi),$$
(3.8)
$$f_2(\xi,\eta) = \xi^{-\alpha}\eta^{\mu+\alpha}\,{}_2F_1(\alpha,\mu+\alpha+\beta;1+\mu+\alpha;\eta/\xi)$$

for $\mu + \alpha$ not an integer. Indeed, the second equation implies $f = \xi^{\mu}h(\eta/\xi)$. Substituting this result in $Pf = 0$ we find h must be a solution of the hypergeometric differential equation. It is clear from this result that the hypergeometric functions are intimately related to the Euler–Darboux equation. Indeed the partial differential equation introduced by Weisner [14] in his group-theoretic treatment of hypergeometric functions is just the Euler–Darboux equation (to within a change of independent variables). Thus, detailed applications of $SL(2,\mathfrak{C})$ symmetry to obtain solutions of the Euler–Darboux equations and transformation properties of these solutions are already contained in [8], [14] and need not be repeated here.

Here we concern ourselves with transformations which map solutions of the Euler–Darboux equation corresponding to (α, β) into solutions of the equation corresponding to (α', β'). To find such transformations we study the operator

$$P^{(1)} = \partial_{\xi\eta} + \frac{1}{\xi-\eta}(t\partial_{t\eta} - u\partial_{u\xi}). \tag{3.9}$$

The solutions $f_{\alpha,\beta}(\xi,\eta)$ of $Pf_{\alpha,\beta} = 0$ correspond to solutions $f(\xi,\eta,t,u)$ of $P^{(1)}f = 0$ such that $f(\xi,\eta,t,u) = f_{\alpha,\beta}(\xi,\eta)t^{\alpha}u^{\beta}$.

To find the symmetry algebra of $P^{(1)}f = 0$ we determine all linear differential operators

$$L = X\partial_{\xi} + Y\partial_{\eta} + T\partial_t + U\partial_u + V \tag{3.10}$$

such that $[L, P^{(1)}] = R(\xi,\eta,t,u)P^{(1)}$. Here, X, Y, T, U and V are functions of ξ,η,t,u to be determined. A tedious computation shows that the symmetry algebra $\mathcal{G}$ is sixteen-dimensional with basis

(3.11)
$$\begin{aligned}
L_1 &= \xi^2\partial_{\xi} + \eta^2\partial_{\eta} + \xi t\partial_t + \eta u\partial_u,\\
L_2 &= \xi\partial_{\xi} + \eta\partial_{\eta} + \tfrac{1}{2}(t\partial_t + u\partial_u), \qquad L_3 = -\partial_{\xi} - \partial_{\eta},\\
L_4 &= t\partial_t - \tfrac{1}{2}, \qquad L_5 = u\partial_u - \tfrac{1}{2}, \qquad L_6 = t\partial_{\xi}, \qquad L_7 = u\partial_{\eta},\\
L_8 &= \xi t\partial_{\xi} + t^2\partial_t, \qquad L_9 = u\eta\partial_{\eta} + u^2\partial_u,\\
L_{10} &= \frac{(\xi-\eta)}{u}\partial_{\xi} + \frac{t}{u}\partial_t + \partial_u - \frac{1}{u},\\
L_{11} &= \frac{(\eta-\xi)}{t}\partial_{\eta} + \partial_t + \frac{u}{t}\partial_u - \frac{1}{t},\\
L_{12} &= \frac{t}{u}(\eta-\xi)\partial_{\xi} - \frac{t^2}{u}\partial_t, \qquad L_{13} = \frac{u}{t}(\xi-\eta)\partial_{\eta} - \frac{u^2}{t}\partial_u,\\
L_{14} &= \frac{\xi}{u}(\xi-\eta)\partial_{\xi} + \frac{\xi t}{u}\partial_t + \eta\partial_u - \frac{\eta}{u},\\
L_{15} &= \frac{\eta}{t}(\eta-\xi)\partial_{\eta} + \xi\partial_t + \frac{\eta u}{t}\partial_u - \frac{\xi}{t}, \qquad E = 1.
\end{aligned}$$

Computing the commutation relations, we find that the operators L_j form a basis for a simple fifteen-dimensional Lie algebra, necessarily isomorphic to $sl(4,\mathfrak{C})$ [15]. The operators $\{L_2, L_4, L_5\}$ span a Cartan subalgebra of $sl(4,\mathfrak{C})$. Thus, $\mathscr{G} \cong sl(4,\mathfrak{C}) \oplus \{E\}$ and we can consider $sl(4,\mathfrak{C})$ as the symmetry algebra of $P^{(1)}$.

The operators L_j map solutions of the Euler–Darboux equation for (α, β) into solutions $(\alpha + \varepsilon, \beta + \rho)$, $\varepsilon, \rho = 0, \pm 1$. Indeed from (3.11) and the remarks following (3.9) we find

$$
\begin{aligned}
&(\xi^2\partial_\xi + \eta^2\partial_\eta + \xi\alpha + \eta\beta)f_{\alpha,\beta} = h_{\alpha,\beta},\\
&-(\partial_\xi + \partial_\eta)f_{\alpha,\beta} = h_{\alpha,\beta}, \qquad \partial_\xi f_{\alpha,\beta} = f_{\alpha+1,\beta}, \qquad \partial_\eta f_{\alpha,\beta} = f_{\alpha,\beta+1},\\
&(\xi\partial_\xi + \alpha)f_{\alpha,\beta} = f_{\alpha+1,\beta}, \qquad (\eta\partial_\eta + \beta)f_{\alpha,\beta} = f_{\alpha,\beta+1},\\
&((\xi - \eta)\partial_\xi + \alpha + \beta - 1)f_{\alpha,\beta} = f_{\alpha,\beta-1},\\
&((\eta - \xi)\partial_\eta + \alpha + \beta - 1)f_{\alpha,\beta} = f_{\alpha-1,\beta},\\
&(\xi(\xi - \eta)\partial_\xi + \alpha\xi + \beta\eta - \eta)f_{\alpha,\beta} = f_{\alpha,\beta-1},\\
&(\eta(\eta - \xi)\partial_\eta + \alpha\xi + \beta\eta - \xi)f_{\alpha,\beta} = f_{\alpha-1,\beta},
\end{aligned}
\tag{3.12}
$$

where $f_{\alpha,\beta}(\xi,\eta)$, $h_{\alpha,\beta}(\xi,\eta)$ are general solutions of the Euler–Darboux equation. (Each of equations (3.12) is independent of the remaining ones.) In addition, the operators L_{12}, L_{13} induce the coupled equations

$$
((\xi - \eta)\partial_\eta - \beta)f_{\alpha,\beta} = f_{\alpha-1,\beta+1}, \qquad ((\eta - \xi)\partial_\xi - \alpha)f_{\alpha-1,\beta+1} = \beta(\alpha - 1)f_{\alpha\beta}.
\tag{3.13}
$$

It is interesting to note that the $so(5,\mathfrak{C})$ operators of the EPD equation do not form a subalgebra of the above $sl(4,\mathfrak{C})$ operators even though the EPD equation is a special case of the Euler–Darboux equation.

To determine the group action of $SL(4,\mathfrak{C})$ we remark that each of the triplets

$$
\begin{aligned}
&\{J^+, J^-, J^3\} \equiv \{L_1, L_3, L_2\}, \qquad \{L_6, L_{15}, -\tfrac{1}{2}L_2 + \tfrac{3}{4}L_4 + \tfrac{1}{4}L_5\},\\
&\{L_7, L_{14}, -\tfrac{1}{2}L_2 + \tfrac{3}{4}L_5 + \tfrac{1}{4}L_4\}, \qquad \{L_8, -L_{11}, \tfrac{1}{2}L_2 + \tfrac{1}{4}L_5 + \tfrac{3}{4}L_4\},\\
&\{L_9, -L_{10}, \tfrac{1}{2}L_2 + \tfrac{1}{4}L_4 + \tfrac{3}{4}L_5\}, \qquad \{L_{13}, L_{12}, \tfrac{1}{2}L_5 - \tfrac{1}{2}L_4\}
\end{aligned}
\tag{3.14}
$$

satisfies the commutation relations

$$
[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^3
$$

and forms a basis for a subalgebra of $sl(4,\mathfrak{C})$ isomorphic to $sl(2,\mathfrak{C})$. Furthermore, each triplet generates a subgroup of $SL(4,\mathfrak{C})$ isomorphic to $SL(2,\mathfrak{C})$ and the six subgroups so obtained generate the full symmetry group $SL(4,\mathfrak{C})$.

A routine calculation shows that $\{L_1, L_3, L_2\}$ generates the group action

$$
[T_1(g)f](\xi,\eta,t,u) = f\left[\frac{a\xi + c}{d + b\xi}, \frac{a\eta + c}{d + b\eta}, \frac{t}{d + b\xi}, \frac{u}{d + b\eta}\right].
\tag{3.15}
$$

For $f(\xi, \eta, t, u) = f_{\alpha,\beta}(\xi, \eta)t^{\alpha}u^{\beta}$, expressions (3.6) and (3.15) agree. The triplet $\{L_6, L_{15}, -\frac{1}{2}L_2 + \frac{3}{4}L_4 + \frac{1}{4}L_5\}$ generates

$$[T_2(g)f](\xi,\eta,t,u) = (a - c\xi/t)^{-1}f\left[d\xi - bt, \frac{\eta t}{at - c(\xi - \eta)}, at - c\xi, \frac{u(at - c\xi)}{at - c(\xi - \eta)}\right] \tag{3.16}$$

while $\{L_7, L_{14}, -\frac{1}{2}L_2 + \frac{3}{4}L_5 + \frac{1}{4}L_4\}$ generates

$$[T_3(g)f](\xi,\eta,t,u) = (a - c\eta/u)^{-1}f\left[\frac{\xi u}{au - c(\eta - \xi)}, d\eta - bu, \frac{t(au - c\eta)}{au - c(\eta - \xi)}, au - c\eta\right]. \tag{3.17}$$

The triplet $\{L_8, -L_{11}, \frac{1}{2}L_2 + \frac{1}{4}L_5 + \frac{3}{4}L_4\}$ generates the action

$$[T_4(g)f](\xi,\eta,t,u) = (a + c/t)^{-1}f\left[\frac{\xi}{d + bt}, \eta(a + c/t) - \xi c/t, \frac{at + c}{d + bt}, u(a + c/t)\right] \tag{3.18}$$

and $\{L_9, -L_{10}, \frac{1}{2}L_2 + \frac{1}{4}L_4 + \frac{3}{4}L_5\}$ generates

$$[T_5(g)f](\xi,\eta,t,u) = (a + c/u)^{-1}f\left[\xi(a + c/u) - \eta c/u, \frac{\eta}{d + bu}, t(a + c/u), \frac{au + c}{d + bu}\right]. \tag{3.19}$$

Finally, $\{L_{13}, L_{12}, \frac{1}{2}L_5 - \frac{1}{2}L_4\}$ generates

$$[T_6(g)f](\xi,\eta,t,u) = f\left[\frac{au\xi - ct\eta}{au - ct}, \frac{dt\eta - bu\xi}{dt - bu}, \frac{tu}{au - ct}, \frac{tu}{dt - bu}\right]. \tag{3.20}$$

Operators (3.15)–(3.20) generate the group action of $SL(4, \mathbb{C})$.

As usual we can determine symmetry adapted solutions of $P^{(1)}f = 0$ by requiring that f be invariant under various one-parameter subgroups of the complete symmetry group. The most important example is the system

$$P^{(1)}f = 0, \qquad L_4 f = (\alpha - \tfrac{1}{2})f, \qquad L_5 f = (\beta - \tfrac{1}{2})f, \qquad L_2 f = (\mu + \alpha/2 + \beta/2)f, \tag{3.21}$$

whose solution space is spanned by

$$\begin{aligned} f_1(\xi,\eta,t,u) &= \xi^{\mu}\,{}_2F_1(-\mu, \beta; 1 - \mu - \alpha; \eta/\xi)t^{\alpha}u^{\beta}, \\ f_2(\xi,\eta,t,u) &= \xi^{-\alpha}\eta^{\mu+\alpha}\,{}_2F_1(\alpha, \mu + \alpha + \beta; 1 + \mu + \alpha; \eta/\xi)t^{\alpha}u^{\beta} \end{aligned} \tag{3.22}$$

for $\mu + \alpha$ not an integer. The significance of the twelve recurrence relations (3.12), (3.13) when applied to these special solutions is revealing. The relations correspond exactly to the twelve differential recurrence relations which raise and lower the parameters a, b, c of ${}_2F_1(a, b; c; z)$.

Remark. If the parameters a, b, c in the hypergeometric equation

$$z(1 - z)\frac{d^2f}{dz^2} + [c - (a + b + 1)z]\frac{df}{dz} - abf = 0 \tag{3.23}$$

are replaced by $q\partial_q$, $r\partial_r$, $s\partial_s$ as in the Euler–Darboux equation, then the resulting partial differential equation is equivalent to $P^{(1)}f = 0$. In this sense the "natural" symmetry group of the hypergeometric equation is $SL(4, \mathfrak{C})$. Past group-theoretic treatments of the ${}_2F_1$ have used only $SL(2, \mathfrak{C})$ symmetry. Use of the full invariance group leads to a variety of new identities via Weisner's method. In this paper we have expressed our results in terms of the Euler–Darboux equation. For the purpose of applying Weisner's method and $SL(4, \mathfrak{C})$ symmetry to derive identities for the ${}_2F_1$ it is more convenient to start with (3.23). Except for a single example, this study will be undertaken in another publication. Here we present some properties of the ${}_2F_1$ which follow immediately from equations (3.15)–(3.20).

From (2.9), (3.15) and (3.22),

$$\begin{aligned}&[T_1(e_0)f_1](\xi,\eta,t,u) = h\\ &= (-1)^\mu \xi^{-\mu-\alpha-\beta} z^{-\beta}\, {}_2F_1(-\mu,\beta;1-\mu-\alpha;z^{-1}) t^\alpha u^\beta, \qquad z = \eta/\xi.\end{aligned} \tag{3.24}$$

Here, h is a solution of

$$\begin{aligned}P^{(1)}h = 0, \qquad L_4 h = (\alpha - \tfrac{1}{2})h, \qquad L_5 h = (\beta - \tfrac{1}{2})h,\\ L_2 h = \left(-\mu - \frac{\alpha}{2} - \frac{\beta}{2}\right)h,\end{aligned} \tag{3.25}$$

that is, α and β are unchanged while μ changes to $-\mu-\alpha-\beta$. Since f_1, f_2 form a basis for the solutions of (3.25) there must exist constants c_1, c_2 such that

$$\begin{aligned}&(-z)^{-\beta}\, {}_2F_1(-\mu,\beta;1-\mu-\alpha;z^{-1})\\ &= c_1\, {}_2F_1(\mu+\alpha+\beta,\beta;1+\mu+\beta;z) + c_2(-z)^{-\mu-\beta}\, {}_2F_1(\alpha,-\mu;1-\mu-\beta;z).\end{aligned} \tag{3.26}$$

The constants are easily computed:

$$c_2 = \frac{\Gamma(1-\mu-\alpha)\Gamma(\beta+\mu)}{\Gamma(\beta)\Gamma(1-\alpha)}, \qquad c_1 = \frac{\Gamma(1-\mu-\alpha)\Gamma(-\mu-\beta)}{\Gamma(-\mu)\Gamma(1-\mu-\alpha-\beta)}, \tag{3.27}$$

where $\Gamma(\alpha)$ is the gamma function [9, vol. I, p. 108]. Similarly, evaluation of $T_4(e_0)f_1$ shows that ${}_2F_1(-\mu, \beta; 1-\mu-\alpha; 1-z)$ is a linear combination of ${}_2F_1(-\mu,\beta;\alpha+\beta;z)$ and $z^{-\alpha-\beta+1}\, {}_2F_1(1-\mu-\alpha-\beta, 1-\alpha; 2-\alpha-\beta; z)$.

The function

$$\begin{aligned}&[T_5(e_0)f_1](\xi,\eta,t,u) = h\\ &= \xi^\mu (1-z)^\mu {}_2F_1\left(-\mu,\beta;1-\mu-\alpha;\frac{z}{z-1}\right) t^\alpha u^{1-\mu-\alpha-\beta}, \qquad z = \eta/\xi,\end{aligned} \tag{3.28}$$

is a solution of (3.21) with $\alpha = \alpha$, $\beta = 1-\mu-\alpha-\beta$, $\mu = \mu$. Furthermore, h is analytic in z at $z = 0$. Therefore, $h = cf_1$ and setting $z = 0$ we see that $c = 1$:

$$\begin{aligned}&(1-z)^\mu\, {}_2F_1\left(-\mu,\beta;1-\mu-\alpha;\frac{z}{z-1}\right)\\ &= {}_2F_1(-\mu, 1-\mu-\alpha-\beta; 1-\mu-\alpha; z).\end{aligned} \tag{3.29}$$

This is one of the transformation formulas for the hypergeometric functions [9]. Similarly, $T_2(e_0)f_1$ leads to the transformation formula

$$(3.30) \qquad (1-z)^{-\beta}\,{}_2F_1\left(-\mu,\beta;1-\mu-\alpha;\frac{z}{z-1}\right) = {}_2F_1(1-\alpha,\beta;1-\mu-\alpha;z)$$

and $T_5(e_0)T_2(e_0)f_1$ leads to

$$(3.31) \qquad \begin{aligned}(1-z)^{\alpha+\beta-1}\,&{}_2F_1(-\mu,\beta;1-\mu-\alpha;z)\\ &= {}_2F_1(1-\alpha,1-\mu-\alpha-\beta;1-\mu-\alpha;z).\end{aligned}$$

Combining these results we can obtain Kummer's 24 solutions of the hypergeometric equation [9, vol. I, p. 105].

There are also non-Lie symmetries of the equation $P^{(1)}f = 0$. For example, the transformation formula (3.31) suggests the symmetry transformation

$$(3.32) \qquad [Sf](\xi,\eta,t,u) = \frac{tu}{\xi-\eta} f\left(\xi,\eta,\frac{\xi-\eta}{u},\frac{\xi-\eta}{t}\right)$$

which can easily be verified directly. The action of S on f_1 leads to an identity equivalent to (3.31). If $f = f_{\alpha\beta}t^\alpha u^\beta$, application of S yields

$$(3.33) \qquad f_{1-\beta,1-\alpha}(\xi,\eta) = (\xi-\eta)^{\alpha+\beta-1} f_{\alpha\beta}(\xi,\eta).$$

This transformation appears not to be a direct consequence of $SL(4,\mathfrak{C})$ symmetry.

In conclusion we present an example of the use of Weisner's method and $SL(4,\mathfrak{C})$ symmetry to obtain generating functions for the ${}_2F_1$ which are not consequences of $SL(2,\mathfrak{C})$ symmetry. (A systematic derivation of possible generating functions will be carried out in another publication.) We follow the method described in § 1 and [11]. Let $f(\xi,\eta,t,u)$ be a solution of the simultaneous equations

$$(3.34) \qquad \begin{gathered}(L_2 - \tfrac{1}{2}L_4 - \tfrac{3}{2}L_5)f = (\gamma+1)f, \qquad (L_2 + \tfrac{1}{2}L_4 - \tfrac{1}{2}L_5)f = \rho f,\\ P^{(1)}f = 0, \qquad (L_{10} + L_{15})f = 0,\end{gathered}$$

which is analytic at $\eta = 0$. The first two equations imply

$$f = u^{-\gamma}t^\rho h\left(\frac{\xi u}{t},\frac{\eta}{\xi}\right)$$

and the third implies

$$h = k\left(\frac{\eta u}{t}\left(\frac{u\xi}{t}-1\right)^{-2}\right).$$

Substituting these results into $P^{(1)}f = 0$ we find

$$(3.35) \quad f = (1-z)^\gamma\,{}_2F_1\left(\frac{-\gamma}{2},\frac{1-\gamma}{2};1-\rho;\frac{-4zw}{(1-z)^2}\right)t^\rho u^{-\gamma}, \qquad z = u\xi/t,\ w = \frac{\eta}{\xi},$$

unique to within a multiplicative constant. Since f can be expanded in a power series in z, (3.21) and the methods of [11] imply

$$(1-z)^{\gamma}\,{}_2F_1\left(\frac{-\gamma}{2},\frac{1-\gamma}{2};1-\rho;\frac{-4zw}{(1-z)^2}\right) = \sum_{n=0}^{\infty} c_n z^n\,{}_2F_1(-n,n-\gamma;1-\rho;w), \qquad |z|<1. \tag{3.36}$$

Setting $w = 0$ in (3.35) we find

$$c_n = \binom{n-\gamma-1}{n}.$$

(Another group-theoretic derivation of (3.36) is given in [11].) We can obtain more identities by applying the group operators of $SL(4, \mathfrak{C})$ to f and expanding the resulting function as a series in the functions f_1. Similar methods applied to the equation $Q^{(1)}f = 0$ enable one to derive generating functions for certain subclasses of the ${}_2F_1$ via $SO(5, \mathfrak{C})$ symmetry.

Acknowledgment. The author wishes to thank the referees for several helpful suggestions.

REFERENCES

[1] G. Birkhoff, *Hydrodynamics*, Princeton University Press, Princeton, 1960.
[2] A. J. A. Morgan, *The reduction by one of the number of independent variables in some systems of partial differential equations*, Quart. J. Math. Oxford Ser. (2), 3 (1952), pp. 250–259.
[3] M. J. Moran and R. A. Gaggioli, *A new systematic formalism for similarity analysis*, J. Engrg. Math., 3 (1969), pp. 151–162.
[4] G. W. Bluman and J. D. Cole, *The general similarity solution of the heat equation*, J. Math. Mech., 18 (1969), pp. 1025–1042.
[5] M. J. Moran and R. A. Gaggioli, *On the reduction of differential equations to algebraic equations*, this Journal, 1 (1970), pp. 37–40.
[6] S. Lie, *Theorie der Transformationsgruppen*, vols. I, II, III, Unter Mitwirkung von F. Engel, Leipzig, 1888, 1890, 1893.
[7] P. M. Cohn, *Lie Groups*, Cambridge University Press, London and New York, 1957.
[8] W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, New York, 1968.
[9] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vols. I, II, McGraw-Hill, New York, 1953.
[10] A. Weinstein, *The generalized radiation problem and the Euler–Poisson–Darboux equation*, Summa Brasiliensis Mathematicae, 3 (1955), pp. 125–146.
[11] W. Miller, Jr., *Lie theory and generalized hypergeometric functions*, this Journal, 3 (1972), pp. 31–44.
[12] L. Weisner, *Generating functions for Hermite functions*, Canad. J. Math., 11 (1959), pp. 141–147.
[13] N. Koshlyakow, M. Smirnow and E. Gliner, *Differential Equations of Mathematical Physics*, North-Holland, Amsterdam, 1964.
[14] L. Weisner, *Group-theoretic origin of certain generating functions*, Pacific J. Math., 5 (1955), pp. 1033–1039.
[15] N. Jacobson, *Lie Algebras*, Interscience, New York, 1962.
[16] L. Ovsjannikov, *Group Properties of Differential Equations*, Academy of Sciences USSR, Novasibirsk, 1962. (In Russian.)
[17] B. Viswanathan, *Generating functions for ultraspherical functions*, Canad. J. Math., 20 (1968), pp. 120–134.

SIAM J. MATH. APPL.
Vol. 4, No. 2, May 1973

CAPACITY AND THE NONLINEAR NAVIER-STOKES EQUATIONS*

VICTOR L. SHAPIRO†

Abstract. A classical result concerning capacity theory and removable sets for harmonic functions of finite energy is extended to solutions of the nonlinear Navier-Stokes equations. The theory of multiple trigonometric series is used in proving the basic lemma, and a new theorem concerning capacity theory and removable sets for first order systems is also established.

1. Introduction. Let Ω be a bounded domain in Euclidean N-space, E_N, $N \geqq 2$, and let $f = (f_1, \cdots, f_N)$ be a fixed vector in $L_1(\Omega)$. Also let $u = (u_1, \cdots, u_N)$ and p represent respectively a vector in $W_2^1(\Omega)$ and a function in $L_2(\Omega)$ (where in $W_k^j(\Omega)$ the j corresponds to the number of derivatives).

We shall deal for the most part in this paper with the nonlinear stationary Navier-Stokes equations [4, p. 115],

$$
\begin{aligned}
&\nu\Delta u_i - u_j\partial u_i/\partial x_j - \partial p/\partial x_i + f_i = 0, \qquad i = 1, \cdots, N,\\
&\partial u_j/\partial x_j = 0,
\end{aligned} \tag{1.1}
$$

where ν is a constant. (In § 5, we deal with the nonlinear nonstationary Navier-Stokes equation.)

From a classical point of view the system (1.1) is equivalent to the following system:

$$
\begin{aligned}
&\nu\Delta u_i - \partial[u_i u_j]/\partial x_j - \partial p/\partial x_i + f_i = 0, \qquad i = 1, \cdots, N,\\
&\partial u_j/\partial x_j = 0.
\end{aligned} \tag{1.1'}
$$

Consequently, and in view of the fact that u is in $W_2^1(\Omega)$ we shall say (u, p) is a distribution solution of (1.1) in Ω_1, an open subset of Ω, if the following holds:

$$
\begin{aligned}
&\int_{\Omega_1} [\nu u_i\Delta\phi + u_i u_j\partial\phi/\partial x_j + p\partial\phi/\partial x_i + \phi f_i]\, dx = 0, \qquad i = 1, \cdots, N,\\
&\int_{\Omega_1} [u_j\partial\phi/\partial x_j]\, dx = 0 \quad \text{for all } \phi \text{ in } \quad C_0^\infty(\Omega_1).
\end{aligned} \tag{1.2}
$$

By capacity in this paper we shall mean ordinary capacity. In particular, if Z is a relatively closed set in Ω, we shall say Z is of capacity zero (or sometimes of ordinary capacity zero in E_N) if

$$\int_Z\int_Z |x - y|^{2-N}\, d\mu(x)\, d\mu(y) = +\infty \quad \text{for} \quad N \geqq 3,$$

$$\int_Z\int_Z \log|x - y|^{-1}\, d\mu(x)\, d\mu(y) = +\infty \quad \text{for} \quad N = 2$$

* Received by the editors December 28, 1971.

† Department of Mathematics, University of California, Riverside, California 92502. This research was sponsored in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant AFOSR 71-2046.

for all nonnegative Borel measures in Ω having their support in Z with $\mu(Z) = 1$.

We shall say (u, p) is in class $\mathscr{A}(\Omega)$ if u and p are as described above, that is, u_i is in $W_2^1(\Omega)$ for $i = 1, \cdots, N$ and p is in $L_2(\Omega)$.

We shall say a relatively closed set $Z \subset \Omega$ is a removable set for the nonlinear stationary Navier–Stokes equations with respect to the class $\mathscr{A}(\Omega)$ if the following holds:

If (u, p) *in* $\mathscr{A}(\Omega)$ *is a distribution solution of* (1.1) *in* $\Omega - Z$, *then* (u, p) *is a distribution solution of* (1.1) *in* Ω.

Motivated somewhat by [2] (which was in turn motivated by [3]), by [1], and by [6], we intend to establish the following result.

THEOREM 1. *If* $Z \subset \Omega$ *is a relatively closed set of capacity zero, then with respect to the class* $\mathscr{A}(\Omega)$, Z *is a removable set for the nonlinear stationary Navier–Stokes equations.*

With $H(\Omega)$ designating the subclass of vectors in $W_2^1(\Omega)$ defined in [4, p. 115], we shall obtain as a corollary to Theorem 1 the following result.

COROLLARY 1. *Let* $Z \subset \Omega$ *be a relatively closed set of capacity zero. Also let* u *be in* $H(\Omega)$, p *be in* $L_2(\Omega)$, *and* f *satisfy a Hölder condition in* Ω. *Suppose that* (u, p) *is a classical solution of* (1.1) *in* $\Omega - Z$. *Then* (u, p) *is a classical solution of* (1.1) *in* Ω *for* $N = 2$ *or* 3.

To be quite explicit, when we say (u, p) is a classical solution of (1.1) in Ω_1, an open subset of Ω, we mean that u_i and p are respectively in $C^2(\Omega_1)$ and $C^1(\Omega_1)$ for $i = 1, \cdots, N$ and satisfy (1.1) at each point x in Ω_1.

We shall say that Ω is a cylindrical domain in E_N if there exists a domain Ω^* in E_{N-1} and an open interval (a, b) of the real line such that $\Omega = \Omega^* \times (a, b)$. Similarly, we shall say Z is a cylindrical set in E_N if there exists a set Z^* in E_{N-1} such that $Z = Z^* \times (a, b)$.

If f is the identically zero vector, we shall refer to (1.1) as the nonlinear stationary Navier–Stokes equations with zero external force.

As a corollary to Theorem 1 and [1, p. 88], we shall also obtain the following result.

COROLLARY 2. *Let* Ω *be a bounded cylindrical domain in* E_N, $N \geqq 3$, *and let* Z *be a relatively closed, cylindrical set contained in* Ω. *Then a necessary and sufficient condition that* Z *be a removable set with respect to the class* $\mathscr{A}(\Omega)$ *for the nonlinear stationary Navier–Stokes equation with zero external force is that* Z *be of capacity zero.*

We shall deal with the analogue of Theorem 1 for the nonlinear nonstationary Navier–Stokes equations in § 5.

We shall use the standard summation conventions in §§ 1, 2, 4 and 5. In § 3, which deals with multiple trigonometric series, we shall not use the convention when dealing with Fourier coefficients.

2. Proof of Theorem 1, Corollary 1 and Corollary 2. We first state Theorem A.

THEOREM A. *Let* $Z \subset \Omega$ *be a relatively closed set of capacity zero. Let* $b_j^q(x)$, $v_q(x)$, *and* $F(x)$ *be respectively functions in* $C^1(\Omega)$, *locally in* $L_2(\Omega)$, *and locally in* $L_1(\Omega)$ *for* $j = 1, \cdots, N$ *and* $q = 1, \cdots, Q$. *Suppose that* $(v_1, \cdots, v_Q)$ *is a distribution solution of*

$$b_j^q \partial v_q / \partial x_j + F = 0 \tag{2.1}$$

in $\Omega - Z$. *Then* $(v_1, \cdots, v_Q)$ *is a distribution solution of* (2.1) *in* Ω.

To be quite explicit, when we say $(v_1, \cdots, v_Q)$ is a distribution solution of (2.1) in Ω_1, an open subset of Ω, we mean

$$\int_{\Omega_1} [v_q \partial(b_j^q \phi)/\partial x_j - F\phi]\, dx = 0 \quad \text{for all } \phi \text{ in } \quad C_0^\infty(\Omega_1). \tag{2.2}$$

Using multiple trigonometric series, we shall prove Theorem A in § 4. Theorem A is strongly motivated by our previous theorem [6, p. 604].

We now deduce Theorem 1 from Theorem A.

First of all we observe from the fact that u_i is in $W_2^1(\Omega)$ for $i = 1, \cdots, N$ that both of the following facts hold:

$$\int_\Omega [u_i \partial\phi/\partial x_j]\, dx = -\int_\Omega [\phi \partial u_i/\partial x_j]\, dx \tag{2.3}$$

$$\text{for } \phi \text{ in } \quad C_0^\infty(\Omega) \quad \text{and} \quad i, j = 1, \cdots, N,$$

and

$$\partial(u_i u_j)/\partial x_j \quad \text{is in} \quad L_1(\Omega) \tag{2.4}$$

and furthermore

$$\int_\Omega [u_i u_j \partial\phi/\partial x_j]\, dx = -\int_\Omega [\phi \partial(u_i u_j)/\partial x_j]\, dx$$

$$\text{for } \phi \text{ in } \quad C_0^\infty(\Omega) \quad \text{and} \quad i = 1, \cdots, N.$$

Consequently on setting

$$v_j^i = \partial u_i/\partial x_j, \tag{2.5}$$

we see from (1.2), (2.3), (2.4), and the hypothesis of the theorem that for fixed i, $(v_1^i, \cdots, v_N^i, p)$ is a distribution solution of

$$\nu \partial v_j^i/\partial x_j - \partial p/\partial x_i + [f_i - \partial(u_i u_j)/\partial x_j] = 0 \tag{2.6}$$

in $\Omega - Z$. But from (2.5) and the hypothesis of Theorem 1, we see that v_j^i and p are in $L_2(\Omega)$ and $[f_i - \partial(u_i u_j)/\partial x_j]$ is in $L_1(\Omega)$. We consequently conclude first from Theorem A that $(v_1^i, \cdots, v_N^i, p)$ is a distribution solution of (2.6) in Ω; and in turn from this last fact, (2.3), and (2.4) that

$$\int_\Omega [\nu u_i \Delta\phi + u_i u_j \partial\phi/\partial x_j + p \partial\phi/\partial x_i + \phi f_i]\, dx = 0 \tag{2.7}$$

$$\text{for all } \phi \text{ in } \quad C_0^\infty(\Omega) \quad \text{and } i = 1, \cdots, N.$$

Next we obtain immediately from Theorem A that since $(u_1, \cdots, u_N)$ is a distribution solution of $\partial u_j/\partial x_j = 0$ in $\Omega - Z$, it is a distribution solution of $\partial u_j/\partial x_j = 0$ in Ω. Consequently,

$$\int_\Omega u_j \partial\phi/\partial x_j = 0 \quad \text{for all } \phi \text{ in } \quad C_0^\infty(\Omega). \tag{2.8}$$

But (2.7) and (2.8) together give Theorem 1, and the proof of Theorem 1 is complete.

To establish Corollary 1, we observe from (1.1), (1.1′), and (1.2) that (u, p) is a distribution solution of (1.1) in $\Omega - Z$. Consequently, we obtain from Theorem 1 that (2.7) and (2.8) hold.

Next, let $\Phi = (\phi_1, \cdots, \phi_N)$, where ϕ_j is in $C_0^\infty(\Omega)$, $j = 1, \cdots, N$, and where

$$\partial\phi_j/\partial x_j = 0. \tag{2.9}$$

Observing from (2.9) that $\int_\Omega p\partial\phi_i/\partial x_i\, dx = 0$ and also that $\int_\Omega u_i \Delta\phi_i\, dx = -\int_\Omega [\partial u_i/\partial x_j\; \partial\phi_i/\partial x_j]\, dx$, we conclude from (2.7) and (2.9) that

$$\int_\Omega [v\partial u_i/\partial x_j\; \partial\phi_i/\partial x_j - u_i u_j \partial\phi_i/\partial x_j]\, dx = \int_\Omega \phi_i f_i\, dx. \tag{2.10}$$

We consequently conclude from (2.10) and [4, p. 115] that u is a generalized solution of (1.1) in Ω. But then it follows from [4, Theorem 6, p. 131] that u_i can be defined at the points of Z so that it is in $C^2(\Omega)$ for $i = 1, \cdots, N$. Now Z is of capacity zero. Since $\partial p/\partial x_i$ is in $C^0(\Omega - Z)$, it then follows from (1.1) that $\partial p/\partial x_i$ can be defined at the points of Z so that it is a continuous function in Ω for $i = 1, \cdots, N$. But this implies that p can be defined at the points of Z so that it is in $C^1(\Omega)$. Consequently (u, p) is a classical solution of (1.1) in Ω, and the proof of Corollary 1 is complete.

The sufficient condition of Corollary 2 follows immediately from Theorem 1. We establish the necessary condition of Corollary 1 by showing that it is a corollary to [1, p. 88]. To do this, we suppose that $\Omega = \Omega^* \times (a, b)$ and $Z = Z^* \times (a, b)$, where (a, b) is a finite open interval and Z^* is a relatively closed subset of the bounded $(N - 1)$-domain Ω^*. Also we suppose that Z is of positive ordinary capacity in E_N. Then, as is well known, this implies that Z^* is of positive ordinary capacity in E_{N-1}. Consequently, there exists a subset of Z^*, call it Z_1^*, which is compact in Ω^*, of positive ordinary capacity in E_{N-1}, and of $(N - 1)$-dimensional Lebesgue measure zero. It follows from [1, p. 88] that there exists a function $v_N(x_1, \cdots, x_{N-1})$ with the following properties: v_N is in $W_2^1(\Omega^*)$, v is harmonic in $\Omega^* - Z_1^*$, v_N is not a distribution solution of Laplace's equation in Ω^*, that is, there exists a ψ in $C_0^\infty(\Omega^*)$ such that $\int_{\Omega^*} v_N \Delta\psi\, dx_1 \cdots dx_{N-1} \neq 0$.

We define the vector $u = (u_1, \cdots, u_N)$ in Ω as follows: $u_j = 0$ for $j = 1, \cdots, N - 1$, $u_N(x_1, \cdots, x_N) = v_N(x_1, \cdots, x_{N-1})$. Also we set $p = 0$ in Ω. Then it follows that (u, p) is in $\mathscr{A}(\Omega)$ and in $C^\infty(\Omega - Z)$. Furthermore, with $Z_1 = Z_1^* \times (a, b)$, it is easy to see that with $f = 0$ in (1.1), (u, p) is a classical solution of (1.1) in $\Omega - Z_1$. Consequently, (u, p) is a distribution solution in $\Omega - Z$ of the nonlinear Navier–Stokes equations with zero external force. If (u, p) were also a distribution solution in Ω of this set of equations, it would follow in particular that $u_N(x)$ would be a distribution solution of Laplace's equation in Ω. But this would imply from Weyl's lemma and the fact that u_N is already harmonic in $\Omega - Z_1$ that v_N could be defined in Z_1^* so that it would be harmonic in Ω^*. But since Z_1^* is of $(N - 1)$-dimensional Lebesgue measure zero, this in turn implies that v_N was originally a distribution solution of Laplace's equation in Ω^*, which is a contradiction. The necessary condition of Corollary 2 is consequently established.

3. Fundamental lemmas. In order to establish Theorem A, we shall need some results in multiple trigonometric series.

We shall use the following notation: $T_N = \{x: -\pi < x_j \leqq \pi, j = 1, \cdots, N\}$; m will designate an integral lattice point; for a function U in $L_1(T_N)$, we shall set

$$\hat{U}(m) = (2\pi)^{-N} \int_{T_N} U(x)\, e^{-i(m,x)}\, dx. \tag{3.1}$$

Also (x, y) will designate the usual inner product $x_1 y_1 + \cdots + x_N y_N$, and $(x, x)^{1/2}$ will be designated by $|x|$.

Given U in $L^1(T_N)$, we shall say U is extended by periodicity to all of E_N if U is defined in all of E_N and is periodic of period 2π in each variable.

We first state some well-known facts in the theory of multiple trigonometric series. (In this section we shall continue to use the summation convention only when dealing with partial derivatives. In particular, we shall not use it when dealing with Fourier coefficients. The situation will be clear from the context.)

LEMMA 1. *Let U be in $L^1(T_N)$ and set*

$$A(U, x, t) = \sum_m \hat{U}(m)\, e^{i(m,x) - |m|t} \quad \textit{for} \quad t > 0. \tag{3.2}$$

Then $\int_{T_N} |A(U, x, t) - U(x)|\, dx \to 0$ *as* $t \to 0$.

For a proof of Lemma 1, we refer the reader to [7, p. 76].

Next, we designate the open N-ball with center x and radius r by $B(x, r)$ and state the following lemma.

LEMMA 2. *Let U be in $L^1(T_N)$ and extended by periodicity to all of E_N. Define $A(U, x, t)$ for x in E_N and $t > 0$ by* (3.2). *Suppose that U is equal almost everywhere in $B(x^0, r_0)$ to a function which is harmonic in the ball $B(x^0, r_0)$, where $0 < r_0 < 1$. Then $\lim_{t \to 0} \Delta A(U, x, t) = 0$ uniformly on compact subsets of $B(x^0, r^0)$.*

For a proof of Lemma 2, in two dimensions, we refer the reader to [6, Lemma 5, p. 609]. A similar proof prevails for $N \geqq 3$.

Next, we establish the following fact.

LEMMA 3. *There are functions $\lambda_j(x)$, $j = 1, \cdots, N$, which are in $C^\infty(E_N)$, periodic of period 2π in each variable, and such that $\partial \lambda_j(x)/\partial x_j = 1$ for x in $B(0, 1)$.*

To establish Lemma 3, choose a function $\lambda(x)$ which is in $C^\infty[B(0, 2)]$, equal to one in $B(0, 1)$, equal to 0 in $B(0, 2) - B(0, 3/2)$, and is such that $\int_{B(0,2)} \lambda(x)\, dx = 0$. Set $\lambda(x) = 0$ in $T_N - B(0, 2)$ and continue $\lambda(x)$ to all of E_N by periodicity of period 2π. Clearly $\lambda(x)$ is in $C^\infty(E_N)$ and has an absolutely convergent Fourier series with $\hat{\lambda}(0) = 0$. In particular,

$$\lambda(x) = \sum_{m \neq 0} \hat{\lambda}(m)\, e^{i(m,x)}, \tag{3.3}$$

where

$$\sum_{m \neq 0} |\hat{\lambda}(m)|\, |m|^k < \infty \quad \text{for} \quad k = 1, 2, \cdots. \tag{3.4}$$

For x in E_N, define $\lambda_j(x)$ as follows for $j = 1, \cdots, N$:

$$\lambda_j(x) = \sum_{m \neq 0} - i m_j \hat{\lambda}(m)\, e^{i(m,x)}/|m|^2. \tag{3.5}$$

From (3.4) and (3.5), it follows that $\lambda_j(x)$ is in $C^\infty(E_N)$ and periodic.

Also it follows from (3.3), (3.4) and (3.5) that

$$\partial\lambda_j(x)/\partial x_j = \sum_{m\neq 0} \hat{\lambda}(m)\, e^{i(m,x)} = \lambda(x). \tag{3.6}$$

But $\lambda(x) = 1$ in $B(0, 1)$, and Lemma 3 is established.

Next, for $j = 1, \cdots, N$ we introduce the functions $H_j(x)$ defined in E_N as follows:

$$H_j(x) = \lim_{t\to 0} \sum_{m\neq 0} im_j\, e^{i(m,x)-|m|t}/|m|^2. \tag{3.7}$$

From [7, p. 72] we obtain that the following properties prevail:

$$\begin{aligned} &H_j(x) \quad \text{is in} \quad L^1(T_N) \quad \text{and} \\ &\hat{H}_j(m) = im_j/|m|^2 \quad \text{for} \quad m \neq 0, \\ &\hat{H}_j(0) = 0. \end{aligned} \tag{3.8}$$

In a similar manner, we introduce the function $H(x)$ defined in E_N as follows:

$$H(x) = \lim_{t\to 0} \sum_{m\neq 0} e^{i(m,x)-|m|t}/|m|^2. \tag{3.9}$$

From [7, p. 72], we also obtain that the following properties prevail:

$$\begin{aligned} &H(x) \quad \text{is in} \quad L^1(T_N) \quad \text{and} \\ &\hat{H}(m) = |m|^{-2} \quad \text{for} \quad m \neq 0, \\ &\hat{H}(0) = 0. \end{aligned} \tag{3.10}$$

Using (3.7) and (3.8), we next establish the following lemma.

LEMMA 4. *Let U_j, $j = 1, \cdots, N$, and V be functions in $L^1(T_N)$ and extended by periodicity to all of E_N. For $t > 0$, define $A(U_j, x, t)$ and $A(V, x, t)$ in a manner analogous to* (3.2). *Suppose $(U_1, \cdots, U_N)$ is a distribution solution in $B(x^0, r_0)$, $0 < r_0 < 1$, of the equation $\partial U_j/\partial x_j + V = 0$. Then*

$$\lim_{t\to 0} [\partial A(U_j, x, t)/\partial x_j + A(V, x, t)] = 0$$

uniformly on compact subsets of $B(x^0, r_0)$.

To establish Lemma 4, we first observe that with no loss in generality we can suppose that $x^0 = 0$. Next we set

$$V'(x) = V(x) - \hat{V}(0) \quad \text{and} \quad U'_j(x) = U_j(x) + \hat{V}(0)\lambda_j(x) \quad \text{for} \quad j = 1, \cdots, N, \tag{3.11}$$

where the functions $\lambda_j(x)$ are defined in Lemma 3. It follows from Lemma 3 and the hypothesis of Lemma 4 that

(3.12) $(U'_1, \cdots, U'_N)$ is distribution solution of $\partial U'_j/\partial x_j + V' = 0$ in $B(0, r_0)$.

Now from Lemma 3, we have that

$$\hat{V}(0) \lim_{t\to 0} \partial A(\lambda_j, x, t)/\partial x_j = \hat{V}(0) \quad \text{uniformly in} \quad B(0, 1). \tag{3.13}$$

We conclude from (3.11) and (3.13) that the lemma will be established if we show

(3.14)
$$\lim_{t\to 0}[\partial A(U_j', x, t)/\partial x_j + A(V', x, t)] = 0$$
uniformly on compact subsets of $B(0, r_0)$.

To establish (3.14), let ϕ be a function in $C_0^\infty[B(0, r_0)]$. Extend ϕ to all of T_N by defining it to be in zero in $T_N - B(0, r_0)$. Then ϕ and all its partial derivatives have absolutely convergent Fourier series, and it follows from (3.11) and (3.12) that

(3.15)
$$\sum_{m\neq 0}\left[\sum_{j=1}^{N} im_j\hat{U}_j'(m) + \hat{V}'(m)\right]\hat{\phi}(-m) = 0.$$

Next, using (3.7), (3.8), (3.9) and (3.10) we define $U(x)$, a function in $L^1(T_N)$, as follows:

(3.16)
$$U(x) = (2\pi)^{-N}\int_{T_N}\left[\sum_{j=1}^{N} U_j'(x-y)H_j(y) + V'(x-y)H(y)\right]dy.$$

We obtain from (3.8), (3.10) and (3.16) that

(3.17)
$$\hat{U}(m) = \left[\sum_{j=1}^{N} im_j\hat{U}_j'(m) + \hat{V}'(m)\right]\Big/|m|^2 \quad \text{for} \quad m \neq 0,$$
$$\hat{U}(0) = 0.$$

In particular from (3.15) and (3.17), we have that

(3.18)
$$\sum_m \hat{U}(m)|m|^2\hat{\phi}(-m) = 0.$$

We conclude from (3.18) that $\int_{T_N} U(x)\Delta\phi(x)\,dx = 0$. But the support of ϕ is contained in $B(0, r_0)$. Consequently,

(3.19)
$$\int_{B(0,r_0)} U(x)\Delta\phi(x)\,dx = 0.$$

Since ϕ was an arbitrary function in $C_0^\infty[B(0, 1)]$, we conclude from Weyl's lemma that $U(x)$ is equal almost everywhere in $B(x^0, r_0)$ to a function which is harmonic in $B(0, r_0)$. But then it follows from Lemma 2 that

(3.20) $\lim_{t\to 0}\Delta A(U, x, t) = 0$ uniformly on compact subsets of $B(0, r_0)$.

From (3.11) and (3.17), we see that

(3.21)
$$-\Delta A(U, x, t) = \partial A(U_j', x, t)/\partial x_j + A(V', x, t).$$

Relations (3.20) and (3.21) together give (3.14), and the proof of the lemma is complete.

LEMMA 5. *Let V be a function in $L_1[B(x^0, r_0)]$, $0 < r_0 < 1$, and let U_j, $j = 1, \cdots, N$, be functions in $L_2[B(x^0, r_0)]$. Also, let Z be a closed set of capacity zero contained in the interior of $B(x^0, r_0)$. Suppose that $U_1, \cdots, U_N$ and V vanish outside of a compact subset of $B(x^0, r_0)$. Also, suppose $(U_1, \cdots, U_N)$ is a distribution* *solution in* $B(x^0, r_0) - Z$ *of the equation*

$$\partial U_j/\partial x_j + V = 0. \tag{3.22}$$

Then $(U_1, \cdots, U_N)$ *is a distribution solution of* (3.20) *in* $B(x^0, r_0)$.

With no loss in generality, we can suppose from the start that $x^0 = 0$. Next, we extend V and $U_1, \cdots, U_N$ to all of T_N by defining these $(N + 1)$-functions to be zero in $T_N - B(0, r_0)$. We then extend them by periodicity of period 2π in each variable to all of E_N.

Next, we set for $t > 0$,

$$A(U_j, x, t) = \sum_m \hat{U}_j(m)\, e^{i(m,x) - |m|t}$$
$$\text{and} \tag{3.23}$$
$$A(V, x, t) = \sum_m \hat{V}(m)\, e^{i(m,x) - |m|t},$$

and observe from Lemma 4 and the Heine–Borel theorem that the following fact holds:

$$\text{If } B \text{ is an open set with } \quad Z \subset B \subset B(0, 1), \quad \text{then}$$
$$\lim_{t \to 0} [\partial A(u_j, x, t)/\partial x_j + A(V, x, t)] = 0 \tag{3.24}$$
$$\text{uniformly in} \quad T_N - B.$$

Next, with $H(x)$ defined by (3.9), we see from [7, p. 72] that we can find a positive constant η_N such that

$$H(x) + \eta_N \geqq 1 \quad \text{for } x \text{ in} \quad E_N. \tag{3.25}$$

We define $G(x)$ to be

$$G(x) = H(x) + \eta_N \quad \text{for } x \text{ in} \quad E_N \tag{3.26}$$

and observe from [7, p. 72] that G has the following properties:

(3.27) (i) G is in $C^\infty[E_N - \bigcup_m \{2\pi m\}]$,

(ii) $\Delta G(x) = 1$ in $E_N - \bigcup_m \{2\pi m\}$,

(iii) there are positive constants α_N and β_N such that for x in $T_N - 0$,

$$|G(x) - \alpha_N |x|^{-(N-1)}| \leqq \beta_N \quad \text{for} \quad N \geqq 3$$

and

$$|G(x) - \alpha_N \log |x|^{-1}| \leqq \beta_N \quad \text{for} \quad N = 2.$$

In particular, it follows from (3.27) that a closed set $Z' \subset B(0, 1)$ is of capacity zero if and only if

$$\int_{Z'} \int_{Z'} G(x - y)\, d\mu(x)\, d\mu(y) = +\infty$$

for all nonnegative Borel measures μ having their support in Z' with $\mu(Z') = 1$.

Let D be a bounded domain, and let $\bar{D}$ designate its closure. Following [5, p. 24], we say $\bar{D}$ satisfies the condition of Poincaré if each point on its boundary is the vertex of a cone of revolution whose interior lies in D. Using the techniques in the theorem given in [5, p. 33], it follows from (3.25), (3.26) and (3.27) that the following fact holds:

(3.28) Let $R_k \subset B(0, 1)$ be the union of a finite number of closed domains each satisfying the condition of Poincaré. Then there exists a unique non-negative Borel measure μ_k having its support in R_k with $\mu(R_k) = 1$ such that $W_k(x) = \int_{R_k} G(x - y)\, d\mu_k(y)$ is a continuous periodic function in E_N and $W_k(x)$ takes a constant value in R_k. This constant value is equal to $\int_{R_k}\int_{R_k} G(x - y)\, d\mu_k(x)\, d\mu_k(y) = I(\mu_k)$.

Now let Z be the set of capacity zero in the hypothesis of the lemma. Then, using (3.28) and standard capacity theory [5, pp. 50–52], it follows that there exists a sequence of closed sets $\{R_k\}_{k=1}^{\infty}$ with the following properties:

(3.29) Each R_k is the union of a finite number of closed domains each of which satisfies the condition of Poincaré.

(3.30) Each R_k is contained in the interior of $B(0, 1)$.

(3.31) $$R_k \supset R_{k+1} \quad \text{for} \quad k = 1, 2, \cdots.$$

(3.32) Z is in the interior of each R_k.

(3.33) If x is in $T_N - Z$, there is an R_k such that x is not in R_k.

(3.34) For each R_k, (3.28) holds. In particular, $W_k(x)/I(\mu_k) = 1$ for x in R_k.

(3.35) $$\lim_{k\to\infty} I(\mu_k) = +\infty.$$

(3.36) $$0 \leqq W_k(x)/I(\mu_k) \leqq 1 \quad \text{for } x \text{ in } T_N \quad \text{and all } k.$$

(3.37) $$\lim_{k\to\infty} W_k(x)/I(\mu_k) = 0 \quad \text{for } x \text{ in } T_N - Z.$$

Next, we set $\hat{\mu}_k(m) = (2\pi)^{-N}\int_{T_N} e^{-i(m,x)}\, d\mu_k(x)$ and observe from (3.26) that

(3.38) $$\hat{G}(m) = |m|^{-2} \quad \text{for} \quad m \neq 0, \qquad \hat{G}(0) = \eta_N.$$

It then follows from (3.28) that

(3.39) $$\hat{W}_k(m) = \hat{G}(m)\hat{\mu}_k(m)(2\pi)^N$$

and that

(3.40) $$\sum_m |\hat{G}(m)|\, |\hat{\mu}_k(m)|^2 = I(\mu_k)/(2\pi)^{2N}$$

From (3.23), (3.39) and (3.40), we next observe from Schwarz's inequality that for $t > 0$ and fixed m^0,

$$\left|(2\pi)^{-N}\int_{T_N} \partial A(U_j, x, t)/\partial x_j \;\; W_k(x)e^{-i(m^0,x)}\, dx\right| \qquad \text{(cont.)}$$

$$(3.41)\qquad = \left| \sum_{j=1}^{N} \sum_{m} i m_j \hat{U}_j(m) \hat{W}_k(m^0 - m)\, e^{-|m|t} \right|$$

$$\leqq \sum_{j=1}^{N} \left(\sum_{m} \left| \hat{U}_j(m) |^2 |m_j|^2 \right| \hat{G}(m^0 - m)| \right)^{1/2} [I(\mu_k)]^{1/2}.$$

To establish the lemma, it is sufficient to show that

$$(3.42)\qquad \sum_{j=1}^{N} i m_j \hat{U}_j(m) = -\hat{V}(m).$$

For suppose that (3.42) is established. Then it follows from (3.23) that

$$(3.43)\qquad \partial A(U_j, x, t)/\partial x_j = -A(V, x, t) \quad \text{for} \quad t > 0.$$

Next, we have from Lemma 1 that as $t \to 0$,

$$(3.44)\qquad \begin{aligned} &\int_{T_N} |A(V, x, t) - V|\, dx \to 0, \\ &\int_{T_N} |A(U_j, x, t) - U_j|\, dx \to 0, \qquad j = 1, \cdots, N. \end{aligned}$$

Let ϕ be a function $C_0^\infty[B(0, r_0)]$. Then from (3.43) and (3.44) we have

$$\begin{aligned} \int_{B(0,r_0)} \phi(x) V(x)\, dx &= \lim_{t \to 0} \int_{B(0,r_0)} \phi(x) A(V, x, t)\, dx \\ &= \lim_{t \to 0} - \int_{B(0,r_0)} \phi(x) \partial A(U_j, x, t)/\partial x_j\, dx \\ &= \lim_{t \to 0} \int_{B(0,r_0)} \partial \phi/\partial x_j \;\; A(U_j, x, t)\, dx \\ &= \int_{B(0,r_0)} \partial \phi/\partial x_j \;\; U_j(x)\, dx, \end{aligned}$$

and the lemma is established.

It remains to show that (3.42) holds. Let m^0 be a fixed lattice point. We shall show that (3.42) holds with m replaced by m^0.

Let $\varepsilon > 0$ be given. Since U_j is in $L^2(T_N)$ for $j = 1, \cdots, N$, it follows from (3.38) that the last sum on the right in (3.41) is finite. Consequently, it follows from (3.35) and (3.41) that there exists a $k_1 > 0$ such that

$$(3.45)\qquad \begin{aligned} &\left| (2\pi)^{-N} \int_{T_N} \partial A(U_j, x, t)/\partial x_j \;\; W_k(x)\, e^{-i(m^0, x)}\, dx \right| \Big/ I[\mu_k] \\ &\leqq \varepsilon \quad \text{for} \quad k \geqq k_1 \quad \text{and} \quad t > 0. \end{aligned}$$

Since Z is of capacity zero and consequently of N-dimensional Lebesgue measure zero, it follows from (3.36) and (3.37) that there exists $k > 0$ such that

$$(2\pi)^{-N}\int_{T_N} |V(x)|\,|W_k(x)|\,dx/I[\mu_k] < \varepsilon \quad \text{for} \quad k \geqq k_2. \tag{3.46}$$

Now set

$$k_3 = \max(k_1, k_2). \tag{3.47}$$

Then it follows from (3.36), (3.44), (3.46) that there exists $t_1 > 0$ such that

$$\begin{aligned}
&(2\pi)^{-N}\int_{T_N} |A(V,x,t)|\,|W_{k_3}(x)|\,dx/I[\mu_{k_3}] \\
&\leqq (2\pi)^{-N}\int_{T_N} |A(V,x,t) - V(x)|\,dx + \varepsilon \\
&\leqq 2\varepsilon \quad \text{for} \quad 0 < t \leqq t_1.
\end{aligned} \tag{3.48}$$

Next, we observe from (3.24), (3.29), (3.30), (3.32) and (3.44) that there exists $t_2 > 0$ such that

$$\int_{T_N - R_{k_3}} |\partial A(U_j,x,t)/\partial x_j + A(V,x,t)|\ \ dx < \varepsilon \quad \text{for} \quad 0 < t \leqq t_2. \tag{3.49}$$

Next we set

$$t_3 = \min(t_1, t_2) \tag{3.50}$$

and observe from (3.23), (3.49), (3.34), (3.36), (3.45), (3.47), (3.48) and (3.50) that for $0 < t \leqq t_3$,

$$\begin{aligned}
&(2\pi)^N\left|\sum_{j=1}^N i m_j^0 \hat{U}_j(m^0) + \hat{V}(m^0)\right| e^{-|m^0|t} \\
&= \left|\int_{T_N} [\partial A(U_j,x,t)/\partial x_j + A(V,x,t)]\, e^{-i(m^0,x)}\,dx\right| \\
&\leqq \varepsilon + \left|\int_{R_{k_3}} [\partial A(U_j,x,t)/\partial x_j + A(V,x,t)]\, e^{-i(m^0,x)}\,dx\right| \\
&\leqq \varepsilon + \left|\int_{R_{k_3}} [\partial A(U_j,x,t)/\partial x_j + A(V,x,t)] W_{k_3}(x)\, e^{-i(m^0,x)}\,dx\right| \Big/ I[\mu_{k_3}] \\
&\leqq 2\varepsilon + \left|\int_{T_N} [\partial A(U_j,x,t)/\partial x_j + A(V,x,t)] W_{k_3}(x)\, e^{-i(m^0,x)}\,dx\right| \Big/ I[\mu_{k_3}] \\
&\leqq [2 + (2\pi)^N]\varepsilon + \left|\int_{T_N} A(V,x,t) W_{k_3}(x)\, e^{-i(m^0,x)}\,dx\right| \Big/ I[\mu_{k_3}] \\
&\leqq [2 + (2\pi)^N]\varepsilon + (2\pi)^N 2\varepsilon.
\end{aligned}$$

We conclude that

(3.51) $$\left|\sum_{j=1}^{N} im_j^0 \hat{U}_j(m^0) + \hat{V}(m^0)\right| \leqq 4\, e^{|m^0|t}\varepsilon \quad \text{for} \quad 0 < t \leqq t_3.$$

Since ε was an arbitrary positive number, we obtain immediately from (3.51) that $\sum_{j=1}^{N} im_j^0 U_j(m^0) = -\hat{V}(m^0)$. Consequently, (3.42) is established, and the proof of the lemma is complete.

4. Proof of Theorem A. Letting Ω_1 designate an open subset of Ω (and returning to the summation convention), we first observe that $(v_1, \cdots, v_Q)$ is a distribution solution of (2.1) in Ω_1 if and only if $(v_1, \cdots, v_Q)$ is a distribution solution in Ω_1 of

(4.1) $$\partial(b_j^q v_q)/\partial x_j + F' = 0, \quad \text{where} \quad F' = F - v_q \partial b_j^q/\partial x_j.$$

Next we observe that $b_j^q v_q$ is locally in $L^2(\Omega)$ and that $F - v_q \partial b_j^q/\partial x_j$ is locally in $L^1(\Omega)$. We conclude that Theorem A will be established once we establish the following theorem.

THEOREM A′. *Let $Z \subset \Omega$ be a relatively closed set of capacity zero. Let $u_j(x)$ be locally in $L^2(\Omega)$ for $j = 1, \cdots, N$ and let $F(x)$ be locally in $L^1(\Omega)$. Suppose that $(u_1, \cdots, u_N)$ is a distribution solution of*

(4.2) $$\partial u_j/\partial x_j + F = 0$$

in $\Omega - Z$. Then $(u_1, \cdots, u_N)$ is a distribution solution of (4.2) in Ω.

To establish Theorem A′, we see, using the notion of partitions of unity, that from the start we can suppose that

(4.3) $$\Omega = B(0, r_0), \quad \text{where} \quad 0 < r_0 < 1.$$

Next, let ϕ be a function in $C_0^\infty[B(0, r_0)]$. In particular, suppose

(4.4) $$\phi(x) = 0 \quad \text{for} \quad B(0, r_0) - B(0, r_4), \quad \text{where} \quad 0 < r_4 < r_0.$$

Theorem A′ will be established if we show

(4.5) $$\int_{B(0,r_0)} [u_j \partial\phi/\partial x_j - F\phi]\, dx = 0.$$

To establish (4.5), we introduce r_1, r_2, and r_3 such that

(4.6) $$0 < r_4 < r_3 < r_2 < r_1 < r_0 < 1,$$

and choose a function $\lambda(x)$ in $C_0^\infty[B(0, r_0)]$ satisfying the following conditions:

(4.7) $$\lambda(x) = \begin{cases} 1 & \text{in} \quad B(0, r_3), \\ 0 & \text{in} \quad B(0, r_0) - B(0, r_2). \end{cases}$$

Next, in $B(0, r_0)$ we define the functions U_j and V as follows:

(4.8) $$U_j = \lambda u_j, \qquad j = 1, \cdots, N,$$
$$V = \lambda F - u_j \partial\lambda/\partial x_j.$$

Also, we define the set

$$Z' = Z \cap \bar{B}(0, r_1), \tag{4.9}$$

where $\bar{B}(0, r_1)$ designates the closure of $B(0, r_1)$, and consider the following equation:

$$\partial U_j/\partial x_j + V = 0. \tag{4.10}$$

We next establish the following:

(4.11) $(U_1, \cdots, U_N)$ is a distribution solution of (4.10) in $B(0, r_0) - Z$.

To see this fact, let ξ be a function in $C_0^\infty[B(0, r_0) - Z]$. Then $\lambda\xi$ is a function in $C_0^\infty[B(0, r_0) - Z]$, and from (4.2) and the hypothesis of Theorem A′ we obtain

$$\begin{aligned} 0 &= \int_{B(0,r_0)} [u_j \partial(\lambda\xi)/\partial x_j - \lambda\xi F]\, dx \\ &= \int_{B(0,r_0)} [\lambda u_j \partial\xi/\partial x_j - (\lambda F - u_j \partial\lambda/\partial x_j)\xi]\, dx \\ &= \int_{B(0,r_0)} [U_j \partial\xi/\partial x_j - V\xi]\, dx. \end{aligned}$$

Consequently (4.11) is established.

Next, we see that

(4.12) $(U_1, \cdots, U_N)$ is a distribution solution of (4.10) in $B(0, r_0) - Z'$.

To establish (4.12), select a function ψ in $C^\infty[B(0, r_0)]$ which has the following properties:

$$\psi(x) = \begin{cases} 1 & \text{in} \quad B(0, r_2), \\ 0 & \text{in} \quad B(0, r_0) - B(0, (r_1 + r_2)/2), \end{cases} \tag{4.13}$$

and let ξ be a function in $C_0^\infty[B(0, r_0) - Z']$. Then it follows immediately from (4.9) and (4.13) that $\xi\psi$ is a function in $C_0^\infty[B(0, r_0) - Z]$. Consequently, we have from (4.11) that

$$\int_{B(0,r_0)} [U_j \partial(\xi\psi)/\partial x_j - V\xi\psi]\, dx = 0. \tag{4.14}$$

But from (4.5), (4.7), (4.8) and (4.13) we have

$$\begin{aligned} &U_j(x)\partial(\xi\psi)(x)/\partial x_j - V(x)\xi(x)\psi(x) \\ &\quad = U_j(x)\partial\xi(x)/\partial x_j - V(x)\xi(x) \quad \text{almost everywhere in} \quad B(0, r_0). \end{aligned} \tag{4.15}$$

But then from (4.14) and (4.15) we have

$$\int_{B(0,r_0)} [U_j \partial\xi/\partial x_j - V\xi]\, dx = 0,$$

and (4.12) is established.

Next, we observe from (4.5) and (4.9) that Z' is a closed set of capacity zero contained in the interior of $B(0, r_0)$. Also, we see from (4.7) and (4.8) that $(U_1, \cdots, U_N)$ and V meet the conditions in the hypothesis of Lemma 5. We consequently conclude from Lemma 5 that

(4.16) $(U_1, \cdots, U_N)$ is a distribution solution of (4.10) in $B(0, r_0)$.

Letting ϕ be the function in $C_0^\infty[B(0, r_0)]$ described in (4.4) and (4.5), we have from (4.16) that

$$\int_{B(0,r_0)} [U_j \partial\phi/\partial x_j - V\phi]\, dx = 0. \tag{4.17}$$

But from (4.4), (4.6), (4.7) and (4.8), we obtain

$$\begin{aligned} &U_j(x)\partial\phi(x)/\partial x_j - V(x)\phi(x) \\ &\quad = u_j \partial\phi(x)/\partial x_j - F(x)\phi(x) \quad \text{almost everywhere in} \quad B(0, r_0). \end{aligned} \tag{4.18}$$

Relations (4.17) and (4.18) together give us (4.5), and the proof of Theorem A′ is complete.

5. The nonlinear nonstationary Navier–Stokes equation. In this section, Ω will be a bounded domain in E_{N+1}, where we now write t for x_{N+1}. Throughout this section we shall assume f_j is in $L^1(\Omega)$, $j = 1, \cdots, N$.

We shall say (u, p) is in the class $\mathscr{B}(\Omega)$, where $u = (u_1, \cdots, u_N)$ if the following holds:

> p and u_j are in $L^2(\Omega)$ for $j = 1, \cdots, N$. Also u_j has first order distribution derivatives in Ω which are such that $\partial u_j/\partial x_k$ is in $L^2(\Omega)$ for $k = 1, \cdots, N$ and $\partial u_j/\partial t$ is in $L^1(\Omega)$.

Classically the nonlinear nonstationary Navier–Stokes equations are given by

$$\begin{aligned} &\partial u_i/\partial t - v\Delta u_i + u_j \partial u_i/\partial x_j + \partial p/\partial x_i - f_i = 0, \qquad i = 1, \cdots, N, \\ &\partial u_j/\partial x_j = 0, \end{aligned} \tag{5.1}$$

where v is a constant (see [4, p. 141]).

As a consequence, we shall say (u, p) in $\mathscr{B}(\Omega)$ is a distribution solution of (5.1) in Ω_1, an open subset of Ω, if the following holds:

$$\begin{aligned} &\int_{\Omega_1} [u_i \partial\phi/\partial t + v u_i \Delta\phi + u_i u_j \partial\phi/\partial x_j + p\partial\phi/\partial x_i + \phi f_i]\, dx \\ &\qquad\qquad = 0 \quad \text{for} \quad i = 1, \cdots, N, \\ &\int_{\Omega_1} u_j[\partial\phi/\partial x_j]\, dx = 0 \quad \text{for all } \phi \text{ in} \quad C_0^\infty(\Omega_1). \end{aligned} \tag{5.2}$$

We shall say a relatively closed set $Z \subset \Omega$ is a removable set for the nonlinear nonstationary Navier–Stokes equations with respect to the class $\mathscr{B}(\Omega)$ if the following holds:

If (u, p) *in* $\mathscr{B}(\Omega)$ *is a distribution solution of* (5.1) *in* $\Omega - Z$, *then* (u, p) *is a distribution solution of* (5.1) *in* Ω.

The following theorem holds.

THEOREM 2. *If* $Z \subset \Omega$ *is a relatively closed set of capacity zero, then with respect to the class* $\mathscr{B}(\Omega)$, Z *is a removable set for the nonlinear nonstationary Navier–Stokes equations.*

The proof of Theorem 2 is very similar to that of Theorem 1; namely, it follows from Theorem A. We leave the details of the proof to the reader.

REFERENCES

[1] L. CARLESON, *Selected Problems on Exceptional Sets*, Van Nostrand, Princeton, 1967.

[2] R. H. DYER AND D. E. EDMUNDS, *Removable singularities of solutions of the Navier–Stokes equations*, J. London Math. Soc. (2), 2 (1970), pp. 535–538.

[3] J. SERRIN, *Local behavior of quasi-linear equations*, Acta Math., 111 (1964), pp. 247–302.

[4] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, Revised 2nd. ed., Gordon & Breach, New York, 1969.

[5] O. FROSTMAN, *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Comm. of the Math. Lund, 3 (1935), pp. 1–118.

[6] V. L. SHAPIRO, *The divergence theorem for discontinuous vector fields*, Ann. of Math. (2), 68 (1958), pp. 604–624.

[7] ———, *Fourier series in several variables*, Bull. Amer. Math. Soc., 70 (1964), pp. 48–93.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

SINGULAR PERTURBATION OF AN IMPROPERLY POSED PROBLEM*

L. E. ADELSON†

Abstract. In this paper we compare the solution of an improperly posed Cauchy problem (assumed to exist) for an elliptic operator having a small coefficient ε multiplying the highest order derivatives with the solution of the appropriately defined Cauchy problem for the elliptic operator resulting from setting ε equal to zero. We prove that if the two solutions belong to the appropriate spaces of functions, then their difference in the $\mathscr{L}^2$-norm over some appropriately defined subdomain is of order ε to some positive power.

1. Introduction. There has been much work done in recent years on singular perturbation for properly posed problems, both in ordinary and partial differential equations. Nearly all of the published papers on this subject investigate the structure of boundary layers and make use of asymptotic expansions in establishing convergence (in some norm) of the solution of the perturbed problem with a small parameter to the solution of the unperturbed problem as the parameter goes to zero. For work in this area see for instance [10], [18]–[22].

The techniques used in studying singular perturbations for well-posed problems do not carry over to improperly posed problems. In fact such problems have been largely ignored in the literature. One result in this area is due to Payne and Sather [14]. They studied a specific case in which a Cauchy problem for an elliptic equation reduced to an initial boundary value problem for the backward heat equation and used a convexity argument to obtain the desired results. Assuming the existence of solutions for the family of "perturbed" problems, these authors have shown that one obtains convergence in $\mathscr{L}^2$ of the perturbed solution to the unperturbed one. Their result is, however, somewhat impractical precisely because of the assumption of existence of solutions for all values of the parameter less than some fixed number. Their results, however, do prove that one may compare the solution of the perturbed problem for a fixed value of the parameter with the solution of the unperturbed problem (parameter = zero).

The question of existence of solutions for all values of ε (the small parameter) in the interval $0 \leqq \varepsilon \leqq \varepsilon_0$ presents no difficulty in most reasonable well-posed problems for partial differential equations or ordinary differential equations. Thus, in those cases one may actually allow ε to go to zero and prove that the perturbed solution converges to the unperturbed solution in some suitable norm.

On the other hand, in improperly posed problems for given data the solution may well fail to exist for some or even all values of ε in the interval. This difficulty can be at least partially overcome by allowing for small variations in the data over the range of values of ε under consideration.

These existence questions are extremely complicated and we do not attempt to answer them in this paper. Our main goal will be to compare the solution of an improperly posed Cauchy problem (assumed to exist) for an elliptic operator

* Received by the editors July 22, 1971, and in revised form February 22, 1972.

† Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506.

with a small coefficient ε multiplying the highest order derivatives with the solution of the appropriately defined Cauchy problem for the elliptic operator resulting from setting ε equal to zero.

One such perturbed problem we consider is

$$\varepsilon bLv + v = u,$$

$$Lu = E(x, \varepsilon, v, u),$$

in an N-dimensional domain D. Here b is a constant which may be positive or negative. On Σ which is a piece of the boundary of D we specify Cauchy data for v, grad v, Lv, and grad (Lv). We require that L be a uniformly elliptic operator and that E satisfy a uniform Lipschitz condition in its last three arguments. We also pose the corresponding unperturbed problem

$$Lw = E(x, 0, w, w)$$

in D with w and grad w specified on Σ. Of course, there are certain compatibility conditions the data must satisfy.

It is well known (see, e.g., Hadamard [5]) that solutions of such improperly posed problems even if they exist will not in general depend continuously on the data. It has been shown, however, by John [6], Pucci [15], Laurentiev [7], [8], and others that if the class of admissible solutions is suitably restricted, then solutions of the type of problem indicated above will in fact depend Hölder continuously on the data. The precise restrictions will be spelled out in the next section.

We prove that if v and w belong to the appropriate spaces of functions, then their difference in the $\mathscr{L}^2$-norm over some appropriately defined subdomain D_α of D is of order ε to some positive power depending on the sign of b and the size of the subdomain. To achieve the result, we use the triangle inequality

$$\|v - w\|_{D_\alpha} \leqq \|v - u\|_{D_\alpha} + \|u - w\|_{D_\alpha}$$

and treat the two terms on the right separately. In fact we use more or less standard techniques on the first term and logarithmic convexity arguments on the second. This is somewhat reminiscent of the methods used by Schaefer [16] in studying a different class of problems.

It is obvious from our results that if we were assured of existence of the solution v (in the appropriate space) for a range of values of ε satisfying $0 < \varepsilon \leqq \varepsilon_0$ and if in addition the corresponding solution w existed, then v would actually converge to w in $\mathscr{L}^2(D_\alpha)$ as $\varepsilon \to 0$.

In this paper we also generalize our result to include the case in which we allow the function E to depend also on grad v and grad u. This assumption essentially cuts the exponent of ε to half of what it is for the corresponding case without the extra dependence.

2. Notation and statement of the problems. Let D be an N-dimensional domain bounded by a closed surface C, and let Σ be that portion of C on which Cauchy data are prescribed. The complement of Σ with respect to C is denoted Σ'. For the purpose of this paper we shall assume $\bar{\Sigma}$ (the closure of Σ) is a C^1-surface.

Let L denote the elliptic operator:

$$Lu = (a_{ij}u_{,i})_{,j},$$

where we have adopted the summation convention over repeated indices and the comma denotes partial differentiation. We also assume that the a_{ij}'s are C^1-functions of the space variables $x = (x_1, \cdots, x_N)$.

Let the operator L be symmetric and strongly elliptic, i.e., the matrix a_{ij} is symmetric and there exists a positive constant a_0 such that for all vectors ξ_i the inequality

$$\frac{1}{a_0}\sum_{i=1}^{N} \xi_i^2 \geqq a_{ij}\xi_i\xi_j \geqq a_0 \sum_{i=1}^{N} \xi_i^2 \tag{2.1}$$

holds at every point in D.

We shall compare solutions v and w of the following set of improperly posed Cauchy problems.

PROBLEM A.

$$\left.\begin{aligned} \varepsilon bLv + v &= u \\ Lu &= E(x, \varepsilon, v, u) \end{aligned}\right\} \quad \text{in} \quad D$$

with

$$L^i v = h_i(x, \varepsilon), \quad \operatorname{grad}(L^i v) = \mathbf{g}_i(x, \varepsilon)$$

on Σ, $i = 0, 1$. (L^i denotes i applications of L.) Here b is a constant and $\mathbf{g}_i(x, \varepsilon)$ denotes for each i a vector-valued function.

We assume that E satisfies a uniform Lipschitz condition in its last three arguments, i.e., there exist constants λ_0, λ_1, and λ_3 such that

$$|E(x, \varepsilon, v, u) - E(x, 0, \bar{v}, \bar{u})| \leqq \lambda_0\varepsilon + \lambda_1|v - \bar{v}| + \lambda_3|u - \bar{u}|. \tag{2.2}$$

Furthermore, we assume that

$$\int_D E^2(0)\, dx \leqq P^2, \tag{2.3}$$

where $E(0) = E(x, 0, 0, 0)$ and P is a constant.

PROBLEM B.

$$Lw = E(x, 0, w, w) \quad \text{in} \quad D$$

with

$$w = h_0(x, 0), \quad \operatorname{grad} w = \mathbf{g}_0(x, 0) \quad \text{on} \quad \Sigma.$$

On Σ we require the Cauchy data $h_i(x, \varepsilon)$ and $\mathbf{g}_i(x, \varepsilon)$ to satisfy

$$\int_\Sigma h_i^2\, ds \leqq \Pi_i^2, \quad \int_\Sigma |\mathbf{g}_i|^2\, ds \leqq \mu_i^2 \tag{2.4}$$

for known constants Π_i and μ_i, $i = 0, 1$, independent of ε. Also we assume

$$|h_0(x, \varepsilon) - h_0(x, 0)| = 0(\varepsilon) \tag{2.5}$$

and

$$|\mathbf{g}_0(x,\varepsilon) - \mathbf{g}_0(x,0)| = 0(\varepsilon). \tag{2.6}$$

To determine the boundary data for u and grad u, we substitute the data for v and its derivatives into the first equation of Problem A. In this way we always know that

$$\|u - v\|_\Sigma = \left(\int_\Sigma (u - v)^2\, ds\right)^{1/2} = 0(\varepsilon)$$

and

$$\|\operatorname{grad}(u - v)\|_\Sigma = 0(\varepsilon).$$

Similarly,

$$\|u - w\|_\Sigma = 0(\varepsilon)$$

and

$$\|\operatorname{grad}(u - w)\|_\Sigma = 0(\varepsilon).$$

We note that if instead of the boundary conditions prescribed for v we imposed

$$\begin{aligned}
&\|v - h_0(x,\varepsilon)\|_\Sigma = 0(\varepsilon),\\
&\|\operatorname{grad} v - \mathbf{g}_0(x,\varepsilon)\|_\Sigma = 0(\varepsilon),\\
&\|Lv - h_1(x,\varepsilon)\|_\Sigma = 0(1),\\
&\|\operatorname{grad}(Lv) - \mathbf{g}_1(x,\varepsilon)\|_\Sigma = 0(1),\\
&\|w - h_0(x,0)\|_\Sigma = 0(\varepsilon),\\
&\|\operatorname{grad} w - \mathbf{g}_0(x,0)\|_\Sigma = 0(\varepsilon),
\end{aligned}$$

in addition to (2.4), (2.5) and (2.6), our results would remain unchanged. One can construct examples where this relaxation is necessary in order that each of the two problems A and B have a solution. As mentioned in the Introduction such a relaxation might result in Problem A having a solution for a range of values of the parameter ε.

We now introduce a class of functions $\tilde{M}$ as follows: a function φ will be said to belong to $\tilde{M}$ if

$$\int_D \varphi^2\, dx \leqq M^2$$

for some prescribed constant M. In addition a function ψ will be said to belong to $\tilde{M}_1$ if

$$\int_D \psi^2\, dx + \int_D |\operatorname{grad}\psi|^2\, dx \leqq M_1^2$$

for some prescribed constant M_1. We shall be concerned with solutions v of problem A and w of problem B which belong either to $\tilde{M}$ or $\tilde{M}_1$. We assume throughout that for the particular value of ε under consideration these solutions exist and belong to the appropriate spaces. We assume further that these solutions are sufficiently differentiable for carrying out the indicated operations. In each case sufficient conditions can be readily found in the literature. Note that we do not require a priori that $u \in \tilde{M}$ or $\tilde{M}_1$.

We propose to prove that if v and w belong to $\tilde{M}$, then the difference of v and w in the $\mathscr{L}^2$-norm for some subdomain of D is of order ε to some positive power. The power depends on the constant b and also on the size of the subdomain.

We shall not be able to compare v and w over all of D, but only over a class of subdomains $D_\alpha \subset D$. We define these subdomains as follows:

Let $f(x) = \text{const.}$ define a set of (not necessarily closed) surfaces. This set is to be so chosen that for each α satisfying

$$0 < \alpha \leqq 1 \tag{2.7}$$

the surface $f(x) = \alpha$ intersects D and forms a closed region D_α whose boundary points consist only of points of Σ and points on the surface $f = \text{const.}$

We require that $f(x)$ have continuous second derivatives in $\bar{D}_1$. We prescribe further that if f satisfies (2.7), then

$$\beta \leqq \gamma \Rightarrow D_\beta \subset D_\gamma, \qquad 0 < \beta \leqq \gamma \leqq 1, \tag{2.8}$$

$$|\operatorname{grad} f| > \delta > 0 \quad \text{in} \quad D_1, \tag{2.9}$$

$$Lf \leqq 0 \quad \text{in} \quad D_1, \tag{2.10}$$

$$|Lf| \leqq a_0 \delta^2 d \quad \text{in} \quad D_1, \tag{2.11}$$

where δ and d are positive constants.

We assume that the surfaces have been so chosen that for α satisfying (2.7), D_α has nonzero measure, but that D_0 has zero measure.

We compare the solutions v and w in the following sense. We show that

$$\|v - w\|_{D_\alpha}^2 \equiv \int_{D_\alpha} (v - w)^2 \, dx = 0(\varepsilon^{\gamma(\alpha)}),$$

where $\gamma(\alpha)$ is a positive function of α for $0 \leqq \alpha < \alpha_1 < 1$ and $\gamma(\alpha_1) = 0$. Thus for $0 \leqq \alpha < \alpha_1 < 1$ our inequality will show that if ε is sufficiently small, v will be arbitrarily close to w in $\mathscr{L}^2$ over D_α.

3. Inequalities and bounds. In this section we shall introduce the mathematical arguments and tools which are required for handling this problem and even more complicated problems, but without the involved detailed arguments required for these other problems. It is hoped that this will permit us to put across the ideas more efficiently. In treating a generalization of this problem, we shall merely have to extend the arguments of this section in various directions. The essence of our methods for handling the indicated classes of singular perturbation will thus be contained in this section.

3.1. Our first objective will be to show that for $0 < \alpha < 1$ the quantity $\|u - v\|_{D_\alpha}^2$ is of order ε in general and under certain conditions is of order ε^2.

We first consider the function $\tau(x)$ defined in $\bar{D}_1$ as

$$\tau(x) = \begin{cases} 1 & \text{in} \quad D_\alpha \cup \Sigma_\alpha, \\ \dfrac{1 - f(x)}{1 - \alpha} & \text{in} \quad \bar{D}_1 - (D_\alpha \cup \Sigma_\alpha), \end{cases} \tag{3.1}$$

where Σ_α is the portion of Σ which lies on the boundary of D_α, and S_α will denote the portion of the surface given by $f(x) = \alpha$ so that the entire boundary of D_α is $\Sigma_\alpha \cup S_\alpha$. Clearly $\tau(x) = 0$ on S_1 and $|\tau(x)| \leqq 1$ in $\bar{D}_1$. Since $f \in C^2(\bar{D}_1)$, $|\tau,_i \tau,_i| \leqq M_3$ and $|\tau,_{ij} \tau,_{ij}| \leqq M_4$ in $\bar{D}_1$ for constants M_3 and M_4.

Thus we have

$$\|u - v\|_{D_\alpha}^2 = \int_{D_\alpha} (u - v)^2 \, dx \leqq \int_{D_1} \tau^s (u - v)^2 \, dx, \tag{3.2}$$

where s is a positive integer to be chosen so large that all subsequent integrals over S_1 vanish.

We now state and prove some lemmas which will allow us to compute the desired results more readily.

LEMMA 3.1. *If b is less than zero and $v \in \tilde{M}$ is a solution to Problem* A, *then*

$$\int_{D_1} \tau^s (u - v)^2 \, dx \leqq 0(\varepsilon^2) + R_0 \varepsilon \int_{D_1} \tau^{s-2} (u - v)^2 \, dx \tag{3.3}$$

for a computable constant R_0.

Proof. From the equations of Problem A and Green's identity we have

$$\begin{aligned} \int_{D_1} \tau^s (u - v)^2 \, dx &= b\varepsilon \int_{D_1} \tau^s (u - v) L v \, dx \\ &= b\varepsilon \int_{D_1} \tau^s (u - v) L(v - u) \, dx + b\varepsilon \int_{D_1} \tau^s (u - v) L u \, dx \\ &= b\varepsilon \int_{\Sigma_1} \tau^s (u - v) \frac{\partial (v - u)}{\partial \nu} \, ds + b\varepsilon \int_{D_1} \tau^s a_{ij} (u - v),_i (u - v),_j \, dx \\ &\quad + b\varepsilon \int_{\Sigma_1} \frac{\partial \tau}{\partial \nu} \frac{(u - v)^2}{2} \, ds - \frac{\varepsilon}{2} \int_{D_1} L\tau^s (u - v)^2 \, dx \\ &\quad + b\varepsilon \int_{D_1} \tau^s (u - v) L u \, dx, \end{aligned} \tag{3.4}$$

where $\partial/\partial \nu$ is the conormal derivative $a_{ij} n_j (\partial/\partial x_i)$ on the boundary Σ_1. Also since we can bound the conormal derivatives in terms of the normal and tangential derivatives, the boundary integrals in (3.4) involve data terms and are $0(\varepsilon^3)$. Because of the boundedness of τ and its derivatives we have $|a_{ij} \tau^s_{,j}| \leqq k_1 \tau^{s-1}$ and $|L\tau^s| \leqq k_2 \tau^{s-2}$ for computable constants k_1 and k_2.

Thus if we use the above bounds in (3.4) we get for a computable constant k_3,

$$\int_{D_1} \tau^s(u - v)^2\, dx \leqq 0(\varepsilon^3) + b\varepsilon \int_{D_1} \tau^s a_{ij}(u - v)_{,i}(u - v)_{,j}\, dx$$

(3.5)
$$+ k_3\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2\, dx + b\varepsilon \int_{D_1} \tau^s(u - v)Lu\, dx.$$

Since $b < 0$ the ellipticity condition (2.1) allows us to drop the second term on the right-hand side (R.H.S.). Since

$$Lu = E(x, \varepsilon, v, u) - E(0) + E(0), \tag{3.6}$$

we may use the Lipschitz condition on the last term of (3.5) to obtain

$$\int_{D_1} \tau^s(u - v)^2\, dx \leqq 0(\varepsilon^3) + k_3\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2\, dx + |b|\varepsilon \int_{D_1} \tau^s|u - v|$$

(3.7)
$$[\lambda_0\varepsilon + \delta_1|v| + \delta_3|u - v|]\, dx + b\varepsilon \int_{D_1} \tau^s(u - v)E(0)\, dx,$$

where $\delta_1 = \lambda_1 + \lambda_3$ and $\delta_3 = \lambda_3$.

We shall in this paper make frequent use of the arithmetic-geometric mean inequality (henceforth abbreviated A-G inequality). Unless we specifically need the constants which enter, we shall use the letters γ_j for the coefficients of the terms we shall subsequently wish to make small and k_j as coefficients of the other terms which are computable and may be large (but will not depend on ε).

Now we employ the A-G inequality on various terms of (3.7) to arrive at

$$\int_{D_1} \tau^s(u - v)^2\, dx \leqq 0(\varepsilon^3) + k_3\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2\, dx + (\gamma_2 + \gamma_3) \int_{D_1} \tau^s(u - v)^2\, dx$$

(3.8)
$$+ k_4\varepsilon^2 \int_{D_1} \tau^s v^2\, dx + k_5\varepsilon \int_{D_1} \tau^s(u - v)^2\, dx$$

$$+ k_6\varepsilon^2 \int_{D_1} \tau^s E^2(0)\, dx.$$

Now by choosing γ_2 and γ_3 small enough, we can solve for the left-hand side (L.H.S.) of (3.8) to obtain

$$\int_{D_1} \tau^s(u - v)^2\, dx \leqq 0(\varepsilon^3) + k_7\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2\, dx + k_8\varepsilon^2 \int_{D_1} \tau^s v^2\, dx$$

(3.9)
$$+ k_9\varepsilon \int_{D_1} \tau^s(u - v)^2\, dx + k_{10}\varepsilon^2 \int_{D_1} \tau^s E^2(0)\, dx.$$

Because $v \in \tilde{M}$ and $E(0)$ satisfies (2.3), we may bound the two terms involving v and $E(0)$ in the following manner:

$$\int_{D_1} \tau^s v^2\, dx \leqq \int_{D_1} v^2\, dx \leqq \int_D v^2\, dx \leqq M^2.$$

However, since both terms have only ε^2 coefficients, the first, third, and fifth terms on the R.H.S. of (3.9) are bounded by a term which is $0(\varepsilon^2)$. We further use the fact that since $\tau \leqq 1$,

$$\int_{D_1} \tau^s(u - v)^2 \, dx \leqq \int_{D_1} \tau^{s-2}(u - v)^2 \, dx.$$

Hence we have completed the proof of Lemma 3.1. Note that the coefficients in the $0(\varepsilon^2)$ term could in fact be computed explicitly.

LEMMA 3.2. *If b is greater than zero and $v \in \tilde{M}_1$ is a solution to Problem* A, *then*

$$\int_{D_1} \tau^s(u - v)^2 \, dx \leqq 0(\varepsilon) + R_1\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2 \, dx \tag{3.10}$$

for some computable constant R_1.

Proof. We proceed exactly as in the proof of Lemma 3.1 up to inequality (3.5), but we must now handle the second term on the R.H.S. differently. We expand it out and use the A-G inequality on the cross terms making use of the ellipticity condition (2.1) to obtain

$$\begin{aligned}\int_{D_1} \tau^s(u - v)^2 \, dx \leqq{}& 0(\varepsilon^3) + k_{11}\varepsilon \int_{D_1} \tau^s a_{ij}u_{,i}u_{,j} \, dx + k_{12}\varepsilon \int_{D_1} \tau^s v_{,i}v_{,i} \, dx \\ &+ k_3\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2 \, dx + b\varepsilon \int_{D_1} \tau^s(u - v)Lu \, dx.\end{aligned} \tag{3.11}$$

We must now deal with the second term on the R.H.S. of (3.11). By Green's identity we have

$$\begin{aligned}\varepsilon \int_{D_1} \tau^s a_{ij}u_{,i}u_{,j} \, dx ={}& \varepsilon \int_{\Sigma_1} \tau^s u \frac{\partial u}{\partial \nu} ds - \frac{\varepsilon}{2} \int_{\Sigma_1} \frac{\partial \tau^s}{\partial \nu} u^2 \, ds \\ &- \varepsilon \int_{D_1} \tau^s u L u \, dx + \frac{\varepsilon}{2} \int_{D_1} L\tau^s u^2 \, dx.\end{aligned} \tag{3.12}$$

By the Lipschitz condition and the A-G inequality we are led to

$$\begin{aligned}\varepsilon \int_{D_1} \tau^s a_{ij}u_{,i}u_{,j} \, dx \leqq{}& 0(\varepsilon) + \varepsilon \int_{D_1} \tau^s |u - v + v|[\lambda_0\varepsilon + \delta_1|v| + \delta_3|u - v|] \, dx \\ &+ \varepsilon \int_{D_1} \tau^s|u - v + v||E(0)| \, dx \\ &+ k_{13}\varepsilon \int_{D_1} \tau^{s-2}(u - v + v)^2 \, dx \\ \leqq{}& 0(\varepsilon) + k_{14}\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2 \, dx + k_{15}\varepsilon \int_{D_1} \tau^{s-2}v^2 \, dx \\ &+ k_{16}\varepsilon \int_{D_1} \tau^s E^2(0) \, dx \\ \leqq{}& 0(\varepsilon) + k_{14}\varepsilon \int_{D_1} \tau^{s-2}(u - v)^2 \, dx.\end{aligned} \tag{3.13}$$

We handle the last term on the R.H.S. of (3.11) as we did in the proof of Lemma 3.1. With the bound for $\int_D v,_i v,_i \, dx$ and the use of (3.13), we complete the proof of Lemma 3.2.

Let us now establish a further lemma.

LEMMA 3.3. *Regardless of the sign of b, if $v \in \tilde{M}$ is a solution to Problem* A, *then for any positive integer ρ such that all integrals over S_1 vanish,*

$$\int_{D_1} \tau^\rho (u - v)^2 \, dx \leqq R \tag{3.14}$$

for some computable constant R.

Proof. By use of the equation in Problem A and Green's identity we have

$$\begin{aligned} \int_{D_1} \tau^\rho (u - v)^2 \, dx &= b\varepsilon \int_{D_1} \tau^\rho (u - v) L v \, dx \\ &= b\varepsilon \int_{\Sigma_1} \tau^\rho (u - v) \frac{\partial v}{\partial \nu} \, ds \\ &\quad - b\varepsilon \int_{D_1} \tau^\rho (u - v),_j a_{ij} v,_i \, dx \\ &\quad - b\varepsilon \int_{D_1} \tau^\rho,_j a_{ij} (u - v) v,_i \, dx, \end{aligned} \tag{3.15}$$

where the boundary integral is $0(\varepsilon^2)$. An expansion of the second term, use of the A-G inequality and combination of like terms yields

$$\begin{aligned} \int_{D_1} \tau^\rho (u - v)^2 \, dx &\leqq 0(\varepsilon^2) + \gamma_4 \varepsilon \int_{D_1} \tau^{\rho+2} a_{ij} u,_i u,_j \, dx \\ &\quad + k_{18} \varepsilon \int_{D_1} \tau^{\rho-2} a_{ij} v,_i v,_j \, dx \\ &\quad + \gamma_5 \int_{D_1} \tau^\rho (u - v)^2 \, dx. \end{aligned} \tag{3.16}$$

Choosing $\gamma_5 < 1$, we may solve for the L.H.S. of (3.16). We must then use Green's identity on the second and third terms of the resulting inequality in which γ_4 is still at our discretion. Thus

$$\begin{aligned} \int_{D_1} \tau^\rho (u - v)^2 \, dx &\leqq 0(\varepsilon) - \gamma_4 \varepsilon \int_{D_1} \tau^{\rho+2} u L u \, dx + \gamma_4 k_{19} \varepsilon \int_{D_1} \tau^\rho u^2 \, dx \\ &\quad - k_{20} \varepsilon \int_{D_1} \tau^{\rho-2} v L v \, dx + k_{21} \varepsilon \int_{D_1} \tau^{\rho-4} v^2 \, dx, \end{aligned} \tag{3.17}$$

where we have already bounded the $L\tau^{\rho+2}$ which results and have kept the same symbol γ_4 for our constant above. It is not exactly the same number as before, but can still be made as small as we like.

We now use the fact that $b\varepsilon Lv = u - v$ to handle the fourth term on the R.H.S. of (3.17), but this means that the resulting term is 0(1), that is,

$$\varepsilon \int_{D_1} \tau^{\rho-2} vLv\, dx \leqq k_{22} \int_{D_1} \tau^{\rho-4} v^2\, dx + \gamma_6 \int_{D_1} \tau^{\rho}(u-v)^2\, dx. \tag{3.18}$$

The second and third terms are dealt with as before, only we must be careful to take small coefficients with $\int_{D_1} \tau^{\rho}(u-v)^2\, dx$ terms. After doing this we again use our bounds for v and $E(0)$ to complete the proof of Lemma 3.3.

We next make use of the lemmas to derive the desired theorems. First we observe that when $b < 0$, we have by iteration of Lemma 3.1,

$$\begin{aligned} \int_{D_1} \tau^{s}(u-v)^2\, dx &\leqq 0(\varepsilon^2) + R_0\varepsilon \int_{D_1} \tau^{s-2}(u-v)^2\, dx \\ &\leqq 0(\varepsilon^2) + R_0^2\varepsilon^2 \int_{D_1} \tau^{s-4}(u-v)^2\, dx. \end{aligned} \tag{3.19}$$

Application of Lemma 3.3 to the last term with $\rho = s - 4$ yields for $s \geqq 9$ the following theorem.

THEOREM 3.4. *If* $b < 0$ *and* $v \in \tilde{M}$ *is a solution to Problem* A, *then*

$$\|u - v\|^2_{D_\alpha} = 0(\varepsilon^2)$$

for α *in the interval* $0 \leqq \alpha < 1$.

Combining Lemmas 3.2 and 3.3 with $\rho = s - 2$ and $s \geqq 7$ we are led to the following.

THEOREM 3.5. *If* $b > 0$ *and* $v \in \tilde{M}_1$ *is a solution to Problem* A, *then*

$$\|u - v\|^2_{D_\alpha} = 0(\varepsilon)$$

for α *in the interval* $0 \leqq \alpha < 1$.

3.2. We now employ the tool of logarithmic convexity to show that $\|u - w\|^2_{D_\alpha}$ is of order ε to some positive power depending on the sign of b.

The following procedure is the same as the one used by L. E. Payne [13] in which he computed bounds for solutions of non-well-posed Cauchy problems for linear elliptic equations. Here we consider the equations

$$\begin{aligned} Lu &= E(x, \varepsilon, v, u), \\ Lw &= E(x, 0, w, w) \end{aligned}$$

with Cauchy data as prescribed in §2 for Problem set (A, B).

Let $\psi = u - w$ and define

$$F(\alpha) = \int_0^{\alpha} (\alpha - \eta) \int_{D_\eta} [a_{ij}\psi_{,i}\psi_{,j} + \psi L\psi]\, dx\, d\eta + Q, \tag{3.20}$$

where

$$Q = m_1 \int_{\Sigma} \psi^2\, ds + m_2 \int_{\Sigma} \psi_{,i}\psi_{,i}\, ds + m_3 \int_{D_{\alpha_1}} (u-v)^2\, dx + m_4 V(D_{\alpha_1})\varepsilon^2 \tag{3.21}$$

and $m_1, \cdots, m_4$ are explicit computable constants. $V(D_{\alpha_1})$ denotes the volume of D_{α_1} which is bounded for $0 < \alpha_1 < 1$.

We show that as a function of α, F satisfies a differential inequality of the form

$$FF'' - (F')^2 \geqq -K_1FF' - K_2F^2 \tag{3.22}$$

for explicit constants K_1 and K_2. The solution of this differential inequality will then lead to the desired bounds.

By exactly the same technique as in [13], it is clear that we can choose the m_i in Q such that F satisfies the continued inequality

$$\frac{d+1}{2}\left[\int_{D_\alpha} r\psi^2\, dx + Q\right] \geqq F(\alpha) \geqq \frac{1}{2}\left[\int_{D_\alpha} r\psi^2\, dx + Q\right], \tag{3.23}$$

where

$$r = a_{ij}f_{,i}f_{,j}. \tag{3.24}$$

To establish (3.22) we require the same two lemmas Payne used in [13]; however, his proofs do not go through verbatim because of the differences in the equations and in the Q. One major difference is the way we bound $\int_0^\alpha\int_{D_\eta}(L\psi)^2\, dx\, d\eta$ and we indicate this here:

$$\begin{aligned}\int_0^\alpha\int_{D_\eta}(L\psi)^2\, dx\, d\eta &= \int_0^\alpha\int_{D_\eta}[E(x,\varepsilon,v,u) - E(x,0,w,w)]^2\, dx\, d\eta\\ &\leqq \int_0^\alpha\int_{D_\eta}[\lambda_0\varepsilon + \delta_1|u-v| + \delta_3|u-w|]^2\, dx\, d\eta\\ &\leqq d_0V(D_\alpha)\varepsilon^2 + d_1\int_0^\alpha\int_{D_\eta}(u-v)^2\, dx\, d\eta\\ &\quad + d_3\int_0^\alpha\int_{D_\eta}(u-w)^2\, dx\, d\eta,\end{aligned} \tag{3.25}$$

where δ_1 and δ_3 are linear combinations of λ_1 and λ_3 of the Lipschitz constants and d_0, d_1 and d_3 are the computable constants we get after applying the A-G inequality to the cross terms which arise from squaring the integrand and combining like terms.

We now state without proofs (which may be found in [1]) the two lemmas.

LEMMA 3.6. *If $F(\alpha)$ is given by* (3.20), *then*

$$|F'| \leqq F' + K_2F$$

for a computable constant K_2.

LEMMA 3.7. *If $F(\alpha)$ is given by* (3.20), *then*

$$\int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\, dx - 2\int_{D_\alpha} r^{-1}[a_{ij}\psi_{,i}f_{,j}]^2\, dx \geqq -K_3F' - K_4F$$

for computable constants K_3 and K_4.

With the use of Lemmas 3.6 and 3.7 and the procedure of Payne [13] we are led to (3.22) with computable constants K_1 and K_2.

It is well known (see, e.g., Levine [9]) that a solution of (3.22) which vanishes for one value of α in the interval $[0, \alpha_1]$ must vanish identically. Thus without loss of generality we may assume that $F(\alpha) > 0$ for all α, $0 \leqq \alpha \leqq \alpha_1 < 1$. Then setting

$$\sigma = e^{-K_1\alpha}, \tag{3.26}$$

we find (regarding F temporarily as a function of σ)

$$\frac{d^2}{d\sigma^2}\{\log [F\sigma^{-K_2/K_1^2}]\} \geqq 0, \tag{3.27}$$

from which it follows by Jensen's inequality that

$$F(\alpha)\sigma^{-K_2/K_1^2} \leqq [F(\alpha_1)\sigma_1^{-K_2/K_1^2}]^{(1-\sigma)/(1-\sigma_1)}[F(0)]^{(\sigma-\sigma_1)/(1-\sigma_1)}. \tag{3.28}$$

Here

$$\sigma_1 = e^{-K_1\alpha_1}$$

and F is now regarded as a function of α. We note by (3.20) that $F(0) \equiv Q$, an expression involving only data terms and from Theorem 3.4 or 3.5 depending on the sign of b, $F(0)$ is either $0(\varepsilon^2)$ or $0(\varepsilon)$.

As has been noted in earlier papers (see e.g., John [6], Pucci [15]), in order to make $F(\alpha)$ small for $0 \leqq \alpha < \alpha_1$, it is not sufficient to make $F(0)$ small. One must be sure at the same time $F(\alpha_1)$ does not become so large that the product is no longer small. To stabilize the problem we therefore assume that the solution w lies in the class $\tilde{M}$.

Since our u plays the role of the φ in Payne's paper we must know that Q is small, but it is either $0(\varepsilon^2)$ or $0(\varepsilon)$ depending on whether $b < 0$ and $v \in \tilde{M}$ or $b > 0$ and $v \in \tilde{M}_1$ respectively. Using the fact that

$$\int_{D_{\alpha_1}} u^2\,dx \leqq 2\int_{D_{\alpha_1}} (u - v)^2\,dx + 2\int_{D_{\alpha_1}} v^2\,dx \leqq 0(\varepsilon) + 2M^2 \leqq M_5 \tag{3.29}$$

for some computable constant M_5, we can compute (using (3.23)) an M_6 such that

$$F(\alpha_1)\sigma_1^{-K_2/K_1^2} \leqq M_6^2. \tag{3.30}$$

Insertion of (3.30) into (3.28) now gives

$$F(\alpha) \leqq \sigma^{K_2/K_1^2}\{M_6^{2(1-\sigma)/(1-\sigma_1)}Q^{(\sigma-\sigma_1)/(1-\sigma_1)}\}. \tag{3.31}$$

Hence we have the following theorem.

THEOREM 3.8. *If v, $w \in \tilde{M}$, w is a solution of Problem* B *and u, v are solutions of Problem* A, *then the difference $u - w$ satisfies the following continuous dependence inequality for α in the range $0 \leqq \alpha < \alpha_1 < 1$:*

$$\|u - w\|_{D_\alpha}^2 \leqq KM_6^{2\nu(\alpha)}Q^{1-\nu(\alpha)}.$$

Here K, M_6, and $\nu(\alpha)$ are computable with $0 \leqq \nu(\alpha) < 1$, and Q is given by (3.21).

Applying the results of Theorems 3.4 and 3.5 to the third term of Q and using the facts that

$$\int_\Sigma (u - w)^2 \, dx = 0(\varepsilon^2)$$

and

$$\int_\Sigma |\text{grad}\,(u - w)|^2 \, ds = 0(\varepsilon^2)$$

we immediately have the following corollary.

COROLLARY 3.9. *With the same hypothesis as in Theorem* 3.8, *we conclude that either*

$$\|u - w\|_{D_\alpha}^2 = 0(\varepsilon^{2(1 - \nu(\alpha))})$$

if $b < 0$, *or*

$$\|u - w\|_{D_\alpha}^2 = 0(\varepsilon^{1 - \nu(\alpha)})$$

if $b > 0$ *and in addition* $v \in \tilde{M}_1$. *Here* α *lies in the interval* $0 \leqq \alpha < \alpha_1 < 1$ *and* $0 \leqq \nu(\alpha) < 1$.

3.3. We now combine the results of § 3.1 and § 3.2 and use the triangle inequality

$$\|v - w\|_{D_\alpha} \leqq \|v - u\|_{D_\alpha} + \|u - w\|_{D_\alpha}$$

to establish a further theorem.

THEOREM 3.10. *If* $v \in \tilde{M}$ *is a solution of Problem* A, $w \in \tilde{M}$ *is a solution of Problem* B, *and* u *is as in Theorem* 3.8, *then the difference* $v - w$ *satisfies one of the following continuous dependence inequalities for* α *in the range* $0 \leqq \alpha < \alpha_1 < 1$ *and* $\nu(\alpha)$ *with* $0 \leqq \nu(\alpha) < 1$:

(i) *If* $b < 0$, *then*

$$\|v - w\|_{D_\alpha}^2 = 0(\varepsilon^{2(1 - \nu(\alpha))}).$$

(ii) *If* $b > 0$ *and* $v \in \tilde{M}_1$, *then*

$$\|v - w\|_{D_\alpha}^2 = 0(\varepsilon^{1 - \nu(\alpha)}).$$

Remark. It is obvious of course that if $b > 0$ and we require v to belong to an even more restrictive space, that is,

$$\int_D [v^2 + |\text{grad}\, v|^2 + (Lv)^2] \, dx \leqq M_2^2$$

for some prescribed constant M_2, we shall again obtain the same order of ε in case (ii) that we established for case (i).

4. Extension of previous results. Up to now we have allowed the function E to depend only on u, v, ε, and the position variables. In this section we allow E to depend also on the derivatives of u and v. Although we still obtain bounds for $\|v - w\|_{D_\alpha}$ which are of order ε to some positive power, as we shall see the power of

ε is essentially half of what it was in the preceding section. Let us now write $E = E(x, \varepsilon, v, v_{,i}, u, u_{,i})$, where $v_{,i}$ and $u_{,i}$ denote general derivatives of v and u respectively and assume that E satisfies a uniform Lipschitz condition in its last five arguments, i.e., there exist constants $\lambda_0, \cdots, \lambda_4$ such that

$$|E(x, \varepsilon, v, v_{,i}, u, u_{,i}) - E(x, 0, \bar{v}, \bar{v}_{,i}, \bar{u}, \bar{u}_{,i})| \leqq \lambda_0 \varepsilon + \lambda_1 |v - \bar{v}| + \lambda_2 |\mathrm{grad}\,(v - \bar{v})| + \lambda_3 |u - \bar{u}| + \lambda_4 |\mathrm{grad}\,(u - \bar{u})|.$$

In this section we compare solutions v and w of the following set of improperly posed Cauchy problems:

PROBLEM A′.

$$\left.\begin{aligned} \varepsilon b L v + v &= u \\ L u &= E(x, \varepsilon, v, v_{,i}, u, u_{,i}) \end{aligned}\right\} \quad \text{in} \quad D$$

with

$$L^i v = h_i(x, \varepsilon), \quad \mathrm{grad}\,(L^i v) = \mathbf{g}_i(x, \varepsilon)$$

on Σ, $i = 0, 1$.

PROBLEM B′.

$$L w = E(x, 0, w, w_{,i}, w, w_{,i}) \quad \text{in} \quad D$$

and

$$w = h_0(x, 0), \quad \mathrm{grad}\, w = \mathbf{g}_0(x, 0) \quad \text{on} \quad \Sigma,$$

where b is a constant. We assume also that (2.3) to (2.6) hold.

4.1. It can be shown by essentially the same arguments as those used in § 3.1 that Theorems 3.4 and 3.5 hold for Problem A′.

We now prove the following lemma which we shall need in the next section.

LEMMA 4.1. *If $v \in \tilde{M}$ is a solution to Problem* A′, *then*

$$\int_{D_1} \tau^{s+2} (u - v)_{,i} (u - v)_{,i}\, dx \leqq 0(\varepsilon) + k_{11} \int_{D_1} \tau^s (u - v)^2\, dx + \frac{1}{a_0} [\gamma_4 + b] \varepsilon \int_{D_1} \tau^{s+2} (Lv)^2\, dx,$$

where k_{11} is a computable constant and γ_4 can be made less than $|b|$ so that if b is negative we may drop the last term of the estimate.

Proof. By the ellipticity condition (2.1) we have

$$\int_{D_1} \tau^{s+2} (u - v)_{,i} (u - v)_{,i}\, dx \leqq \frac{1}{a_0} \int_{D_1} \tau^{s+2} a_{ij} (u - v)_{,i} (u - v)_{,j}\, dx. \tag{4.1}$$

We proceed making use of Green's theorem to obtain

$$\begin{aligned}
&\int_{D_1} \tau^{s+2} a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx \\
&= \int_{\Sigma_1} \tau^{s+2}(u-v)\frac{\partial(u-v)}{\partial \nu}\,ds - \int_{\Sigma_1} \frac{\partial \tau^{s+2}}{\partial \nu}\frac{(u-v)^2}{2}\,ds \\
&\quad - \int_{D_1} \tau^{s+2}(u-v)L(u-v)\,dx + \int_{D_1} L\tau^{s+2}\frac{(u-v)^2}{2}\,dx.
\end{aligned} \tag{4.2}$$

Since the boundary terms are $0(\varepsilon^2)$ and we can bound $L\tau^s$, we have

$$\begin{aligned}
&\int_{D_1} \tau^{s+2} a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx \\
&\leqq 0(\varepsilon^2) + k_1 \int_{D_1} \tau^s (u-v)^2\,dx + \int_{D_1} \tau^{s+2}(u-v)Lv\,dx \\
&\quad - \int_{D_1} \tau^{s+2}(u-v)Lu\,dx.
\end{aligned} \tag{4.3}$$

We now employ the equations of Problem A′, and the Lipschitz condition to produce

$$\begin{aligned}
&\int_{D_1} \tau^{s+2} a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx \\
&\leqq 0(\varepsilon^2) + k_1 \int_{D_1} \tau^s (u-v)^2\,dx + b\varepsilon \int_{D_1} \tau^{s+2}(Lv)^2\,dx \\
&\quad + \int_{D_1} \tau^{s+2}|u-v|[\lambda_0\varepsilon + \delta_1|v| + \delta_2|\operatorname{grad} v| \\
&\quad + \delta_3|u-v| + \delta_4|\operatorname{grad}(u-v)|]\,dx - \int_{D_1} \tau^{s+2}(u-v)E(0)\,dx.
\end{aligned} \tag{4.4}$$

We next use the A-G inequality after appropriate substitution of $|b|\varepsilon|Lv|$ for $|u-v|$ to get

$$\begin{aligned}
&\int_{D_1} \tau^{s+2} a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx \\
&\leqq 0(\varepsilon^2) + k_1 \int_{D_1} \tau^s (u-v)^2\,dx + b\varepsilon \int_{D_1} \tau^{s+2}(Lv)^2\,dx \\
&\quad + k_2\varepsilon \int_{D_1} \tau^{s+2} v^2\,dx + k_3 \int_{D_1} \tau^{s+2}(u-v)^2\,dx \\
&\quad + \gamma_1 \int_{D_1} \tau^{s+2} a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx + k_4\varepsilon \int_{D_1} \tau^{s+2} a_{ij} v_{,i} v_{,j}\,dx \quad \text{(cont.)}
\end{aligned} \tag{4.5}$$

$$+ \gamma_2\varepsilon \int_{D_1} \tau^{s+2}(Lv)^2\,dx + k_5 \int_{D_1} \tau^{s+2}(u-v)^2\,dx$$

$$+ k_6\varepsilon \int_{D_1} \tau^{s+2}E^2(0)\,dx.$$

We observe that

$$\varepsilon \int \tau^{s+2}a_{ij}v_{,i}v_{,j}\,dx \leqq 0(\varepsilon) + \gamma_3\varepsilon \int \tau^{s+2}(Lv)^2\,dx + k_7\varepsilon \int \tau^s v^2\,dx, \tag{4.6}$$

and by choosing γ_1 small enough so that we may solve for the L.H.S. of (4.5) we obtain

$$\int_{D_1} \tau^{s+2}a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx$$

$$\leqq 0(\varepsilon) + k_8 \int_{D_1} \tau^s(u-v)^2\,dx + [\gamma_2 + \gamma_3 + b]\varepsilon \int_{D_1} \tau^{s+2}(Lv)^2\,dx \tag{4.7}$$

$$+ k_9\varepsilon \int_{D_1} \tau^s v^2\,dx + k_{10}\varepsilon \int_{D_1} E^2(0)\,dx.$$

We again make use of the bounds for v and $E(0)$ to establish the lemma with $\gamma_4 = \gamma_2 + \gamma_3$ still at our discretion.

Remark. Since we eventually only use Lemma 4.1 with $b < 0$, we could have obtained the desired result more easily. It can be shown directly that

$$\int_{D_1} \tau^s(u-v)^2\,dx - \frac{b}{2}\varepsilon \int \tau^s a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx = 0(\varepsilon^2).$$

And since $\int_{D_1} \tau^s(u-v)^2 = 0(\varepsilon^2)$ it follows that

$$\int_{D_1} \tau^s a_{ij}(u-v)_{,i}(u-v)_{,j}\,dx = 0(\varepsilon).$$

4.2. Let us now turn our attention to the problem of bounding $\|u - w\|^2_{D_\alpha}$ in the case $E = E(x, \varepsilon, v, v_{,i}, u, u_{,i})$. The problem is basically the same as in § 3.2, namely, we consider the equations

$$Lu = E(x, \varepsilon, v, v_{,i}, u, u_{,i}),$$
$$Lw = E(x, 0, w, w_{,i}, w, w_{,i})$$

with Cauchy data as prescribed at the beginning of § 4 for Problem set (A′, B′).

Again we let $\psi = u - w$ and make use of logarithmic convexity arguments with

$$F(\alpha) = \int_0^\alpha (\alpha - \eta) \int_{D_\eta} [a_{ij}\psi_{,i}\psi_{,j} + \psi L\psi]\,dx\,d\eta + Q, \tag{4.8}$$

where

$$Q = m_1 \int_\Sigma \psi^2\, ds + m_2 \int_\Sigma \psi_{,i}\psi_{,i}\, ds + m_3 \int_{D_{\alpha_1}} (u - v)^2\, dx$$
$$+ m_4 \int_{D_{\alpha_1}} (u - v)_{,i}(u - v)_{,i}\, dx + m_5 V(D_{\alpha_1})\varepsilon^2 \tag{4.9}$$

and $m_1, \cdots, m_5$ are explicit constants which may be chosen so that the analogue of (3.23) holds.

We proceed exactly as in § 3.2 up to Lemma 3.6 which we must now re-prove in the following form.

LEMMA 4.2. *If* $F(\alpha)$ *is given by* (4.8), *then*

$$|F'| \leqq K_1 F' + K_2 F$$

for computable constants K_1 *and* K_2.

Proof. Differentiating (4.8) we have

$$F'(\alpha) = \int_0^\alpha \int_{D_\eta} [a_{ij}\psi_{,i}\psi_{,j} + \psi L\psi]\, dx\, d\eta$$

from which it follows that

$$|F'| \leqq F' + 2\left|\int_0^\alpha \int_{D_\eta} \psi L\psi\, dx\, d\eta\right|. \tag{4.10}$$

Now using the A-G inequality we have

$$\left|\int_0^\alpha \int_{D_\eta} \psi L\psi\, dx\, d\eta\right| \leqq \frac{1}{2}\int_0^\alpha \int_{D_\eta} \psi^2\, dx\, d\eta + \frac{1}{2}\int_0^\alpha \int_{D_\eta} (L\psi)^2\, dx\, d\eta. \tag{4.11}$$

Thus

$$\int_0^\alpha \int_{D_\eta} (L\psi)^2\, dx\, d\eta = \int_0^\alpha \int_{D_\eta} [E(x, \varepsilon, v, v_{,i}, u, u_{,i}) - E(x, 0, w, w_{,i}, w, w_{,i})]^2\, dx\, d\eta$$
$$\leqq \int_0^\alpha \int_{D_\eta} [\lambda_0 \varepsilon + \delta_1|u - v| + \delta_2|\mathrm{grad}\,(u - v)| + \delta_3|u - w|$$
$$+ \delta_4|\mathrm{grad}\,(u - w)|]^2\, dx\, d\eta$$
$$\leqq d_0\varepsilon^2 V(D_\alpha) + d_1 \int_0^\alpha \int_{D_\eta} (u - v)^2\, dx\, d\eta \tag{4.12}$$
$$+ d_2 \int_0^\alpha \int_{D_\eta} (u - v)_{,i}(u - v)_{,i}\, dx\, d\eta$$
$$+ d_3 \int_0^\alpha \int_{D_\eta} (u - w)^2\, dx\, d\eta$$
$$+ d_4 \int_0^\alpha \int_{D_\eta} (u - w)_{,i}(u - w)_{,i}\, dx\, d\eta.$$

Here we have used the Lipschitz condition and the A-G inequality and summed like terms to obtain the computable constants $d_0, \cdots, d_4$. The last term on the R.H.S. of (4.11) requires some manipulation. Using the ellipticity condition (2.1), we obtain

$$\begin{aligned} d_4 \int_0^\alpha \int_{D_\eta} (u-w)_{,i}(u-w)_{,i}\,dx\,d\eta &\leqq \frac{d_4}{a_0}\int_0^\alpha \int_{D_\eta} a_{ij}\psi_{,i}\psi_{,j}\,dx\,d\eta \\ &\leqq \frac{d_4}{a_0}F' + \frac{d_4}{a_0}\left|\int_0^\alpha \int_{D_\eta} \psi L\psi\,dx\,d\eta\right| \\ &\leqq \frac{d_4}{a_0}F' + \frac{d_4}{2a_0\gamma}\int_0^\alpha \int_{D_\eta} (L\psi)^2\,dx\,d\eta \\ &\quad + \frac{d_4\gamma}{2a_0}\int_0^\alpha \int_{D_\eta} \psi^2\,dx\,d\eta, \end{aligned}$$

where we have used the A-G inequality with constant γ. We choose $\gamma = d_4/a_0$, substitute into (4.12) and solve for the term on the L.H.S to obtain

$$\begin{aligned} \int_0^\alpha \int_{D_\eta} (L\psi)^2\,dx\,d\eta &\leqq 2d_0\varepsilon^2 V(D_\alpha) + 2d_1\int_0^\alpha \int_{D_\eta} (u-v)^2\,dx\,d\eta \\ &+ 2d_2\int_0^\alpha \int_{D_\eta} (u-v)_{,i}(u-v)_{,i}\,dx\,d\eta + \left(2d_3 + \frac{d_4^2}{a_0^2}\right)\int_0^\alpha \int_{D_\eta} \psi^2\,dx\,d\eta + \frac{2d_4}{a_0}F'. \end{aligned} \tag{4.13}$$

Using (4.11) and substituting (4.13) into (4.10) we have

$$\begin{aligned} |F'| &\leqq \left(1 + \frac{2d_4}{a_0}\right)F' + 2d_1\int_0^\alpha \int_{D_\eta} (u-v)^2\,dx\,d\eta \\ &\quad + 2d_2\int_0^\alpha \int_{D_\eta} (u-v)_{,i}(u-v)_{,i}\,dx\,d\eta \\ &\quad + \left(1 + 2d_3 + \frac{d_4^2}{a_0^2}\right)\int_0^\alpha \int_{D_\eta} \psi^2\,dx\,d\eta + 2d_0\varepsilon^2 V(D_\alpha) \\ &\leqq K_1F' + K_2F \end{aligned} \tag{4.14}$$

(by analogue of (3.23)) for computable constants K_1 and K_2.

We now prove the analogue of Lemma 3.7.

LEMMA 4.3. *If* $F(\alpha)$ *is given by* (4.8), *then*

$$\int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\,dx - 2\int_{D_\alpha} r^{-1}[a_{ij}\psi_{,i}f_{,j}]^2\,dx \geqq -K_3F' - K_4F \tag{4.15}$$

for computable constants K_3 *and* K_4.

Proof. The proof is exactly the same as that for Lemma 3.7 except for the way we bound $\int_{D_\alpha}(\alpha-\eta)(L\psi)^2\,dx$; but it follows from (4.13) with the help of the analogue of (3.23) that

$$\int_{D_\alpha} (\alpha-\eta)(L\psi)^2\,dx \leqq C_5F + C_6F' \tag{4.16}$$

for computable constants C_5 and C_6.

We now form

$$
(4.17)\quad \begin{aligned} FF'' - (F')^2 \geqq & \left\{\frac{1}{2}\int_{D_\alpha} r\psi^2\,dx \int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\,dx - \left(\int_{D_\alpha} a_{ij}\psi_{,i}\psi f_{,j}\,dx\right)^2\right\} \\ & + F\int \psi L\psi\,dx - 2|F'|\left|\int_0^\alpha \int_{\Sigma_\eta} \psi\frac{\partial\psi}{\partial\nu}\,ds\,d\eta\right|, \end{aligned}
$$

where we have dropped a number of nonnegative terms on the right.

Because E depends on $u_{,i}$ we are unable to bound $\int_{D_\alpha}\psi L\psi\,dx$ by the A-G inequality; hence, we must resort to a different procedure. Call the term in braces in (4.17) G; then

$$
(4.18)\quad \left\{\frac{1}{2}\int_{D_\alpha} r\psi^2\,dx \int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\,dx\right\}^{1/2} = \left\{G + \left(\int_{D_\alpha} a_{ij}\psi_{,i}\psi f_{,j}\,dx\right)^2\right\}^{1/2}.
$$

We now have

$$
(4.19)\quad \begin{aligned} \int_{D_\alpha} \psi L\psi\,dx &= \int_{D_\alpha} \psi[E(x,\varepsilon,v,v_{,i},u,u_{,i}) - E(x,0,w,w_{,i},w,w_{,i})]\,dx \\ &\leqq \int_{D_\alpha} |u-w|[\lambda_0\varepsilon + \delta_1|u-v| + \delta_2|\mathrm{grad}\,(u-v)| + \delta_3|u-w| \\ &\qquad + \delta_4|\mathrm{grad}\,(u-w)|]\,dx \\ &\leqq \left\{\int_{D_\alpha} \psi^2\,dx \int_{D_\alpha} (u-w)_{,i}(u-w)_{,i}\,dx\right\}^{1/2} + m_6F \\ &\leqq m_7\left\{\frac{1}{2}\int_{D_\alpha} r\psi^2\,dx \int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\,dx\right\}^{1/2} + m_6F \end{aligned}
$$

by the Lipschitz condition on E, the A-G inequality, Schwarz's inequality, the analogue of (3.23), and the ellipticity condition. Here m_6 and m_7 are computable constants. Hence

$$
(4.20)\quad \int_{D_\alpha} \psi L\psi\,dx \geqq -m_7\left\{\frac{1}{2}\int_{D_\alpha} r\psi^2\,dx \int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\,dx\right\}^{1/2} - m_6F.
$$

Substitution of (4.20) into (4.17) gives

$$
(4.21)\quad \begin{aligned} FF'' - (F')^2 &\geqq G - F\left\{m_7\left(\frac{1}{2}\int_{D_\alpha} r\psi^2\,dx \int_{D_\alpha} a_{ij}\psi_{,i}\psi_{,j}\,dx\right)^{1/2} - m_6F\right\} \\ &\quad - 2|F'|\left|\int_0^\alpha \int_{\Sigma_\eta} \psi\frac{\partial\psi}{\partial\nu}\,ds\,d\eta\right| \\ &\geqq G - m_7F\left\{G + \left(\int_{D_\alpha} a_{ij}\psi_{,i}f_{,j}\psi\,dx\right)^2\right\}^{1/2} \\ &\quad - m_6F^2 - 2|F'|\left|\int_0^\alpha \int_{\Sigma_\eta} \psi\frac{\partial\psi}{\partial\nu}\,ds\,d\eta\right|. \end{aligned}
$$

We next observe that G is a continuous function of α; thus if we consider G pointwise we see that when $G > 0$,

$$\left\{G + \left(\int_{D_\alpha} a_{ij}\psi_{,i}f_{,j}\psi\, dx\right)^2\right\}^{1/2} \leqq \sqrt{G} + \left|\int_{D_\alpha} a_{ij}\psi_{,i}f_{,j}\psi\, dx\right| \leqq \sqrt{G} + |F| + \left|\int_0^\alpha \int_{\Sigma_\eta} \psi \frac{\partial \psi}{\partial \nu}\, ds\, d\eta\right| \tag{4.22}$$

by (3.25). Therefore,

$$FF'' - (F')^2 \geqq G - m_7 F\sqrt{G} - m_7 F|F'| - m_7 F\left|\int_0^\alpha \int_{\Sigma_\eta} \psi \frac{\partial \psi}{\partial \nu}\, ds\, d\eta\right| - m_6 F^2 - 2|F'|\left|\int_0^\alpha \int_{\Sigma_\eta} \psi \frac{\partial \psi}{\partial \nu}\, ds\, d\eta\right|. \tag{4.23}$$

Completing the square for the first two terms yields

$$FF'' - (F')^2 \geqq \left(\sqrt{G} - \frac{m_7 F}{2}\right)^2 - \frac{m_7^2 F^2}{4} - m_7 F|F'| - m_6 F^2 - (m_7 F + 2|F'|)\left|\int_0^\alpha \int_{\Sigma_\eta} \psi \frac{\partial \psi}{\partial \nu}\, ds\, d\eta\right|. \tag{4.24}$$

It can be shown (see [1]) that

$$\left|\int_0^\alpha \int_{\Sigma_\eta} \psi \frac{\partial \psi}{\partial \nu}\, ds\, d\eta\right| \leqq K_6 F \tag{4.25}$$

for a computable constant K_6. Using Lemma 4.2 we have for $G > 0$ that

$$FF'' - (F')^2 \geqq -\left(\frac{m_7^2}{4} + m_7 K_2 + m_6 + 2K_2K_6 + m_7 K_6\right)F^2 - (m_7 K_1 + 2K_1K_6)FF' \geqq -K_7 F^2 - K_8 FF'. \tag{4.26}$$

Now when $G \leqq 0$,

$$\left\{G + \left(\int_{D_\alpha} a_{ij}\psi_{,i}f_{,j}\psi\, dx\right)^2\right\}^{1/2} \leqq \left|\int_{D_\alpha} a_{ij}\psi_{,i}f_{,j}\psi\, dx\right| \leqq |F'| + \left|\int_0^\alpha \int_{\Sigma_\eta} \psi \frac{\partial \psi}{\partial \nu}\, ds\, d\eta\right| \leqq |F'| + K_6 F, \tag{4.27}$$

so that

$$FF'' - (F')^2 \geqq G - m_7 F|F'| - m_7 K_6 F^2 - m_6 F^2 - 2|F|K_6 F \geqq G - (m_7 K_2 + m_5 K_6 + m_6 + 2K_2K_6)F^2 - (m_7 K_1 + 2K_1K_6)FF'. \tag{4.28}$$

Using Schwarz's inequality on G and Lemma 4.3 we obtain for $G \leqq 0$,

$$
\begin{aligned}
FF'' - (F')^2 &\geqq -(m_9 K_4 + m_7 K_2 + m_7 K_6 + m_6 + 2K_2 K_6)F^2 \\
&\quad - (m_8 K_3 + m_7 K_1 + 2K_1 K_6)FF' \\
&\geqq -K_9 F^2 - K_{10} FF'.
\end{aligned} \tag{4.29}
$$

Thus regardless of the sign of G, combining (4.26) and (4.29) we have

$$
\begin{aligned}
FF'' - (F')^2 &\geqq -(\max(K_7, K_9))F^2 - (\max(K_8, K_{10}))FF' \\
&\geqq -K_{11} F^2 - K_{12} FF'.
\end{aligned} \tag{4.30}
$$

Therefore we have established the analogue of (3.22) with computable constants K_{11} and K_{12}.

By the same argument as at the end of § 3.2, we are led to the analogue of (3.31), that is,

$$F(\alpha) \leqq \sigma^{K_{11}/K_{12}^2}\{M_2^{2(1-\sigma)/(1-\sigma_1)} Q^{(\sigma-\sigma_1)/(1-\sigma_1)}\}. \tag{4.31}$$

But our Q has one term that did not appear in § 3.2. We must therefore show that the extra term is $0(\varepsilon)$ in order to obtain the desired theorem.

Since

$$\int_{D_{\alpha_1}} (u - v)_{,i}(u - v)_{,i}\, dx \leqq \int_{D_1} \tau^{s+2}(u - v)_{,i}(u - v)_{,i}\, dx, \tag{4.32}$$

we have by Lemma 4.1 that

$$\int_{D_{\alpha_1}} (u - v)_{,i}(u - v)_{,i}\, dx \leqq 0(\varepsilon) + k_{11} \int_{D_1} \tau^s (u - v)^2\, dx, \tag{4.33}$$

where we have assumed $b < 0$ and dropped the last term involving $\varepsilon \int_{D_1} \tau^{s+2}(Lv)^2\, dx$. Hence, by the analogue of Lemma 3.1, we have

$$\int_{D_{\alpha_1}} (u - v)_{,i}(u - v)_{,i}\, dx \leqq 0(\varepsilon) + R_0 k_{11} \varepsilon \int_{D_1} \tau^{s-2}(u - v)^2\, dx \tag{4.34}$$

and by the analogue of Lemma 3.3 we have

$$\int_{D_{\alpha_1}} (u - v)_{,i}(u - v)_{,i}\, dx \leqq 0(\varepsilon) + R_0 k_{11} R \varepsilon = 0(\varepsilon). \tag{4.35}$$

Thus, if $b < 0$ we have shown that the extra term in Q is indeed $0(\varepsilon)$. Note we could still show that Q is $0(\varepsilon)$ even if $b > 0$. However, then we would have to assume that $\int_D (Lv)^2\, dx$ is bounded (independent of ε) which does not seem to be a practical hypothesis.

We have established the following results.

THEOREM 4.4. *If $v, w \in \tilde{M}$, w is a solution of Problem* B′ *and u, v are solutions of Problem* A′, *then the difference $u - w$ satisfies the following continuous dependence inequality for α in the range* $0 \leqq \alpha < \alpha_1 < 1$:

$$\|u - w\|_{D_\alpha}^2 \leqq R M_7^{2\nu(\alpha)} Q^{1-\nu(\alpha)}.$$

Here R, M_7, and $\nu(\alpha)$ are computable, with $0 \leqq \nu(\alpha) < 1$, *and Q is given by* (4.9).

COROLLARY 4.5. *With the same hypotheses as in Theorem* 4.5, *and with* $b < 0$, *we conclude that*

$$\|u - w\|_{D_\alpha}^2 = 0(\varepsilon^{1-\nu(\alpha)})$$

for α *in the range* $0 \leqq \alpha < \alpha_1 < 1$ *and* $\nu(\alpha)$ *satisfying* $0 \leqq \nu(\alpha) < 1$.

4.3. We now combine the results of § 4.1 and § 4.2 and use the triangle inequality

$$\|v - w\|_{D_\alpha} \leqq \|v - u\|_{D_\alpha} + \|u - w\|_{D_\alpha}$$

to establish the following theorem.

THEOREM 4.6. *If* $v \in \tilde{M}$ *is a solution of Problem* A′, $w \in M$ *is a solution of Problem* B′, u *is as in Theorem* 4.4, *and* $b < 0$, *then the difference* $v - w$ *satisfies the following continuous dependence inequality for* α *in the range* $0 \leqq \alpha < \alpha_1 < 1$ *and* $\nu(\alpha)$ *with* $0 \leqq \nu(\alpha) < 1$:

$$\|v - w\|_{D_\alpha}^2 = 0(\varepsilon^{1-\nu(\alpha)}).$$

Remark. The order of ε here is the same as that in Theorem 3.10(ii) of the preceding section or half of the order we computed in the previous section when we assumed $b < 0$.

Acknowledgment. The author is very grateful to Professor L. E. Payne of Cornell University for his helpful advice and suggestions.

REFERENCES

[1] L. E. ADELSON, *Singular perturbations in a class of improperly posed problems*, Doctoral thesis, Cornell University, Ithaca, N.Y., 1970.

[2] S. AGMON, *Unicité et convexité dans les problèmes différentiels*, Sem. Math. Sup. (1965), University of Montreal Press, 1966.

[3] S. AGMON AND L. NIRENBERG, *Lower bounds and uniqueness theorems for solutions of differential equations in a Hilbert space*, Comm. Pure Appl. Math., 20 (1967), pp. 207–229.

[4] J. CONLAN AND G. N. TRYTTEN, *Pointwise bounds in the Cauchy problem for elliptic systems of partial differential equations*, Arch. Rational Mech. Anal., 22 (1966), pp. 143–152.

[5] J. HADAMARD, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale University Press, New Haven, Conn., 1923.

[6] F. JOHN, *Continuous dependence on the data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math., 13 (1960), pp. 551–585.

[7] M. M. LAURENTIEV, *Some Improperly Posed Problems of Mathematical Physics*, Springer Tracts in Natural Philosophy, vol. II, Springer, Berlin, 1967.

[8] ———, *On the Cauchy problem for the Laplace equation*, Izvest. Akad. Nauk. SSSR, Ser. Mat., 20 (1965), pp. 819–842.

[9] H. LEVINE, *Logarithmic convexity and the Cauchy problem for some abstract second order differential inequalities*, J. Differential Equations, 8 (1970), pp. 34–55.

[10] N. LEVINSON, *The first boundary value problem for* $\varepsilon\Delta u + Au_x + Bu_y + Cu = D$ *for small* ε, Ann. of Math., 51 (1950), pp. 121–139.

[11] L. E. PAYNE, *Bounds in the Cauchy problem for the Laplace equation*, Arch. Rational Mech. Anal., 5 (1960), pp. 35–45.

[12] ———, *On some non-well-posed problems for partial differential equations*, Numerical Solutions of Nonlinear Differential Equations, John Wiley, New York, 1966, pp. 239–263.

[13] ———, *On a priori bounds in the Cauchy problem for elliptic equations*, this Journal, 1 (1970), pp. 82–89.

[14] L. E. PAYNE AND D. SATHER, *On singular perturbation in non well posed problems*, Ann. Mat. Pura Appl., (IV), 75 (1967), pp. 219–230.

[15] C. PUCCI, *Discussione del problema di Cauchy per le equazione di tipo ellitico*, Ibid., 46 (1958), pp. 131–153.

[16] P. W. SCHAEFER, *On the Cauchy problem for elliptic systems*, Arch. Rational Mech. Anal., 20 (1965), pp. 391–412.

[17] G. N. TRYTTEN, *Pointwise bounds for solutions in the Cauchy problem for elliptic equations*, Ibid., 13 (1963), pp. 222–244.

[18] M. I. VIŠIK AND L. A. LIUSTERNIK, *Solutions of some perturbation problems in the case of matrices and self-adjoint or non-self-adjoint differential equations*, Russian Math. Surveys, 15 (1960), no. 3, pp. 1–73.

[19] ———, *Regular degeneration for linear differential equations with small parameter*, Amer. Math. Soc. Transl., 20 (1962), pp. 293–364.

[20] W. WASOW, *Asymptotic solution of boundary value problems for the differential equation* $\Delta U + \lambda(\partial U/\partial x) = f(x, y)$, Duke Math. J., 11 (1944), pp. 405–415.

[21] ———, *Singular perturbation problems of systems of second order analytic differential equations*, Arch. Rational Mech. Anal., 14 (1963), pp. 61–80.

[22] M. ZLAMAL, *The parabolic equation as a limiting case of a certain elliptic equation*, Ann. Mat. Pura Appl., 57 (1962), pp. 143–150.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

EXPLICIT INTEGRAL TRANSFORM PROOFS OF SOME TRANSPLANTATION THEOREMS FOR THE HANKEL TRANSFORM*

SUSAN SCHINDLER†

Abstract. The transplantation theorem for Hankel transforms states that

$$\int_0^\infty |g_\mu(x)|^p x^{\alpha p}\,dx \cong \int_0^\infty |g_\nu(x)|^p x^{\alpha p}\,dx,$$

where

$$g_\mu(x) = \int^\infty G(z)(xz)^{1/2}J_\mu(xz)\,dz, \qquad \mu,\nu \geqq -\tfrac{1}{2}, \qquad 1 < p < \infty, \quad \text{and} \quad -1/p < \alpha < 1 - 1/p.$$

It was proved, in a very elegant way, by D. L. Guy in 1960. His proof is, however, an indirect one. It does not completely illuminate (i) the role played by the singular integral transform which arises, (ii) for what values of μ, ν the singularity disappears, and (iii) when one may expect an L^1 theorem. Our proof explicitly gives the relationship, via an integral transform, between $g_\mu(x)$ and $g_\nu(x)$, and immediately answers these questions. To find the kernel of this integral transform, we first suppose g_ν is a "good" function. The Weber–Schafheitlin formula is used and in the course of justifying its applicability we obtain a δ-function. The kernel also involves hypergeometric functions for which the needed estimates are derived. We then extend the representation obtained to more general g_ν; the mapping theorems then follow easily.

1. Introduction. In 1960, D. L. Guy proved the first of the "transplantation" theorems for classical expansions. His result is the existence of a constant $A(\mu, \nu, p, \alpha)$ such that

$$\int_0^\infty |g_\mu(x)|^p x^{\alpha p}\,dx \leqq A \int_0^\infty |g_\nu(x)|^p x^{\alpha p}\,dx,$$

where $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, $\mu, \nu \geqq -\frac{1}{2}$, $g_\mu(x) = \int_0^\infty G(t)(xt)^{1/2}J_\mu(xt)\,dt$, and $g_\nu(x)$ is given analogously. His proof is elegant, but rather indirect. It uses Fourier transforms and convolutions, the Weber–Schafheitlin formula and the M. Riesz multiplier theorem. We shall rely on the Weber–Schafheitlin formula and fairly elementary techniques to find explicitly the kernel of the integral transform which maps g_ν to g_μ. The above L^p estimates will then follow from the Riesz conjugate function theorem, and estimates, which we shall derive, for the hypergeometric function. We shall arrive at the formula

$$g_\mu(x) = \int_0^{x^-} + \int_{x^+}^\infty g_\nu(y)\tilde{I}_{\mu,\nu}(x,y)\,dy + \left[\cos\left(\frac{\mu\pi}{2} - \frac{\nu\pi}{2}\right)\right] g_\nu(x),$$

where for $0 < y < x$,

$$\tilde{I}_{\mu,\nu}(x,y) = \frac{2(xy)^{1/2}}{\Gamma(\nu+1)} \frac{\Gamma((\mu+\nu+2)/2)}{\Gamma((\mu-\nu)/2)} \left(\frac{y}{x}\right)^\nu \frac{1}{x^2 - y^2} \cdot {}_2F_1\left(\frac{\nu-\mu}{2}, \frac{\nu+\mu}{2}; \nu+1; \frac{y^2}{x^2}\right),$$

and for $y > x$, $\tilde{I}_{\mu,\nu}(x,y) = \tilde{I}_{\nu,\mu}(y,x)$.

* Received by the editors December 9, 1971.

† Department of Mathematics, The Bernard M. Baruch College of The City University of New York, New York, New York 10010.

When $\mu = \nu + 2k$, $k = 1, 2, \cdots$, this transformation is also bounded for $p = 1$, $\alpha < \nu + \frac{1}{2}$ and for $p > 1$ for a different range of α than Guy's theorem gives. Also, $\cos(\mu\pi/2 - \nu\pi/2) = 0$ when $\mu - \nu = \pm 1$ and here one gets the generalized functions of Muckenhoupt and Stein [3].

For $\mu \geqq -\frac{1}{2}$, and $G \in L^1(0, \infty)$ the Hankel μ-transform of G is defined by:

$$g_\mu(x) = \int_0^\infty G(z)(xz)^{1/2} J_\mu(xz)\, dz. \tag{1.1}$$

We shall first find a relationship between g_μ and g_ν when G is a fixed "good" function ($\mu, \nu > -\frac{1}{2}$ from here on). Denote by $C_c^\infty(0, \infty)$ the set of infinitely differentiable functions of compact support in $(0, \infty)$. Let $\mathscr{H}_\nu = \{G \in L^1 \cap C[0, \infty): g_\nu \in C_c^\infty(0, \infty)\}$.

2. The μ-transform of a function $G \in \mathscr{H}_\nu$ as a sum of integral transforms of g_ν. If $G \in \mathscr{H}_\nu$,

$$G(z) = \int_0^\infty g_\nu(y)(yz)^{1/2} J_\nu(yz)\, dy, \qquad z \geqq 0.$$

(This is fairly easy to show from the fact that the inversion formula is valid for g_ν.) Hence, for $G \in \mathscr{H}_\nu$,

$$g_\mu(x) = \int_0^\infty \int_0^\infty g_\nu(y)(yz)^{1/2} J_\nu(yz)(xz)^{1/2} J_\mu(xz)\, dy\, dz.$$

We may view the inner integral as a Riemann integral. By the kind of Riemann argument to be invoked in the lemmas, the inner integral may be split into two Riemann integrals, each of which is Lebesgue integrable, that is, for $\delta > 0$,

$$g_\mu(x) = \int_0^\infty \left\{ \int_{|x-y| \geqq \delta} + \int_{|x-y| \leqq \delta} \right\} g_\nu(y)(yz)^{1/2} J_\nu(yz)(xz)^{1/2} J_\mu(xz)\, dy\, dz$$

or

$$g_\mu(x) = \int_0^\infty \int_{|x-y| \geqq \delta} g_\nu(y)(yz)^{1/2} J_\nu(yz)(xz)^{1/2} J_\mu(xz)\, dy\, dz + C_{\mu,\nu}(G; \delta; x). \tag{2.1}$$

The second term will approach a constant times $g_\nu(x)$. We cannot directly apply Fubini's theorem; we apply Lemmas 1–4 instead to a *fixed* $G \in \mathscr{H}_\nu$.

LEMMA 1. *With $C_{\mu,\nu}(\delta; x) = C_{\mu,\nu}(G; \delta; x)$, there is a constant $C'_{\mu,\nu} = C'_{\nu,\mu}$ (independent of G) such that*

$$\lim_{\delta \downarrow 0} C_{\mu,\nu}(\delta; x) = C'_{\mu,\nu} g_\nu(x), \qquad x > 0.$$

LEMMA 2. *Let*

$$g_{\mu,\rho}(x) = \int_0^\infty \frac{G(z)}{z^\rho}(xz)^{1/2} J_\mu(xz)\, dz.$$

Then

$$\lim_{\rho\downarrow 0} g_{\mu,\rho}(x) = g_\mu(x).$$

(Note that $g_{\mu,\rho}(x)$ is both a Lebesgue and a Riemann integral.)

LEMMA 3. *Let*

$$C_{\mu,\nu}(\delta;\rho;x) = \int_0^\infty \frac{1}{z^\rho}\int_{|x-y|\leqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz.$$

Then $C_{\mu,\nu}(\delta;\rho;x)$ *is a Riemann integral with* $\lim_{\rho\downarrow 0} C_{\mu,\nu}(\delta;\rho;x) = C_{\mu,\nu}(\delta;x)$.

LEMMA 4.

$$\lim_{c\downarrow 0}\int_0^\infty \frac{e^{-cz}}{z^\rho}\int_{|x-y|\leqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz = C_{\mu,\nu}(\delta;\rho;x).$$

Proof of Lemma 1. The function $w^{1/2}J_\alpha(w)$ is bounded for all w if $\alpha \geqq -\frac{1}{2}$ (see [5, p. 16]). Then

$$\left|\int_0^1\int_{|x-y|\leqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz\right| \leqq M\int_0^1\int_{|x-y|\leqq\delta}|g_\nu(y)|\,dy\,dz$$
$$\leqq M\delta\|g_\nu\|_\infty,$$

and so this contribution to $C_{\mu,\nu}(\delta;x)$ tends to zero as $\delta\downarrow 0$. The contribution from $1\leqq z<\infty$ is somewhat more complicated.

First we expand the function g_ν about x: $g_\nu(y) = g_\nu(x) + (x-y)h_\nu(y)$, where $h_\nu(y)$ is infinitely differentiable. To prove Lemma 1 it is enough to prove the following facts:

$$\lim_{\delta\downarrow 0}\int_0^\infty\int_{|x-y|\leqq\delta}(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz = C'_{\mu,\nu} \tag{2.2}$$

exists, and

$$\lim_{\delta\downarrow 0}\int_0^\infty\int_{|x-y|\leqq\delta}(x-y)h_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz = 0. \tag{2.3}$$

We use the expansion [5, p. 14, (1.71.8)]:

$$w^{1/2}J_\alpha(w) = \sqrt{\frac{2}{\pi}}\cos\left(w-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\sum_{k=0}^{p-1}a_k w^{-2k} + O(w^{-2p})$$
$$+\sqrt{\frac{2}{\pi}}\sin\left(w-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\sum_{k=0}^{p-1}b_k w^{-2k-1}$$

for $w\to\infty$. Take

$$\begin{aligned}\beta_\alpha &= \frac{-\alpha\pi}{2}-\frac{\pi}{4},\quad \sigma_1 = x-\delta,\quad \sigma_2 = x+\delta,\\ C_1 &= \beta_\mu+\beta_\nu,\quad C_2 = \beta_\mu-\beta_\nu.\end{aligned} \tag{2.4}$$

Then

$$\frac{2}{\pi}\int_1^\infty \cos(xz+\beta_\mu)\int_{|x-y|\leqq\delta}\cos(yz+\beta_\nu)\,dy\,dz$$
$$=\frac{1}{\pi}\int_1^\infty \frac{\sin([2x+\delta]z+\beta_\mu+\beta_\nu)-\sin([2x-\delta]z+\beta_\mu+\beta_\nu)\,dz}{z}$$
$$+\frac{1}{\pi}\int_1^\infty \frac{\sin(\delta z+\beta_\mu-\beta_\nu)-\sin(-\delta z+\beta_\mu-\beta_\nu)\,dz}{z}.$$

The second integral becomes

$$\frac{1}{\pi}\int_\delta^\infty \frac{\sin(u+C_2)-\sin(-u+C_2)}{u}\,du=\frac{2\cos C_2}{\pi}\int_\delta^\infty \frac{\sin u}{u}\,du,$$

which approaches $\cos(\mu\pi/2-\nu\pi/2)$ as $\delta\downarrow 0$. We shall show that all other integrals contributing to $C_{\mu,\nu}(\delta;x)$ tend to zero with δ, that is,

$$C'_{\mu,\nu}=\cos\left(\frac{\mu\pi}{2}-\frac{\nu\pi}{2}\right)=C'_{\nu,\mu}. \tag{2.5}$$

Now

$$\int_1^\infty \frac{\sin([2x+\delta]z+C_1)-\sin([2x-\delta]z+C_1)}{z}\,dz$$
$$=\frac{\cos(2x+\delta+C_1)}{2x+\delta}-\frac{\cos(2x-\delta+C_1)}{2x-\delta}$$
$$-\int_1^\infty \frac{1}{z^2}\left\{\frac{\cos([2x+\delta]z+C_1)}{2x+\delta}-\frac{\cos([2x-\delta]z+C_1)}{2x-\delta}\right\}dz.$$

The integrated term clearly approaches zero; so, too, does the integral, by the dominated convergence theorem.

Next, we take the remainder after one term in our expansion for $(xz)^{1/2}J_\mu(xz)$ and we examine

$$\int_1^\infty\int_{|x-y|\leqq\delta} O\left(\frac{1}{xz}\right)(yz)^{1/2}J_\nu(yz)\,dy\,dz.$$

By using the standard differentiation formula: $D_w w^\alpha J_\alpha(w)=w^\alpha J_{\alpha-1}(w)$, [1, p. 11, (50)], this can be written as:

$$\int_1^\infty O(z^{-2})\int_{|x-y|\leqq\delta}(yz)^{-\nu-1/2}(yz)^{\nu+1}J_\nu(yz)z\,dy\,dz$$
$$=\int_1^\infty O(z^{-2})\{(\sigma_2 z)^{1/2}J_{\nu+1}(\sigma_2 z)-(\sigma_1 z)^{1/2}J_{\nu+1}(\sigma_1 z)\}\,dz$$
$$+(\nu+\tfrac{1}{2})\int_1^\infty O(z^{-2})\int_{|x-y|\leqq\delta}\frac{1}{y}(yz)^{1/2}J_{\nu+1}(yz)\,dy\,dz.$$

Both integrals approach zero, by dominated convergence.

We must still work with the product of the leading term for $(xz)^{1/2}J_\mu(xz)$ and the remainder after one term of the expansion for $(yz)^{1/2}J_\nu(yz)$. First

$$\int_1^\infty \cos(xz+\beta_\mu)\int_{|x-y|\leqq\delta}\frac{\sin(yz+\beta_\nu)}{yz}\,dy\,dz$$

$$\int_1^\infty \frac{\cos(xz+\beta_\mu)}{z^2}\left\{\frac{-\cos(yz+\beta_\nu)}{y}\bigg|_{y=\sigma_1}^{\sigma_2} - \int_{|x-y|\leqq\delta}\frac{\cos(yz+\beta_\nu)}{y^2}\,dy\right\}dz.$$

The bracketed expression is bounded, approaches zero, and the usual reasoning shows that the double integral has limit zero.

We next have to evaluate

$$\int_1^\infty \cos(xz+\beta_\mu)\int_{|x-y|\leqq\delta} O\left(\frac{1}{(yz)^2}\right)dy\,dz.$$

(The "O" term is the remainder after 2 terms in the expansion for $(yz)^{1/2}J_\nu(yz)$.) This is easily seen to approach zero with δ by the dominated convergence theorem, completing the proof of (2.2).

Considering next (2.3), it is easy to see that

$$\int_0^1\int_{|x-y|\leqq\delta} h_\nu(y)(x-y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz\to 0$$

as $\delta\downarrow 0$, again, by the dominated convergence theorem. To find the contribution for $1\leqq z<\infty$, we first replace $(yz)^{1/2}J_\nu(yz)$ by $\cos(yz+\beta_\nu)$. Then

$$\begin{aligned}
&\int_{|x-y|\leqq\delta} h_\nu(y)(x-y)\cos(yz+\beta_\nu)\,dy\\
&\quad=\frac{1}{z}\left\{h_\nu(y)(x-y)\sin(yz+\beta_\nu)\bigg|_{y=\sigma_1}^{\sigma_2}\right\}\\
&\qquad-\frac{1}{z}\int_{|x-y|\leqq\delta} D_y[h_\nu(y)(x-y)]\sin(yz+\beta_\nu)\,dy\\
&\quad=\frac{1}{z}\left\{h_\nu(y)(x-y)\sin(yz+\beta_\nu)\bigg|_{y=\sigma_1}^{\sigma_2}\right\}\\
&\qquad+\frac{1}{z^2}D_y[h_\nu(y)(x-y)]\cos(yz+\beta_\nu)\bigg|_{y=\sigma_1}^{\sigma_2}\\
&\qquad-\frac{1}{z^2}\int_{|x-y|\leqq\delta} D_y^2[h_\nu(y)(x-y)]\cos(yz+\beta_\nu)\,dy.
\end{aligned}$$

Now the last two terms are bounded by a constant over z^2 and both approach zero as δ approaches zero. Hence their contribution to

$$\int_1^\infty\int_{|x-y|\leqq\delta} h_\nu(y)(x-y)(xz)^{1/2}J_\mu(xz)\cos(yz+\beta_\nu)\,dy\,dz$$

has limit zero. We have to show that

$$\lim_{\delta\downarrow 0}\int_1^\infty \frac{(xz)^{1/2}J_\mu(xz)}{z}\left\{h_\nu(y)(x-y)\sin(yz+\beta_\nu)\Big|_{y=\sigma_1}^{\sigma_2}\right\}dz=0.$$

Now $(xz)^{1/2}J_\mu(xz)=\sqrt{2/\pi}\cos(xz+\beta_\mu)+O(1/xz)$ and by the usual reasoning,

$$\lim_{\delta\downarrow 0}\int_1^\infty O\left(\frac{1}{xz}\right)\frac{1}{z}\left\{h_\nu(y)(x-y)\cdot\sin(yz+\beta_\nu)\Big|_{y=\sigma_1}^{\sigma_2}\right\}dz=0.$$

Our next task is to show that

$$\int_1^\infty \frac{\cos(xz+\beta_\mu)}{z}h_\nu(y)(x-y)\sin(yz+\beta_\nu)\Big|_{y=\sigma_1}^{\sigma_2}dz\to 0.$$

This expression equals

$$-\frac{\delta}{2}\int_1^\infty \frac{\sin([2x+\delta]z+C_1)h_\nu(\sigma_2)+\sin([2x-\delta]z+C_1)h_\nu(\sigma_1)}{z}dz$$
$$+\frac{\delta}{2}\int_1^\infty \frac{\sin(-\delta z+C_2)h_\nu(\sigma_2)+\sin(\delta z+C_2)h_\nu(\sigma_1)}{z}dz.$$

The first term equals

$$\frac{\delta}{2}\left\{\frac{\cos([2x+\delta]z+C_1)h_\nu(\sigma_2)}{(2x+\delta)z}+\frac{\cos([2x-\delta]z+C_1)h_\nu(\sigma_1)}{(2x-\delta)z}\Big|_{z=1}^{\infty}\right\}$$
$$+\frac{\delta}{2}\int_1^\infty\frac{1}{z^2}\left\{\frac{\cos([2x+\delta]z+C_1)h_\nu(\sigma_2)}{2x+\delta}+\frac{\cos([2x-\delta]z+C_1)h_\nu(\sigma_1)}{2x-\delta}\right\}dz.$$

Inspection and the dominated convergence theorem show that this approaches zero. By integration by parts, the second integral equals

$$\frac{1}{2}\left(\frac{\cos(-\delta z+C_2)h_\nu(\sigma_2)}{z}-\frac{\cos(\delta z+C_2)h_\nu(\sigma_1)}{z}\right)\Bigg|_{z=1}^{\infty}$$
$$+\frac{1}{2}\int_1^\infty\frac{\cos(-\delta z+C_2)h_\nu(\sigma_2)-\cos(\delta z+C_2)h_\nu(\sigma_1)}{z^2}dz,$$

and, again, the two terms have limit zero. We now have shown that

$$\lim_{\delta\downarrow 0}\int_1^\infty (xz)^{1/2}J_\mu(xz)\int_{|x-y|\leq\delta}h_\nu(y)(x-y)\cos(yz+\beta_\nu)\,dy\,dz=0.$$

The second term in the expansion of $(yz)^{1/2}J_\nu(yz)$ is a constant times

$$\frac{\sin(yz+\beta_\nu)}{yz}.$$

Now

$$\left|\int_1^\infty\frac{(xz)^{1/2}J_\mu(xz)}{z}\int_{|x-y|\leq\delta}\frac{h_\nu(y)}{y}(x-y)\sin(yz+\beta_\nu)\,dy\,dz\right|\leq\int_1^\infty\frac{M}{z^2}O(\delta)\,dz$$

by integration by parts of the inner integral. It therefore has limit zero. The remainder in the expansion of $(yz)^{1/2}J_\nu(yz)$ is $O\{(yz)^{-2}\}$ and the dominated convergence theorem is once again sufficient. The proof of (2.3), and hence of Lemma 1, is complete.

Lemma 2 is obvious, since G is continuous and in $L^1(0, \infty)$.

Proof of Lemma 3. We must show $\lim_{\rho\downarrow 0} C_{\mu,\nu}(\delta;\rho;x) = C_{\mu,\nu}(\delta;0;x) = C_{\mu,\nu}(\delta;x)$. Now

$$\int_{|x-y|\leqq\delta} g_\nu(y)(yz)^{1/2}J_\nu(yz)\,dy$$
$$= \frac{1}{z}g_\nu(y)(yz)^{1/2}J_{\nu+1}(yz)\Big|_{y=\sigma_1}^{\sigma_2}$$
$$-\frac{1}{z}\int_{|x-y|\leqq\delta} D_y\{g_\nu(y)(yz)^{-\nu-1/2}\}(yz)^{\nu+1}J_{\nu+1}(yz)\,dy.$$

Therefore,

$$\int_{|x-y|\leqq\delta} g_\nu(y)(yz)^{1/2}J_\nu(yz)\,dy$$
$$= \frac{1}{z}g_\nu(y)(yz)^{1/2}J_{\nu+1}(yz)\Big|_{y=\sigma_1}^{\sigma_2}$$
$$-\frac{1}{z}\int_{|x-y|\leqq\delta} g_\nu'(y)(yz)^{1/2}J_{\nu+1}(yz)\,dy$$
$$+(\nu+\tfrac{1}{2})\int_{|x-y|\leqq\delta} g_\nu(y)(yz)^{-1/2}J_{\nu+1}(yz)\,dy.$$

Let us first work with the integrated term, evaluated at σ_1. We multiply it by $(xz)^{1/2}J_\mu(xz)$, divide by z^ρ and integrate with respect to z. By the Weber–Schafheitlin integral [6, p. 401, (2)], we obtain, for $\mu+\nu+2>\rho>0$,

$$g_\nu(\sigma_1)(x\sigma_1)^{1/2}\int_0^\infty \frac{J_\mu(xz)J_{\nu+1}(\sigma_1 z)}{z^\rho}\,dz$$
$$= g_\nu(\sigma_1)(x\sigma_1)^{1/2}\frac{(\sigma_1)^{\nu+1}\Gamma((\mu+\nu+2-\rho)/2)}{2^\rho x^{-\rho}\Gamma(\nu+2)\Gamma((\mu-\nu+\rho)/2)}$$
$$\cdot {}_2F_1\left(\frac{\mu+\nu+2-\rho}{2},\frac{\nu+2-\mu-\rho}{2};\nu+2;\left(\frac{\sigma_1}{x}\right)^2\right).$$

We have x and σ_1 fixed so this expression is continuous at $\rho=0$. Evaluation of the integrated term at $y=\sigma_2$ leads us to consider the other noncritical case of the Weber–Schafheitlin integral. We see that the contribution of the integrated term to $C_{\mu,\nu}(\delta;\rho;x)$ is continuous at $\rho=0$.

The unintegrated terms are both $O(z^{-1+(1/2)+\nu+1}) = O(z^{(1/2)+\nu})$ for $z\to 0$. Since $\frac{1}{2}+\nu>0$ and $(xz)^{1/2}J_\mu(xz)$ is bounded, we see that the integral of these terms $\int_0^1 \cdots dz/z^\rho$ are continuous functions at $\rho = 0$. For $1 \leqq z < \infty$, we have

$$-\frac{1}{z}\int_{|x-y|\leqq\delta}\left\{g_\nu'(y) - \left(\nu + \frac{1}{2}\right)\frac{g_\nu(y)}{y}\right\}(yz)^{1/2}J_{\nu+1}(yz)\,dy,$$

and one more integration by parts shows that this expression is $O(z^{-2})$. The dominated convergence theorem finishes the argument.

Proof of Lemma 4. Clearly

$$\int_0^1 \frac{e^{-cz}}{z^\rho}\int_{|x-y|\leqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}\cdot J_\nu(yz)\,dy\,dz$$

has the desired limit, and integration of

$$\int_{|x-y|\leqq\delta} g_\nu(y)(yz)^{1/2}J_\nu(yz)\,dy$$

by parts and reference to the dominated convergence theorem shows that

$$\int_1^\infty \frac{e^{-cz}}{z^\rho}\int_{|x-y|\leqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz$$

tends to the same limit.

We now work with

$$g_{\mu,\rho}(x) = \int_0^\infty \frac{1}{z^\rho}\int_{|x-y|\geqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz + C_{\mu,\nu}(\delta;\rho;x).$$

Note that as a consequence of the proof of Lemma 4 both terms here are Lebesgue integrals (dz). Clearly,

$$\lim_{c\downarrow 0}\int_0^\infty \frac{e^{-cz}}{z^\rho}G(z)(xz)^{1/2}J_\mu(xz)\,dz = \int_0^\infty \frac{G(z)}{z^\rho}(xz)^{1/2}J_\mu(xz)\,dz$$

and so by Lemma 4,

$$\int_0^\infty \frac{1}{z^\rho}\int_{|x-y|\geqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz$$
$$= \lim_{c\downarrow 0}\int_0^\infty \frac{e^{-cz}}{z^\rho}\int_{|x-y|\leqq\delta} g_\nu(y)(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)\,dy\,dz.$$

Now the inner integral is bounded so that Fubini's theorem applies and we have

$$\lim_{c\downarrow 0}\int_{|x-y|\geqq\delta} g_\nu(y)(xy)^{1/2}\int_0^\infty e^{-cz}\frac{J_\mu(xz)J_\nu(yz)}{z^{-1+\rho}}\,dz\,dy.$$

Let

$$I_{c,\rho}(x,y) = \int_0^\infty e^{-cz}\frac{J_\mu(xz)J_\nu(yz)}{z^{-1+\rho}}\,dz. \tag{2.6}$$

We need a lemma concerning $I_{c,\rho}$.

LEMMA 5. *Let* $x > 0$. *The function* $g_\nu(y)(xy)^{1/2}I_{c,\rho}(x,y)$ *is bounded by a function which is in* $L^1(dy)$ *on* $|x-y| \geqq \delta$ *and* $\lim_{c\downarrow 0} I_{c,\rho}(x,y) = I_{0,\rho}(x,y)$.

Proof. The essence of the argument is contained in [6, pp. 399–401]. For the proof of the second part of the statement, note that the integral for $I_{0,\rho}(x, y)$ converges ($x \neq y$) and

$$I_{0,\rho}(x, y) = \lim_{R\uparrow\infty} \int_0^R \frac{J_\mu(xz)J_\nu(yz)}{z^{-1+\rho}}\, dz$$

$$= \lim_{R\uparrow\infty} \lim_{c\downarrow 0} \int_0^{R} e^{-cz} \frac{J_\mu(xz)J_\nu(yz)}{z^{-1+\rho}}\, dz.$$

Replacement of either $(xz)^{1/2}J_\mu(xz)$ or $(yz)^{1/2}J_\nu(yz)$ by the remainder after a one-term expansion for large z gives something uniformly convergent in R as $c \downarrow 0$. For the leading terms we have

$$(xy)^{-1/2} \lim_{c\downarrow 0} \int_0^R \frac{e^{-cz}}{z^\rho} \cos(xz + \beta_\mu)\cos(yz + \beta_\nu)\, dz$$

$$= \frac{(xy)^{-1/2}}{2} \lim_{c\downarrow 0} \int_0^R \frac{e^{-cz}}{z^\rho}[\cos(xz + yz + \beta_\mu + \beta_\nu) + \cos(xz - yz + \beta_\mu - \beta_\nu)]\, dz.$$

Consideration of

$$\int_0^R \frac{e^{-cz}}{z^\rho} \cos wz\, dz$$

for $w \neq 0$ shows that the limit as $c \downarrow 0$ is uniform in R, finishing the proof for the second part of the lemma.

We first prove the asserted boundedness for $y < x - \delta$. With

$$A_1 = \max\{|\mu + \nu - \rho + 2|, |\nu - \mu - \rho + 1|\}, \qquad 0 < c \leqq C < \delta,$$

$X = (C^2)/(x^2 + C^2)$, we have, according to Watson [6], that $|I_{c,p}|$ is bounded by the sum of two series $I'_{c,\rho}$ and $I''_{c,\rho}$ and

$$I'_{c,\rho}(x, y) \leqq \frac{(1/2)^{\mu+\nu}\Gamma(1/2)(1 - \sqrt{X})^{-A_1} y^\nu}{(x^2 + C^2)^{(\mu+\nu-\rho+2)/2}}$$

$$\cdot \sum_{m=0} \frac{\Gamma(\mu + \nu - \rho + 2 + 2m)}{m!\Gamma(\nu + 1 + m)|\Gamma((\mu - \nu + \rho)/2 - m)|\Gamma((\mu + \nu - \rho + 3)/2 + m)}$$

$$\cdot \left[\left(\frac{y}{2} \frac{1}{\sqrt{x^2 + C^2} - C}\right)^2\right]^m.$$

The power series has radius of convergence $\frac{1}{4}$ and

$$\frac{y}{2[\sqrt{x^2 + C^2} - C]} \leqq \frac{(x - \delta)(\sqrt{x^2 + \delta^2} + \delta)}{2x^2} < \frac{1}{2},$$

and so $I'_{c,\rho}(x, y) \leqq L'(\mu, \nu, x)y^\nu$. We also obtain the same kind of bound for $I''_{c,\rho}$. Since $\nu > -\frac{1}{2}$ this establishes the existence of the required L^1 bound on

$$0 \leqq y \leqq x - \delta.$$

The proof for $x + \delta \leqq y$ is easier. Taking

$$A_2 = \max\{|\mu + \nu - \rho + 2|, |-\nu + \mu - \rho + 1|\},$$

letting $0 < c \leqq C < \delta$, $Y = C^2/(y^2 + C^2)$, reversing the roles of μ, ν and x, y, we see that $|I_{c,\rho}|$ is bounded by the sum of two series, which we again call $I'_{c,\rho}$ and $I''_{c,\rho}$. We have

$$I'_{c,\rho}(x, y) \leqq \frac{(1/2)^{\nu+\mu}\Gamma(1/2)(1 - \sqrt{Y})^{-A_2}x^\mu}{(y^2 + C^2)^{(\nu+\mu-\rho+2)/2}}$$

$$\cdot \sum_{m=0}^{\infty} \frac{\Gamma(\nu + \mu - \rho + 2 + 2m)}{m!\Gamma(\mu + 1 + m)|\Gamma((\nu - \mu + \rho)/2 - m)|\Gamma((\nu + \mu - \rho + 3)/2 + m)}$$

$$\cdot \left[\left(\frac{x}{2(\sqrt{y^2 + C^2} - C)}\right)^2\right]^m.$$

The radius of convergence is 1/4. Now

$$\frac{x}{2(\sqrt{y^2 + C^2} - C)} \leqq \frac{(y - \delta)[\sqrt{y^2 + \delta^2} + \delta]}{2y^2} < \frac{1}{2}$$

for each y. Hence, there is an $\eta, 0 < \eta < \frac{1}{2}$, such that for $x + \delta \leqq y \in \operatorname{supp} g_\nu$, $x/(2(\sqrt{y^2 + C^2} - C)) \leqq \eta$. This implies that

$$I'_{c,\rho}(x, y) \leqq \frac{L'(\mu, \nu, x)(1 - \sqrt{Y})^{-A_2}}{(y^2 + C^2)^{(\nu+\mu-\rho+2)/2}}$$

on $x + \delta \leqq y \in \operatorname{supp} g_\nu$. Now $((1 - \sqrt{Y})^{-A_2})/((y^2 + C^2)^{(\nu+\mu-\rho+2)/2})$ is bounded for these values of y, and so $g_\nu(y)(xy)^{1/2}I'_{c,\rho}(x, y)$ has the needed L^1 bound on $x + \delta \leqq y$.

Lemma 5 allows us to take $\lim_{c\downarrow 0}$ under the integral sign, that is,

$$\lim_{c\downarrow 0} \int_{|x-y|\geqq\delta} g_\nu(y)(xy)^{1/2}I_{c,\rho}(x, y)\,dy = \int_{|x-y|\geqq\delta} g_\nu(y)(xy)^{1/2}I_{0,\rho}(x, y)\,dy$$

and

$$g_{\mu,\rho}(x) = \int_{|x-y|\geqq\delta} g_\nu(y)(xy)^{1/2}I_{0,\rho}(x, y)\,dy + C_{\mu,\nu}(\delta; \rho; x)$$

$$= \left(\int_0^{x-\delta} + \int_{x+\delta}^{\infty}\right) g_\nu(y)(xy)^{1/2}I_{0,\rho}(x, y)\,dy + C_{\mu,\nu}(\delta; \rho; x).$$

We need to see how the integrals behave as $\rho \downarrow 0$. By the Weber–Schafheitlin integral [6, p. 401, (2)],

$$I_{0,\rho}(x, y) = \frac{2y^\nu((\mu + \nu + 2 - \rho)/2)}{2^\rho x^{\nu+2-\rho}\Gamma(\nu + 1)\Gamma((\mu - \nu + \rho)/2)}$$

$$\cdot {}_2F_1\left(\frac{\mu + \nu + 2 - \rho}{2}, \frac{\nu - \mu + 2 - \rho}{2}; \nu + 1; \frac{y^2}{x^2}\right)$$

for $y \leqq x - \delta$. This is continuous in ρ and y on $\{0 \leqq \rho \leqq \rho_0, 0 \leqq y \leqq x - \delta\}$, a compact set, so $I_{0,\rho}(x, y)$ is bounded there. The dominated convergence theorem implies that

$$\lim_{\rho\downarrow 0}\int_0^{x-\delta} g_\nu(y)(xy)^{1/2}I_{0,\rho}(x,y)\,dy = \int_0^{x-\delta} g_\nu(y)(xy)^{1/2}\lim_{\rho\downarrow 0} I_{0,\rho}(x,y)\,dy.$$

For $y \geqq x + \delta$, we have

$$I_{0,\rho}(x,y) = \frac{2x^\mu\Gamma((\mu+\nu+2-\rho)/2)}{2^\rho y^{\mu+2-\rho}\Gamma(\mu+1)\Gamma((\nu-\mu+\rho)/2)} \cdot {}_2F_1\left(\frac{\nu+\mu+2-\rho}{2},\frac{\mu-\nu+2-\rho}{2};\mu+1,\frac{x^2}{y^2}\right).$$

Now ${}_2F_1((\nu+\mu+2-\rho)/2, (\mu-\nu+2-\rho)/2; \mu+1; w)$ is continuous in (ρ, w) on $\{0 \leqq \rho \leqq \rho_0,\ 0 \leqq w \leqq w_0 < 1\}$. In consequence, it is bounded there. Hence $|I_{0,\rho}(x,y)| \leqq (Bx^\mu)/(y^{\mu+2-\rho})$, which implies that $|g_\nu(y)(xy)^{1/2}I_{0,\rho}(x,y)|$ is bounded, independent of $\rho \sim 0$, by a function in $L^1(dy)$ on $x+\delta \leqq y < \infty$, and so

$$\lim_{\rho\downarrow 0}\int_{x+\delta}^{\infty} g_\nu(y)(xy)^{1/2}I_{0,\rho}(x,y)\,dy = \int_{x+\delta}^{\infty} g_\nu(y)(xy)^{1/2}\lim_{\rho\downarrow 0} I_{0,\rho}(x,y)\,dy.$$

Letting $I(x,y) = \lim_{\rho\downarrow 0} I_{0,\rho}(x,y)$, and recalling Lemma 3, we see that

$$g_\mu(x) = \int_0^{x-\delta} + \int_{x+\delta}^{\infty} g_\nu(y)(xy)^{1/2}I(x,y)\,dy + C_{\mu,\nu}(\delta;x). \tag{2.7}$$

3. The integral kernels $\tilde I_{\mu,\nu}(x,y)$. We shall separate the singular part of $I(x,y)$. From the Weber–Schafheitlin integral, we have for $0 \leqq y < x$,

$$\begin{aligned} I(x,y) &= \frac{2\Gamma((\mu+\nu+2)/2)}{\Gamma(\nu+1)\Gamma((\mu-\nu)/2)}\frac{y^\nu}{x^{\nu+2}}\,{}_2F_1\left(\frac{\mu+\nu+2}{2},\frac{\nu-\mu+2}{2};\nu+1;\frac{y^2}{x^2}\right)\\ &= \frac{2\Gamma((\mu+\nu+2)/2)}{\Gamma(\nu+1)\Gamma((\mu-\nu)/2)}\left(\frac{y}{x}\right)^\nu\frac{1}{x^2-y^2}\,{}_2F_1\left(\frac{\nu-\mu}{2},\frac{\nu+\mu}{2};\nu+1;\frac{y^2}{x^2}\right),\end{aligned} \tag{3.1}$$

by Euler's identity [1, vol. I, p. 64, (23)]. Similarly, for $x < y < \infty$,

$$I(x,y) = \frac{2\Gamma((\mu+\nu+2)/2)}{\Gamma(\mu+1)\Gamma((\nu-\mu)/2)}\left(\frac{x}{y}\right)^\mu\frac{1}{y^2-x^2}\,{}_2F_1\left(\frac{\mu-\nu}{2},\frac{\nu+\mu}{2};\mu+1;\frac{x^2}{y^2}\right). \tag{3.2}$$

We next introduce some notations:

$$\tilde I_{\mu,\nu}(x,y) = (xy)^{1/2}\lim_{\rho\downarrow 0}\int_0^\infty \frac{(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\nu(yz)}{z^\rho}\,dz, \tag{3.3}$$ [1]

and

$$\alpha(\mu,\nu) = \frac{4}{\Gamma((\mu-\nu)/2)\Gamma((\nu-\mu)/2)(\nu-\mu)}. \tag{3.4}$$

Then

$$\tilde I_{\mu,\nu}(x,y) = \tilde I_{\nu,\mu}(y,x) \tag{3.5}$$

[1] Previously, this would have been called $(xy)^{1/2}I(x,y)$.

and

$$\alpha(\mu, \nu) = -\alpha(\nu, \mu). \tag{3.6}$$

We now rewrite (3.1) after letting $\delta \downarrow 0$, to obtain, for $G \in \mathscr{H}_\nu$,

$$\begin{aligned} g_\mu(x) = &\left\{\int_0^{x/2} + \int_{2x}^{\infty}\right\} g_\nu(y)\tilde{I}_{\mu,\nu}(x, y)\,dy \\ &+ \int_{x/2}^{2x} g_\nu(y)\left[\tilde{I}_{\mu,\nu}(x, y) - \frac{\alpha(\mu, \nu)x}{x^2 - y^2}\right] dy \\ &+ CPV\int_{x/2}^{2x} g_\nu(y)\frac{\alpha(\mu, \nu)x}{x^2 - y^2}\,dy + C'_{\mu,\nu}g_\mu(x). \end{aligned} \tag{3.7}$$

We shall see that the integrals over $0 \leqq y \leqq x/2$ and $2x \leqq y < \infty$ and the integral of the difference exist as Lebesgue integrals.

To prove the mapping from g_ν to g_μ is bounded on the space $L^{p,\alpha}(0, \infty)$, we note the estimates, easily obtained from (3.1) and (3.2):

$$|\tilde{I}_{\mu,\nu}(x, y)| \leqq \begin{cases} \dfrac{A(\mu, \nu)}{x} & \text{for} \quad 0 < y \leqq x/2, \\ \dfrac{A(\mu, \nu)}{y} & \text{for} \quad 2x < y < \infty. \end{cases} \tag{3.8}$$

We also need the following results.

LEMMA 6.

$$\tilde{I}_{\mu,\nu}(x, y) - \frac{\alpha(\mu, \nu)x}{x^2 - y^2} = O\left(\frac{|\log(1 - y^2/x^2)|}{x}\right), \qquad x/2 \leqq y < x,$$

and

$$\tilde{I}_{\mu,\nu}(x, y) - \frac{\alpha(\mu, \nu)y}{x^2 - y^2} = O\left(\frac{|\log(1 - x^2/y^2)|}{y}\right), \qquad x \leqq y < 2x.$$

Proof. We shall only consider the case $x/2 \leqq y \leqq x$. The other case follows from (3.5) and (3.6). We have

$$\begin{aligned} &\tilde{I}_{\mu,\nu}(x, y) - \frac{\alpha(\mu, \nu)x}{x^2 - y^2} \\ &= \frac{(xy)^{1/2}}{x^2 - y^2}\left(\frac{y}{x}\right)^\nu\left[\frac{2\Gamma((\mu + \nu + 2)/2)}{\Gamma(\nu + 1)\Gamma((\mu - \nu)/2)} \cdot {}_2F_1\left(\frac{\nu - \mu}{2}, \frac{\nu + \mu}{2}; \nu + 1; \frac{y^2}{x^2}\right)\right. \\ &\quad \left. - \alpha(\mu, \nu)\right] + \alpha(\mu, \nu)\left[\frac{(xy)^{1/2}}{x^2 - y^2}\frac{y^\nu - x^\nu}{x^\nu} + \frac{(xy)^{1/2} - x}{x^2 - y^2}\right] \\ &= \frac{(xy)^{1/2}}{x^2 - y^2}\left(\frac{y}{x}\right)^\nu\left[\frac{2\Gamma((\mu + \nu + 2)/2)}{\Gamma(\nu + 1)\Gamma((\mu - \nu)/2)} \cdot {}_2F_1\left(\frac{\nu - \mu}{2}, \frac{\nu + \mu}{2}; \nu + 1; \frac{y^2}{x^2}\right)\right. \\ &\quad \left. - \alpha(\mu, \nu)\right] + O\left(\frac{1}{x}\right), \end{aligned}$$

since

$$\frac{(xy)^{1/2}}{x^2 - y^2}\,\frac{y^\nu - x^\nu}{x^\nu} = \frac{(xy)^{1/2}}{x + y}\,\frac{O(x^{\nu-1})}{x^\nu},$$

unless $\nu = 0$, in which case the term in question does not appear, and also $(xy)^{1/2} - x = x^{1/2}(y^{1/2} - x^{1/2}) = O(1)(x - y)$.

To estimate the expression in square braces, let $w = y^2/x^2$ and call the resulting expression $s(w)$. The hypergeometric function satisfies the appropriate condition for Gauss's theorem [4, p. 28, (1.7.6)]; hence, from (3.4), $\lim_{w\uparrow 1} s(w) = 0$. We may therefore apply L'Hospital's rule to $s(w)/(1 - w)\log(1 - w) - (1 - w)$. This gives

$$\begin{aligned}\lim_{w\uparrow 1}&\frac{s(w)}{(1 - w)[\log(1 - w) - 1]}\\ &= \lim_{w\uparrow 1}\frac{2\Gamma((\mu + \nu + 2)/2)\cdot((\nu - \mu)/2)((\nu + \mu)/2)}{\Gamma(\nu + 1)\Gamma((\mu - \nu)/2)(\nu + 1)}\\ &\quad\cdot\frac{{}_2F_1((\nu - \mu + 2)/2, (\nu + \mu + 2)/2; \nu + 2; w)}{-\log(1 - w)}.\end{aligned}$$

Since $(\nu - \mu + 2)/2 + (\nu + \mu + 2)/2 = \nu + 2$, this limit exists [4, p. 29, (1.7.8)]. Hence for some constant $A(\mu, \nu)$, we have $|s(w)/((1 - w)[\log(1 - w) - 1])| \leqq A(\mu, \nu)$ for $\frac{1}{2} \leqq w \leqq 1$, and so

$$\begin{aligned}&\left|\frac{(xy)^{1/2}}{x^2 - y^2}\left(\frac{y}{x}\right)^\nu\left[\frac{2\Gamma((\mu + \nu + 2)/2)}{\Gamma(\nu + 1)\Gamma((\mu - \nu)/2)}\cdot {}_2F_1\left(\frac{\nu - \mu}{2}, \frac{\nu + \mu}{2}; \nu + 1; \frac{y^2}{x^2}\right) - \alpha(\mu, \nu)\right]\right|\\ &\quad\leqq A(\mu, \nu)\frac{(xy)^{1/2}}{x^2 - y^2}\left(\frac{y}{x}\right)^\nu\left|\left(1 - \frac{y^2}{x^2}\right)\log\left(1 - \frac{y^2}{x^2}\right) - \left(1 - \frac{y^2}{x^2}\right)\right|\\ &\quad\leqq \frac{A'(\mu, \nu)}{x}\left|\log\left(1 - \frac{y^2}{x^2}\right)\right|.\end{aligned}$$

The lemma is proved.

In brief, a measurable function f is in $L^{p,\alpha}$, $1 \leqq p < \infty$, if

$$\|f\|_{p,\alpha} = \left\{\int_0^\infty |f(x)|^p x^{\alpha p}\,dx\right\}^{1/p} < \infty.$$

We consider the mapping $T_{\mu,\nu}$ defined by

$$(3.9)\qquad (T_{\mu,\nu}h)(x) = \int_0^\infty h(y)\tilde{I}_{\mu,\nu}(x, y)\,dy + C'_{\mu,\nu}h(x).$$

To see that $T_{\mu,\nu}$ is a bounded operator on $L^{p,\alpha}$ for $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, we have only to apply the boundedness of the following operators:

$$f \to \int_0^\infty \frac{f(y)}{x + y}\,dy,$$

$$f \to \frac{1}{x}\int_{x/2}^x f(y)\left|\log\left(1 - \frac{y^2}{x^2}\right)\right|dy + \frac{1}{x}\int_x^{2x} f(y)\left|\log\left(1 - \frac{x^2}{y^2}\right)\right|dy$$

and

$$f \to \int_0^\infty \frac{f(y)}{y - x}\, dy.$$

Rewriting $T_{\mu,\nu}h$ as in (3.7), using the estimates in (3.8) and the identities $x/(x^2 - y^2) = \frac{1}{2}[1/(x + y) + 1/(x - y)]$, $y/(x^2 - y^2) = \frac{1}{2}[1/(x - y) - 1/(x + y)]$, we obtain the following theorem.

THEOREM 1. *If $T_{\mu,\nu}$ is defined by (3.9), then $T_{\mu,\nu}$ is a bounded operator on $L^{p,\alpha}(0, \infty)$ whenever $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$.*

4. Extension to $G \in L^1$ with $g_\nu \in L^{p,\alpha}$. Finally we want to show that $T_{\mu,\nu}$ does carry ν-transforms in $L^{p,\alpha}$ into corresponding μ-transforms. First we need two lemmas.

LEMMA 7. *Let $G \in \mathscr{H}_\mu$, that is, $G \in L^1 \cap C[0, \infty)$ and $g_\mu \in C_c^\infty(0, \infty)$. Then $g_\nu \in L^1$ and consequently $G(z)$ equals the Lebesgue integral*

$$\int_0^\infty g_\nu(y)(yz)^{1/2} J_\nu(yz)\, dy.$$

Proof. Since $G \in L^1$, g_ν is continuous on $[0, \infty)$. It is thus enough to show that $g_\nu(x) = O(x^{-2})$ for $x \to \infty$. From (3.7), with ν and μ interchanged, we see that for $x \geqq 2 \max\{t : g_\mu(t) \neq 0\}$, $g_\nu(x) = \int_0^{x/2} g_\mu(y)\tilde{I}_{\nu,\mu}(x, y)\, dy$. Hence by (3.1), $|g_\nu(x)| \leqq (M\|g_\mu\|_\infty)/(x^2)$, and $g_\nu \in L^1$. Since $G \in \mathscr{H}_\mu$, it is infinitely differentiable, and this finally guarantees that ν-inversion holds for G.

LEMMA 8. *Let $1 < p < \infty$ and $-1/p < \alpha < 1 - 1/p$. Then the adjoint of $T_{\mu,\nu}$, defined on $L^{q,-\alpha}$ $(1/p + 1/q = 1)$, is $T_{\nu,\mu}$.*

Proof. Let $H \in \mathscr{H}_\nu$ and $G \in \mathscr{H}_\mu$. Then

$$\begin{aligned} \int (T_{\mu,\nu}h_\nu)g_\mu &= \int_0^\infty h_\mu(x)g_\mu(x)\, dx \\ &= \int_0^\infty \int_0^\infty H(y)(xy)^{1/2}J_\mu(xy)g_\mu(x)\, dy\, dx \\ &= \int_0^\infty H(y)G(y)\, dy = \int h_\nu g_\nu = \int h_\nu T_{\nu,\mu}g_\mu, \end{aligned}$$

since $H, g_\mu, g_\nu \in L^1$ and $\int_0^\infty g_\mu(x)(xy)^{1/2}J_\mu(xy)\, dx = G(y) = \int_0^\infty g_\nu(x)(xy)^{1/2}J_\nu(xy)\, dx$, by the definition of $\mathscr{H}_\nu$ and Lemma 7. Since $C_c^\infty(0, \infty)$ is dense in $L^{p,\alpha}$ and $T_{\mu,\nu}$ is bounded, h_ν may be replaced by any $h \in L^{p,\alpha}$. Hence $T_{\mu,\nu}^* g_\mu = T_{\nu,\mu}g_\mu$. Continuity of the operators, again, implies that $T_{\mu,\nu}^* = T_{\nu,\mu}$.

We can now prove our $L^{p,\alpha}$ integral representation theorem.

THEOREM 2. *Let $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, $H \in L^1$, $h_\nu \in L^{p,\alpha}$. Then $T_{\mu,\nu}h_\nu = h_\mu$.*

Proof. First, by Lemma 8, $\int (T_{\mu,\nu}h_\nu)g_\mu = \int h_\nu g_\nu$ for arbitrary $G \in \mathscr{H}_\mu$. Now $H \in L^1$ and by Lemma 7, so is g_ν; hence by Fubini's theorem, and the inversion part of Lemma 7,

$$\int_0^\infty h_\nu(z)g_\nu(z)\,dz = \int_0^\infty\int_0^\infty H(y)g_\nu(z)(yz)^{1/2}J_\nu(yz)\,dy\,dz$$
$$= \int_0^\infty H(y)G(y)\,dy.$$

Also since $H, g_\mu \in L^1$, this integral equals, by Fubini's theorem, $\int h_\mu g_\mu$. The g_μ's run over the dense (in $L^{p,\alpha}$) set $C_c^\infty(0,\infty)$ and so $T_{\mu,\nu}h_\nu = h_\mu$, as required.

It should be noted that if either of μ or ν equals $-\frac{1}{2}$, and the other is greater, the proofs become simpler, and Theorems 1 and 2 continue to hold.

5. The case $\mu = \nu + 2k$ $(k = \pm 1, \pm 2, \cdots)$. We turn now to the special case $\mu = \nu + 2k, k = 1, 2, \cdots$, or $(\nu - \mu)/2 = -k$, and so, by (3.2), $\tilde{I}_{\nu+2k,\nu}(x, y) = 0$ whenever $y > x$. By the equality preceding (3.1), we have for $y < x$,

$$\tilde{I}_{\nu+2k,\nu}(x,y) = \frac{2\Gamma(\nu+k+1)}{\Gamma(\nu+1)\Gamma(k)}(xy)^{1/2}\frac{y^\nu}{x^{\nu+2}}\cdot {}_2F_1\left(\nu+k+1, -k+1; \nu+1; \frac{y^2}{x^2}\right).$$

The hypergeometric function is a polynomial of degree $k-1$ in y^2/x^2. (In fact, it equals

$$\frac{\Gamma(\nu+2)\Gamma(k)}{\Gamma(\nu+k+1)}P_{k-1}^{(\nu+1,0)}(1-(2y^2)/x^2); \qquad P_{k+1}^{(\nu+1,0)} \text{ is a Jacobi polynomial.)}$$

$$\tilde{I}_{\nu+2k,\nu}(x,y) = \frac{1}{x}\sum_{j=0}^{k-1}\gamma_j\left(\frac{y}{x}\right)^{2j+\nu+1/2},$$

where $\gamma_j = \gamma_j(\nu, k)$ is constant and $\gamma_0 = 2\Gamma(\nu+k+1)/\Gamma(\nu+1)\Gamma(k)$. Hence if $G \in L^1$ and $g_\nu \in$ some $L^{p,\alpha}$, Theorem 2 gives

$$g_{\nu+2k}(x) = \frac{1}{x}\sum_{j=0}^{k-1}\gamma_j\int_0^x\left(\frac{y}{x}\right)^{2j+\nu+1/2}g_\nu(y)\,dy + (-1)^k g_\nu(x).$$

(We also used (2.5).) Now $G \in C_c^\infty(0,\infty)$ implies $g_\nu \in L^{2,0}$ and so for $G \in C_c^\infty(0,\infty)$,

$$\int_0^\infty G(z)(xz)^{1/2}J_{\nu+2k}(xz)\,dz$$
$$= \frac{1}{x}\sum_{j=0}^{k-1}\gamma_j\int_0^x\left(\frac{y}{x}\right)^{2j+\nu+1/2}\cdot\int_0^\infty G(z)(yz)^{1/2}J_\nu(yz)\,dz\,dy$$
$$+(-1)^k\int_0^\infty G(z)(xz)^{1/2}J_\nu(xz)\,dz.$$

For $G \in C_c(0,\infty)$, Fubini's theorem applies where relevant.

$$\int_0^\infty G(z)(xz)^{1/2}J_{\nu+2k}(xz)\,dz$$
$$= \frac{1}{x}\int_0^\infty G(z)\sum_{j=0}^{k-1}\gamma_j\int_0^\infty\left(\frac{y}{x}\right)^{2j+\nu+1/2}(yz)^{1/2}J_\nu(yz)\,dy\,dz \qquad \text{(cont.)}$$

$$+(-1)^k \int_0^\infty G(z)(xz)^{1/2} J_\nu(xz)\,dz.$$

Since $C_c^\infty(0, \infty)$ is dense in $C_c(0, \infty)$, that is, continuous functions of compact support in $(0, \infty)$, this implies that

$$(5.1) \qquad (xz)^{1/2} J_{\nu+2k}(xz) = \frac{1}{x}\sum_{j=0}^{k-1} \gamma_j \int_0^\infty \left(\frac{y}{x}\right)^{2j+\nu+1/2} (yz)^{1/2} J_\nu(yz)\,dy + (-1)^k (xz)^{1/2} J_\nu(xz).$$

(*Note*. One can prove (5.1) for $k = 1$ by the recursion formulas and extend it to $k = 2, 3, \cdots$ by induction. Also, formula (5.1) is valid for all real ν.) We obtain, for any $G \in L^1$,

$$(5.2) \qquad g_{\nu+2k}(x) = \frac{1}{x}\sum_{j=0}^{k-1} \gamma_j \int_0^x \left(\frac{y}{x}\right)^{2j+\nu+1/2} g_\nu(y)\,dy + (-1)^k g_\nu(x),$$

all real ν, but we are really only interested in the case $\nu \geqq -1$.

For $j = 0, 1, \cdots, k-1$, let S_j be the operator formally defined by

$$(5.3) \qquad (S_j f)(x) = \frac{1}{x}\int_0^x \left(\frac{y}{x}\right)^{2j+\nu+1/2} f(y)\,dy.$$

We shall prove a simple lemma about S_j and use it to obtain some stronger results about $T_{\nu+2k,\nu}$.

LEMMA 9. *Suppose* $1 \leqq p \leqq \infty$ *and* $\delta \neq 2j + \nu + 3/2$ *for* $p = \infty$, $\delta \neq 2j + \nu + 1/2$ *for* $p = 1$. *Then there exists a constant* $A_p = A_p(\delta, j, \nu)$ *such that* $\|S_j f(x) x^\delta\|_p \leqq A_p \|f(y) y^\delta\|_\infty$ *whenever* $\int_0^\infty f(y) y^{2j+\nu+1/2}\,dy = 0$.

Proof. First let $p = \infty$. Then if $\delta < 2j + \nu + 3/2$,

$$|(S_j f)(x) x^\delta| \leqq \frac{1}{x^{2j+\nu+3/2-\delta}} \|f(y)y^\delta\|_\infty \cdot \int_0^x y^{2j+\nu+1/2-\delta}\,dy = \frac{1}{2j+\nu+3/2-\delta}\|f(y)y^\delta\|_\infty.$$

If $\delta > 2j + \nu + 3/2$, the condition on f implies that

$$|(S_j f)(x) x^\delta| \leqq \frac{1}{x^{2j+\nu+3/2-\delta}} \|f(y)y^\delta\|_\infty \int_x^\infty y^{2j+\nu+1/2-\delta}\,dy = \frac{1}{|2j+\nu+3/2-\delta|}\|f(y)y^\delta\|_\infty.$$

Now let $p = 1$. If $\delta < 2j + \nu + 1/2$,

$$\int_0^\infty |(S_j f)(x)| x^\delta\,dx \leqq \int_0^\infty \frac{1}{x^{2j+\nu+3/2-\delta}} \int_0^x y^{2j+\nu+1/2-\delta} |f(y)| y^\delta\,dy\,dx = \frac{1}{2j+\nu+1/2-\delta}\int_0^\infty |f(y)| y^\delta\,dy$$

(by reversing the order of integration). If $\delta > 2j + \nu + 1/2$, we use the condition on f and reverse the order of integration to get

$$\int_0^\infty |(S_j f)(x)| x^\delta \, dx \leqq \int_0^\infty |f(y)| y^\delta y^{2j+\nu+1/2-\delta} \int_0^y x^{-2j-\nu-3/2+\delta} \, dx \, dy$$

$$= \frac{1}{|2j + \nu + 1/2 - \delta|} \int_0^\infty |f(y)| y^\delta \, dy.$$

The lemma then follows for $1 < p < \infty$ by interpolation; if $\delta = 2j + \nu + 1/2$ or $2j + \nu + 3/2$, a change of measure argument is required too.

We note that if $\nu > -1/2$, then the lemma holds for $1 \leqq p \leqq \infty$ and $\delta < \nu + 1/2$, and in particular for $\delta = 0$, with no extra condition on f. Finally, the lemma and (5.2) give two theorems.

THEOREM 3. *Let $\delta < \nu + 1/2$. If $1 \leqq p \leqq \infty$ there is a constant $B_p = B_p(k, \nu, \delta)$, $k = 1, 2, \cdots$, such that for $G \in L^1$, $\|g_{\nu+2k}(x) x^\delta\|_p \leqq B_p \|g_\nu(y) y^\delta\|_p$.*

THEOREM 3'. *Let $1 \leqq p \leqq \infty$ and $\delta \neq 2j + \nu + 3/2$ for $p = \infty$, $\delta \neq 2j + \nu + 1/2$ for $p = 1$, and $k = 0, 1, \cdots$, $j = 0, 1, \cdots, k - 1$. Then there is a constant $B_p = B_p(k, \nu, \delta)$ such that if $G \in L^1$, then $\|g_{\nu+2k}(x) x^\delta\|_p \leqq B_p \|g_\nu(y) y^\delta\|_p$ whenever*

$$\int_0^\infty g_\nu(y) y^{2j+\nu+1/2} \, dy = 0$$

for all $j = 0, 1, \cdots, k - 1$.

Note that for $k = 1$, the condition on g_ν in Theorem 3′ states roughly that $G(0) = 0$, or that

$$\int_0^\infty g_\nu(y) (yz)^{1/2} J_\nu(yz) \, dy \to 0$$

as $z \downarrow 0$. For higher k, the condition would state something about derivatives of G at 0. Inspection of the proof shows that the conditions are necessary to obtain the two-sided δ-range.

Note that if $\mu = \nu - 2k$, $k = 1, 2, \cdots$, then

$$\tilde{I}_{\nu-2k,\nu}(x, y) = \tilde{I}_{\nu,\nu-2k}(y, x) = \tilde{I}_{\mu+2k,\mu}(y, x)$$

$$= \begin{cases} \dfrac{2\Gamma(\mu + k + 1)}{\Gamma(\mu + 1)\Gamma(k)} \dfrac{1}{y} \dfrac{x^{\mu+1/2}}{y^{\mu+1/2}} \cdot {}_2F_1\left(\mu + k + 1; -k + 1; \mu + 1; \dfrac{x^2}{y^2}\right) & \text{for } y > x, \\ 0 & \text{for } y < x, \end{cases}$$

and, as before, for all ν, and $G \in L^1$,

$$g_{\nu-2k}(x) = \sum_{j=0}^{k-1} \gamma'_j \int_x^\infty \frac{1}{y} \left(\frac{x}{y}\right)^{2(j-k)+\nu+1/2} g_\nu(y) \, dy + (-1)^k g_\nu(x).$$

For completeness, we state the following pair of theorems. The proofs would obviously be along the lines of those of the last pair.

THEOREM 4. *Let $k = 1, 2, \cdots$, and $\delta > 2k - \nu - 1/2$. If $1 \leqq p \leqq \infty$, there is a constant $B_p = B_p(k, \nu, \delta)$ such that if $G \in L^1$, $\|g_{\nu-2k}(x) x^\delta\|_p \leqq B_p \|g_\nu(y) y^\delta\|_p$.*

THEOREM 4$'$. *Let* $k = 1, 2, \cdots$, $1 \leqq p \leqq \infty$ *and* $\delta \neq 2k - 2j - \nu - 1/2$ *for* $p = \infty$, $\delta \neq 2k - 2j - \nu - 3/2$ *for* $p = 1$, $j = 0, 1, \cdots, k = 1$. *Then there is a constant* $B_p = B_p(k, \nu, \delta)$ *such that if* $G \in L^1$, $\|g_{\nu-2k}(x)x^\delta\|_p \leqq B_p\|g_\nu(y)y^\delta\|_p$ *whenever*

$$\int_0^\infty g_\nu(y)y^{2k-2j-\nu-3/2}\,dy = 0, \qquad j = 0, 1, \cdots, k - 1.$$

Note that for $1 \leqq p < \infty$, Theorem 3 can be restated as: "*Let* $k = 1, 2, \cdots$ *and* $\alpha < \nu + 3/2$. *Then there is a constant* $A_p(k, \nu, \alpha)$ *such that if* $G \in L^1$, $\|g_{\nu-2k}\|_{p,\alpha} \leqq A_p\|g_\nu\|_{p,\alpha}$." Theorems 3$'$, 4 and 4$'$ can be restated similarly, to give an easy comparison with the more general result of Theorem 1.

We may also note that we had, in general, by (3.1) and (3.2) and the proof of Lemma 6,

$$\tilde{I}_{\mu,\nu}(x, y) \leqq \begin{cases} \dfrac{A(\mu, \nu)}{x}\left(\dfrac{y}{x}\right)^{\nu+1/2} & \text{for} \quad y < x/2, \\[2ex] \dfrac{A(\mu, \nu)}{y}\left(\dfrac{x}{y}\right)^{\mu+1/2} & \text{for} \quad y > 2x, \end{cases}$$

and

$$\tilde{I}_{\mu,\nu}(x, y) - \frac{\alpha(\mu, \nu)x}{x^2 - y^2} = O\left(\frac{|\log(1 - (y^2/x^2))|}{x}\right)\left(\frac{y}{x}\right)^{\nu+1/2}, \qquad x/2 \leqq y < x,$$

and

$$\tilde{I}_{\mu,\nu}(x, y) - \frac{\alpha(\mu, \nu)y}{x^2 - y^2} = O\left(\frac{|\log(1 - (x^2/y^2))|}{x}\right)\left(\frac{x}{y}\right)^{\mu+1/2}, \qquad x < y < 2x.$$

Of the terms in these expansions, the only one which does not yield L^1 or L^∞ results of the kind we have in Theorems 3, 3$'$, 4, 4$'$ is that involving $\alpha(\mu, \nu)$. This term vanishes only when $\mu - \nu$ is an even integer. Thus we may say that our theorems give best possible results on L^1 and L^∞, and we evidently have stronger L^p results $(1 < p < \infty)$ if $\mu - \nu =$ an even integer.

Acknowledgment. The author would like to thank Professor Richard A. Askey for his helpful suggestions in the preparation of this paper.

REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vols. I, II, McGraw-Hill, New York, 1953.
[2] D. L. GUY, *Hankel multiplier transformations and weighted p-norms*, Trans. Amer. Math. Soc., 95 (1960), no. 1, pp. 137–189.
[3] B. MUCKENHOUPT AND E. M. STEIN, *Classical expansions and their relation to conjugate harmonic functions*, Ibid., 118 (1965), no. 6, pp. 17–92.
[4] L. SLATER, *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1966.
[5] G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, American Mathematical Society, New York, 1939.
[6] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1966.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

SOME MONOTONICITY PROPERTIES OF BESSEL FUNCTIONS*

LEE LORCH†, M. E. MULDOON† AND PETER SZEGO‡

Abstract. It is proved that the sequence

$$\left\{\int_{c_{\nu k}}^{c_{\nu,k+1}} t^{\gamma-1}|\mathscr{C}_\nu(t)|\,dt\right\}_{k=\varkappa}^{\infty}$$

is decreasing for all ν, for $-\infty < \gamma < \frac{3}{2}$, and for suitable $\varkappa$, where $\mathscr{C}_\nu(t)$ is an arbitrary Bessel function of order ν and $c_{\nu k}$ its kth positive zero. This subsumes and unifies results obtained by G. Szegö and R. G. Cooke, extending and sharpening some. For one of his results Szegö used a Sturm comparison theorem which is shown here to permit the requisite generalization and to incorporate and extend other results originally proved by quite different methods. Auxiliary results are derived. Various remarks are collected in the final section.

1. Introduction and results. G. Szegö has proved [2, p. 104] that the sequence of areas

$$\left\{\int_{c_{\nu k}}^{c_{\nu,k+1}} |\mathscr{C}_\nu(t)|\,dt\right\}_{k=\varkappa}^{\infty} \tag{1}$$

under the successive arches of an arbitrary Bessel (cylinder) function $\mathscr{C}_\nu(t)$, with kth positive zero $c_{\nu k}$, form a decreasing sequence when $\varkappa$ is selected properly (see Corollary 3 below) and, using a different approach, that [5, p. 281, (19)]

$$\int_0^{j_{\alpha,2k}} t^{-\alpha}J_\alpha(t)\,dt > 0, \qquad k = 2, 3, 4, \cdots, \tag{2}$$

where $J_\alpha(t)$ is the Bessel function of the first kind, $j_{\alpha k}$ is its kth positive zero, and α is the unique value satisfying

$$\int_0^{j_{\alpha 2}} t^{-\alpha}J_\alpha(t)\,dt = 0. \tag{3}$$

He showed that $-\frac{1}{2} < \alpha \leqq 0$ and mentioned that D. R. Snow had computed α to be $-0.2693885\ldots$.

Using the Sonin integral [5, p. 279, (12)], [11, p. 373, (1)] as indicated by Szegö [5, p. 280, (16) ff.], it can be shown that (2) remains valid when α is replaced by any larger value, and consequently that

$$\int_0^{z} t^{-\nu}J_\nu(t)\,dt > 0, \qquad z > 0, \nu > \alpha. \tag{2'}$$

The inequalities (2) and (2′), with accompanying discussion and inferences, are found in the Notes which G. Szegö appended to a posthumous [5, p. 275, first footnote] paper of Ervin Feldheim in the course of preparing it for publication.

* Received by the editors February 8, 1972, and in revised form April 17, 1972. This research was supported by the National Research Council of Canada.

† Department of Mathematics, York University, Downsview 463, Ontario, Canada.

‡ Research Department, Ampex Corporation, Redwood City, California 94063.

Our purpose here is to connect the Szegö results with one another so that each emerges from the method he used to establish the first; his own proof of (2) follows a quite different line of reasoning. This unification yields (Corollary 2) a sharper and more general formulation for (2′). Also, a lemma due to R. G. Cooke [3, p. 282] can be subsumed and generalized by this approach (Corollary 4). Remarks on these results are collected in § 6; §§ 2, 3, 4, 5 are devoted to proofs.

The Szegö and Cooke results are contained in the following.

THEOREM 1. *For all* ν, *and for* $-\infty < \gamma < \frac{3}{2}$, *the sequence*

$$\left\{\int_{c_{\nu k}}^{c_{\nu,k+1}} t^{\gamma-1}|\mathscr{C}_\nu(t)|\,dt\right\}_{k=\varkappa}^{\infty} \tag{4}$$

is decreasing, where $\varkappa$ *is the smallest integer such that*

$$c_{\nu\varkappa} \geqq \lambda(\nu;\gamma) \equiv \begin{cases} \frac{1}{3}[2\gamma(\gamma^2 - 9\nu^2)/(3 - 2\gamma)]^{1/2}, & 0 < \gamma < \frac{3}{2}, \quad |\nu| < \frac{1}{3}\gamma, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

Thus, $\varkappa = 1$ *when* $|\nu| \geqq \frac{1}{2}$ *or when* $\gamma \leqq 0$.

If $0 < \gamma \leqq 1$, *then* $\varkappa \leqq 2$ *and* $\varkappa = 1$ *when* $|\nu| \geqq \frac{1}{3}\gamma$. *If* $\mathscr{C}_\nu(t) \equiv J_\nu(t)$ *(still with* $\gamma \leqq 1$*), then* $\varkappa = 1$ *for any* ν.

The final assertion in the theorem can be generalized. This is recorded separately as Corollary 1 because its proof is rather more tedious (and is based on less elementary arguments) than the other proofs required here and we do not wish it to obscure the main line of argument. The extension is not required, for example, for our proof of Szegö's positivity results (2) and (2′), nor for our generalization of Cooke's lemma (Corollary 4).

COROLLARY 1. *If* $\gamma \leqq 1$ *and* $c_{\nu 1} \geqq y_{\nu 1}$ *(for example, when* $\mathscr{C}_\nu(t) \equiv Y_\nu(t)$*), then* $\varkappa = 1$ *in Theorem* 1 *even for the remaining range* $|\nu| < \frac{1}{3}\gamma$.

Here $y_{\nu 1}$ denotes the first positive zero of $Y_\nu(t)$, the Bessel function of the second kind.

That this corollary does subsume the final sentence of the theorem follows from the inequality $j_{\nu 1} > y_{\nu 1}$ valid for $\nu > -\frac{1}{2}$ (see [7 (a), p. 364, (i) for $\nu \geqq 0$, and Corollary, p. 366, for $0 > \nu > -\frac{1}{2}$ (after replacing ν by $-\nu$), whereby $j_{\nu 1} > y_{-\nu,1} > y_{\nu 1}$], since $y_{\nu 1}$ increases with $\nu > -\frac{1}{2}$ [11, p. 508, (3)]).

Upon choosing $\gamma = 1 - \nu$, Theorem 1 becomes (in view of (3)) a sharper and more general version of (2′). The precise result is as follows.

COROLLARY 2. *For* $\nu > -\frac{1}{2}$, *the sequence*

$$\left\{\int_{c_{\nu k}}^{c_{\nu,k+1}} t^{-\nu}|\mathscr{C}_\nu(t)|\,dt\right\}_{k=\varkappa}^{\infty} \tag{6}$$

is decreasing, where $\varkappa$ *is the smallest integer such that*

$$c_{\nu\varkappa} \geqq \lambda(\nu; 1 - \nu) \equiv \begin{cases} \frac{1}{3}[2(1 - \nu)(1 - 4\nu)]^{1/2} & \text{if} \quad -\frac{1}{2} < \nu < \frac{1}{4}, \\ 0 & \text{if} \quad \nu \geqq \frac{1}{4}. \end{cases} \tag{7}$$

Here $\varkappa \leqq 2$ *for all* $\nu > -\frac{1}{2}$, *and* $\varkappa = 1$ *when* $\nu \geqq \frac{1}{4}$ *or when* $\mathscr{C}_\nu(t) \equiv J_\nu(t)$. *The Szegö inequalities* (2) *and* (2′) *follow on choosing* $\nu = \alpha$ *and* $\mathscr{C}_\nu(t) \equiv J_\nu(t)$ *in* (6).

In Corollary 2, $\varkappa = 1$ also for $\mathscr{C}_\nu(t) \equiv Y_\nu(t)$ (or, more generally, for any $\mathscr{C}_\nu(t)$ for which $c_{\nu 1} \geqq y_{\nu 1}$), at least when $\nu \geqq \mu$, where μ is defined by the equation $y_{\mu 1} = \lambda(\mu; 1 - \mu)$. The root μ is unique, since $y_{\nu 1}$ increases for $\nu > -\frac{1}{2}$ while

$\lambda(\nu; 1-\nu)$, as defined by (7), decreases. Moreover, $-\frac{1}{3} < \mu < 0$, since $\lambda(-\frac{1}{3};\frac{4}{3}) = \frac{2}{9}\sqrt{14} > .38 > y_{-1/3,1}$ (the last inequality is obtained from [11, p. 714]) and $\lambda(0,1) = \frac{1}{3}\sqrt{2} < .89 < y_{01}$ [11, p. 748].

Using the York University computer facilities, Dr. Marian Shepherd has calculated μ to be $-0.1866\ldots$, where there is uncertainty about the last digit. We thank her most cordially.

The Szegö result on (1) [2, p. 104] follows on choosing $\gamma = 1$ in the theorem; we have added only the two last sentences in the next corollary.

COROLLARY 3. *For all* ν, *the sequence of areas* (1) *is decreasing, where* $\varkappa$ *is the smallest integer such that*

$$c_{\nu\varkappa} \geqq \lambda(\nu;1) = \begin{cases} \frac{1}{3}[2(1-9\nu^2)]^{1/2} & \text{if} \quad |\nu| < \frac{1}{3}, \\ 0 & \text{if} \quad |\nu| \geqq \frac{1}{3}. \end{cases} \tag{8}$$

Here $\varkappa \leqq 2$. *When* $|\nu| \geqq \frac{1}{3}$, *or when* (*for* $|\nu| < \frac{1}{3}$), $c_{\nu 1} \geqq y_{\nu 1}$ (*for example, if* $\mathscr{C}_\nu(t) \equiv J_\nu(t)$ *or* $\mathscr{C}_\nu(t) \equiv Y_\nu(t)$), *then* $\varkappa = 1$.

The following result reduces to the Cooke lemma [3, p. 282] in the special case $\mathscr{C}_\nu(t) \equiv J_\nu(t)$; it is obtained from Theorem 1 on putting $\gamma = 2 - \nu$. (Cf. § 6 (xi) for a further generalization.)

COROLLARY 4. *If* $\nu > \frac{1}{2}$, *then*

$$\left\{ \int_{c_{\nu k}}^{c_{\nu,k+1}} t^{1-\nu} |\mathscr{C}_\nu(t)|\, dt \right\}_{k=1}^{\infty} \tag{9}$$

is a decreasing sequence.

In view especially of Corollary 1, it is of some interest to characterize those nontrivial Bessel functions

$$\mathscr{C}_\nu(t) = AJ_\nu(t) + BY_\nu(t), \qquad |\nu| < \tfrac{1}{3},$$

for which $c_{\nu 1} \geqq y_{\nu 1}$, in the following way.

THEOREM 2. *For* $0 \leqq \nu < 1$, $c_{\nu 1} \geqq y_{\nu 1}$ *when and only when* $AB \leqq 0$; *for* $-\frac{1}{2} < \nu < 0$, $c_{\nu 1} \geqq y_{\nu 1}$ *if and only if* $A = 0$ *or* $B/A \leqq -\tan \nu\pi$. (It is assumed throughout that A and B are not both zero.)

In the latter case ($-\frac{1}{2} < \nu < 0$), $\tan \nu\pi < 0$, so that $c_{\nu 1} \geqq y_{\nu 1}$ whenever A and B have opposite signs, or one is zero, and also for certain values in which they have the same sign, such as $J_{-1/4}(t) + Y_{-1/4}(t)$ which equals $\sqrt{2}J_{1/4}(t)$. A change of behavior occurs at $\nu = -\frac{1}{2}$.

2. Proof of Theorem 1. This is based on an application of a Sturm-type lemma, formulated by G. N. Watson [11, p. 518], sharpened and applied in greater detail by E. Makai [8], to the differential equation

$$y'' + \varphi(x)y = 0, \tag{10}$$

where $\varphi(x)$ is monotonic. In particular, this lemma states that the areas under successive arches of the graph of $y(x)$ decrease (increase) when $\varphi(x)$ increases (decreases).

Here, as in [2, p. 104], we choose

$$\varphi(x) = \tfrac{1}{4}(1 - 4\beta^2\nu^2 + 4\beta^2 x^{2\beta})x^{-2}; \quad y(x) = x^{1/2}\mathscr{C}_\nu(x^\beta), \qquad \beta > 1. \tag{11}$$

Regardless of the value of ν, $\varphi(x)$ is an increasing function for all $x > 0$ for which

$$x^{2\beta} > \tfrac{1}{4}(1 - 4\beta^2\nu^2)\beta^{-2}(\beta - 1)^{-1}. \tag{12}$$

Hence, for such x, the areas bounded by successive arches of $y(x)$ form a decreasing sequence. Now with $t = x^\beta$,

$$\int_{x_k}^{x_{k+1}} |y(x)|\, dx = \beta^{-1} \int_{c_{\nu k}}^{c_{\nu,k+1}} t^{\gamma-1}|\mathscr{C}_\nu(t)|\, dt,$$

where $\gamma = 3/(2\beta)$ and $x_k = c_{\nu k}^{1/\beta}$ is the kth positive zero of $y(x)$.

We note that $0 < \gamma < \frac{3}{2}$, since $\infty > \beta > 1$. Conversely, given any $\gamma, 0 < \gamma < \frac{3}{2}$, there is a unique $\beta > 1$, namely $\beta = 3/(2\gamma)$, which can be used in (11). Substituting this value for β in (12) yields (5) and the main part of the theorem is proved for $0 < \gamma < \frac{3}{2}$.

If $\gamma \leqq 0$, pick δ such that $0 \leqq \lambda(\nu;\delta) < c_{\nu 1}$, $0 < \delta < \frac{3}{2}$, and write

$$\int_{c_{\nu k}}^{c_{\nu,k+2}} t^{\gamma-1}\mathscr{C}_\nu(t)\, dt = \int_{c_{\nu k}}^{c_{\nu,k+2}} t^{\gamma-\delta}\{t^{\delta-1}\mathscr{C}_\nu(t)\}\, dt,$$

$k = 1, 2, \cdots$. Here $\gamma - \delta < 0$ so that $t^{\gamma-\delta}$ is a positive decreasing function. Hence the second mean value theorem shows that the integral on the left, being equal to the one on the right, has the same sign as

$$\int_{c_{\nu k}}^{c_{\nu,k+2}} t^{\delta-1}\mathscr{C}_\nu(t)\, dt, \qquad k = 1, 2, \cdots,$$

and the principal assertion of the theorem is proved for all $\gamma \leqq 0$ as well as for $0 < \gamma < \frac{3}{2}$.

The proof shows that $\varkappa = 1$ for all $\mathscr{C}_\nu(t)$ when $\gamma \leqq 0$, in conformity with definition (5) of $\lambda(\nu;\gamma)$.

To prove the assertions concerning the range $0 < \gamma \leqq 1$, it is sufficient to consider $\gamma = 1$, since $\lambda(\nu;\gamma) \leqq \lambda(\nu;1)$, $0 < \gamma \leqq 1$, for all ν.

That $\varkappa \leqq 2$ for such γ follows from the inequalities $\lambda(\nu;\gamma) \leqq \lambda(\nu;1) \leqq \frac{1}{3}\sqrt{2} < 1$ and $c_{\nu 2} > 1$ [6, p. 1254, Remark (i)], valid for all ν.

When $|\nu| \geqq \frac{1}{3}\gamma$, (5) requires $\lambda(\nu;1) = 0$ and so $\varkappa = 1$ for such ν and $0 < \gamma \leqq 1$.

Thus, only the remaining range $|\nu| < \frac{1}{3}\gamma$ need be considered for the final case in which $\mathscr{C}_\nu(t) \equiv J_\nu(t)$, with, still, $0 < \gamma \leqq 1$. Here $j_{\nu 1} > j_{-1/2,1} = \frac{1}{2}\pi > 1$, since $j_{\nu 1}$ increases with ν for $\nu > -1$ [11, p. 508], so that $j_{\nu 1} > \lambda(\nu;1) \geqq \lambda(\nu;\gamma)$, $0 < \gamma \leqq 1$.

The theorem is proved.

3. Proof of Corollary 1. Without loss of generality it can be assumed that $\mathscr{C}_\nu(t) > 0$ for $0 < t < y_{\nu 1}$, since $c_{\nu 1} \geqq y_{\nu 1}$. Using again the observation, already made in the course of proving the parts of Theorem 1 concerned with the range $0 < \gamma \leqq 1$, that $\lambda(\nu;\gamma) \leqq \lambda(\nu;1)$ for $0 < \gamma \leqq 1$, it follows that $\varkappa = 1$ if $y_{\nu 1} > \lambda(\nu;1)$.

Hence, it suffices to show that

$$y_{\nu 1} \geqq \tfrac{1}{3}[2(1 - 9\nu^2)]^{1/2}, \qquad -\tfrac{1}{3} < \nu < \tfrac{1}{3}. \tag{13}$$

Now [11, p. 748], $y_{01} = 0.89\ldots > \frac{1}{3}\sqrt{2}$ and $y_{\nu 1}$ is an increasing function of ν for $\nu > -\frac{1}{2}$ [11, pp. 508–509], so that (13) holds for $0 \leqq \nu < \frac{1}{3}$. Since $y_{\nu 1}$ increases with ν and the right member of (13) is even, this can subsumed in the substantially more complicated proof for $0 > \nu > -\frac{1}{3}$ which follows.

From [11, p. 714] it is clear that

$$.38 > y_{-1/3,1} > .36 > \tfrac{1}{3}.$$

Moreover, $y_{01} > \frac{8}{9}$. Writing $y(\nu) \equiv y_{\nu 1}$, we have [11, p. 508, (3)]

$$\frac{d^2}{d\nu^2}[\log y(\nu)] = 4\int_0^\infty K_0'(2y(\nu)\sinh t)y'(\nu)(\sinh t)\,e^{-2\nu t}\,dt$$
$$-4\int_0^\infty K_0(2y(\nu)\sinh t)te^{-2\nu t}\,dt < 0,$$

since $K_0(t) > 0$ and $K_0'(t) < 0$ for $0 < t < \infty$. Thus, the increasing function $\log y(\nu)$ is concave down on the interval $-\frac{1}{2} < \nu < \infty$. The respective unattained lower bounds $\frac{1}{3}$ and $\frac{8}{9}$ for $y(-\frac{1}{3})$ and $y(0)$ show that

$$\log y(\nu) > 3\nu \log \tfrac{8}{3} + \log \tfrac{8}{9}, \qquad -\tfrac{1}{3} \leqq \nu \leqq 0.$$

The result (13) will follow, for this range of ν, if

$$3\nu \log \tfrac{8}{3} + \log \tfrac{8}{9} \geqq -\log 3 + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\log(1 - 9\nu^2), \qquad -\tfrac{1}{3} < \nu \leqq 0.$$

This is equivalent to

$$(64/9)^{3\nu+1} \geqq 2(1 - 9\nu^2), \qquad -\tfrac{1}{3} \leqq \nu \leqq 0,$$

which, in turn, follows from the sequence of inequalities

$$(64/9)^{3\nu+1} \geqq 7^{3\nu+1} = e^{(3\nu+1)\log 7} \geqq e(\log 7)(3\nu + 1)$$
$$\geqq 4(3\nu + 1) \geqq 2(1 - 9\nu^2), \qquad -\tfrac{1}{3} \leqq \nu \leqq 0.$$

This completes the proof of (13) and that of Corollary 1.

4. Proofs of Corollaries 2, 3, 4. Corollary 2, except for the last two sentences, follows on specializing γ to be $1 - \nu$ in (4) and (5); this gives (6) and (7) respectively.

To show that $\varkappa \leqq 2$, it suffices to recall that $c_{\nu 2} > 1$, all ν [6, p. 1254, Remark (i)] since $\lambda(\nu; 1 - \nu) < 1$, $-\frac{1}{2} < \nu < \frac{1}{4}$.

To show that $\varkappa = 1$ when $\mathscr{C}_\nu(t) = J_\nu(t)$ we recall that $j_{\nu 1}$ increases with ν for $\nu > -1$ [11, p. 508]. Thus, from (7),

$$j_{\nu 1} > j_{-1/2,1} = \tfrac{1}{2}\pi > 1 > \lambda(\nu; 1 - \nu), \qquad \nu > -\tfrac{1}{2}.$$

The Szegö inequality follows as described on pairing arches. Corollary 3 follows from Theorem 1 and Corollary 1 on putting $\gamma = 1$.

Corollary 4 follows on putting $\gamma = 2 - \nu$ in the theorem and noting that (5) gives

$$\lambda(\nu; 2 - \nu) = 0, \qquad \nu > \tfrac{1}{2}.$$

5. Proof of Theorem 2. We may put $A = 1$ without loss of generality, the case $A = 0$ being trivial. The Wronskian of $f(t), g(t)$ is defined, as usual, to be $\mathscr{W}(f, g; t) = f(t)g'(t) - f'(t)g(t)$. From [11, p. 76, (1)], with $\mathscr{C}_\nu(t) = J_\nu(t) + BY_\nu(t)$,

$$\mathscr{W}(\mathscr{C}_\nu, Y_\nu; t) = \mathscr{W}(J_\nu, Y_\nu; t) = \frac{2}{\pi t} > 0, \qquad t > 0.$$

The definition (for noninteger ν)

$$Y_\nu(t) = (\csc \nu\pi)[(\cos \nu\pi)J_\nu(t) - J_{-\nu}(t)]$$

implies (again for noninteger ν)

$$\mathscr{C}_\nu(t) = (1 + B \cot \nu\pi)J_\nu(t) - (B \csc \nu\pi)J_{-\nu}(t).$$

With the familiar asymptotic equation

$$J_\nu(t) \cong [2^\nu\Gamma(1 + \nu)]^{-1}t^\nu, \quad \nu > -1, \quad \text{as } t \to 0+$$

in mind, we divide the proof of Theorem 2 into four parts.

(i) *If* $0 \leqq \nu < 1$ *and* $B \leqq 0$, *then* $c_{\nu 1} > y_{\nu 1}$.

For $B = 0$ the result is known [7(a), p. 364, (i)].

If $B < 0$, then $\mathscr{C}_\nu(0) = +\infty$, since $Y_\nu(0) = -\infty$. Furthermore, $0 < \mathscr{W}(\mathscr{C}_\nu, Y_\nu; c_{\nu 1}) = -\mathscr{C}'_\nu(c_{\nu 1})Y_\nu(c_{\nu 1})$, whence $Y_\nu(c_{\nu 1}) > 0$ and $c_{\nu 1} > y_{\nu 1}$ as asserted.

(ii) *If* $-\frac{1}{2} < \nu < 0$ *and* $B \leqq -\tan \nu\pi$, *then again* $c_{\nu 1} > y_{\nu 1}$.

Here $\mathscr{C}_\nu(t) > 0$, $0 < t < c_{\nu 1}$, and the reasoning of (i) applies to yield the desired result.

(iii) *If* $0 \leqq \nu < 1$ *and* $B > 0$, *then* $y_{\nu 1} > c_{\nu 1}$.

Here $\mathscr{C}_\nu(t) < 0$, $0 < t < c_{\nu 1}$, with $0 < \mathscr{W}(\mathscr{C}_\nu, Y_\nu; y_{\nu 1}) = \mathscr{C}_\nu(y_{\nu 1})Y'_\nu(y_{\nu 1})$. Thus, $\mathscr{C}_\nu(y_{\nu 1}) > 0$ and $y_{\nu 1} > c_{\nu 1}$.

(iv) *If* $-\frac{1}{2} < \nu < 0$ *and* $B > -\tan \nu\pi$, *then again* $y_{\nu 1} > c_{\nu 1}$.

As in (iii), $\mathscr{C}_\nu(t) < 0$, $0 < t < c_{\nu 1}$, $\mathscr{C}_\nu(y_{\nu 1}) > 0$, so that $y_{\nu 1} > c_{\nu 1}$.

6. Remarks. (i) In Theorem 1, the range of γ cannot be extended. For $\gamma = \frac{3}{2}$ the areas become equal when $|\nu| = \frac{1}{2}$ and increase [8] when $|\nu| < \frac{1}{2}$. For the same reason ν cannot be extended in Corollary 2 to the value $\nu = -\frac{1}{2}$, nor in Corollary 4 to the value $\nu = \frac{1}{2}$.

(ii) Szegö's monotonicity result on areas (Corollary 3) cannot, in general, be extended to include the area of the first arch, that is, the one beginning at $t = 0$ and terminating at $t = c_{\nu 1}$ (a relevant question when $\varkappa = 1$), since [11, p. 394, (8)]

$$\int_0^\infty Y_0(t)\, dt = 0.$$

For this case ($\nu = 0$, $\mathscr{C}_0(t) \equiv Y_0(t)$), we have $\varkappa = 1$, since $y_{01} = .89\ldots > \frac{1}{3}\sqrt{2} = \lambda(0; 1)$. Here the first arch bounds a smaller area than does the second arch. This is true also for $Y_\nu(t)$, $-\frac{1}{2} < \nu < 0$, as can be inferred from [11, p. 394, (7)], together with Corollary 1.

(iii) When $\mathscr{C}_\nu(t) \equiv J_\nu(t)$, $\nu > -1$, on the contrary, the first arch does begin the sequence of arches with decreasing areas. This was proved by R. G. Cooke [4]. The example in (ii) shows also that Cooke's theorem cannot be extended from $J_\nu(t)$ to arbitrary $\mathscr{C}_\nu(t)$, not even to $Y_\nu(t)$.

(iv) A particularly simple proof of Cooke's theorem [4] has been devised by J. Steinig [9].

For the range $|\nu| > \frac{1}{2}$, E. Makai proved [8] results of Cooke type [4] in a more general setting, in a very neat way, using the differential equation method employed here in § 2.

(v) There is a misprint in [2, p. 104] in describing Szegö's result on (1). There, the value of $\lambda(\nu; 1)$ in (8) is printed with the square root replaced by the cube root. This slip was noticed also by J. Steinig, as we learned from correspondence with him.

(vi) Theorem 1 and Corollaries 1, 2 and 3 give conditions under which $\varkappa \leqq 2$ in (4). However, for $|\nu| < \frac{1}{2}$, $\varkappa$ can be arbitrarily large, when γ $(<\frac{3}{2})$ is sufficiently close to $\frac{3}{2}$. More precisely, given $\varkappa$ and ν, where $|\nu| < \frac{1}{2}$, by choosing $\gamma < \frac{3}{2}$ properly we can have at least the first $\varkappa$ elements of the sequence (4) *increase* before the sequence begins to decrease. This is a consequence of the Sturmian lemma used in § 2. All that is needed is to pick β in (11) so that $\varphi(x)$ is a decreasing function for $0 < x < c_{\nu\varkappa}$, for example, by choosing γ so that, in (5), $\lambda(\nu; \gamma) = c_{\nu\varkappa}$, $|\nu| < \frac{1}{2}$.

(vii) R. Askey, in the final paragraph of [1], advances an interesting conjecture related to (2′) and (3), by allowing the exponent of the factor t^{-1} in the integrand to differ from the order α of $J_\alpha(t)$.

(viii) J. Steinig utilizes integrals similar to those occurring in (4) in his study of the sign of Lommel functions [10].

(ix) The differential equation (10), with $\varphi(x)$ defined by (11), can be used also to prove Theorem 5.4 of [6, p. 1253]. Doing so eliminates the need to separate the cases $|\nu| \geqq \frac{1}{2}$ and $\frac{1}{3} \leqq |\nu| < \frac{1}{2}$ (as was done in [6]), since $\varphi'(x)$ is completely monotonic, $0 < x < \infty$, for $|\nu| \geqq \frac{1}{3}$ when $\beta = \frac{3}{2}$.

(x) Theorem 5.4 [6, p. 1253] makes more precise the result of Corollary 3 when $|\nu| \geqq \frac{1}{3}$ since that theorem shows, for such ν, that the sequence of areas is completely monotonic. A similar partial (in ν) extension of Theorem 1 of this note can be effectuated by the method of proof of Theorem 5.4 of [6].

(xi) Corollary 4 (Cooke) can be strengthened similarly [6, Theorem 5.1, p. 1251; Theorem 5.2, p. 1252]. The sequence (9) is completely monotonic, not merely decreasing.

Acknowledgment. The authors wish to thank the referee whose careful reading led to the clarification of several points. Similar thanks go to Professor J. Steinig.

REFERENCES

[1] RICHARD ASKEY, *Positive Jacobi polynomial sums*, Tôhoku Math. J., 24 (1972), pp. 109–119.

[2] I. BIHARI, *Oscillation and monotonicity theorems concerning non-linear differential equations of the second order*, Acta Math. Acad. Sci. Hungar., 9 (1958), pp. 83–104.

[3] RICHARD G. COOKE, *On the sign of Lommel's function*, J. London Math. Soc., 7 (1932), pp. 281–283.

[4] ———, *A monotonic property of Bessel functions*, Ibid., 12 (1937), pp. 180–185.

[5] ERVIN FELDHEIM, *On the positivity of certain sums of ultraspherical polynomials*, J. Analyse Math., 11 (1963), pp. 275–284 (edited with additional notes by G. Szegö, pp. 279–284).

[6] LEE LORCH, M. E. MULDOON AND PETER SZEGO, *Higher monotonicity properties of certain Sturm–Liouville functions. III*, Canad. J. Math., 22 (1970), pp. 1238–1265.

[7] LEE LORCH AND DONALD J. NEWMAN, (a) *A supplement to the Sturm separation theorem, with applications*, Amer. Math. Monthly, 72 (1965), pp. 359–366; (b) *Acknowledgment of priority*, Ibid., 72 (1965), p. 980.

[8] E. MAKAI, *On a monotonic property of certain Sturm–Liouville functions*, Acta Math. Acad. Sci. Hungar., 3 (1952), pp. 165–172.

[9] JOHN STEINIG, *On a monotonicity property of Bessel functions*, Math. Z., 122 (1971), pp. 363–365.

[10] ———, *The sign of Lommel's function*, Trans. Amer. Math. Soc., 163 (1972), pp. 123–129.

[11] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.

SIAM J. MATH. ANAL.
Vol. 4, No. 2, May 1973

AVERAGED INTEGRAL TRANSFORMS*

D. M. ROGNLIE† AND B. C. CARLSON‡

Abstract. An integral transform $\bar{f}(s) = \int f(t)\varphi(s, t)\,dt$ can be averaged over the convex hull of $\{s_1, \cdots, s_k\}$ to produce an analogous function $\bar{F}(s_1, \cdots, s_k)$ of several real or complex variables. The question arises whether it is legitimate to take the average under the integral sign, so that $\bar{F}(s_1, \cdots, s_k) = \int f(t)\Phi(s_1, \cdots, s_k; t)\,dt$, where Φ is the corresponding average of φ. Conditions for the validity of this equation are of interest in the theory of special functions because the kernel Φ may be a Bessel function, elliptic integral, or other hypergeometric function when φ is the kernel of a Laplace, Fourier, or Stieltjes transformation. Sufficient conditions of validity are established in the case of these three transformations and the inverse Laplace, Fourier, and Mellin transformations. The averages have some but not all of the operational properties of the corresponding ordinary transforms. Some examples are given involving various special functions.

1. Introduction. The one-sided Laplace transformation,

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt, \tag{1.1}$$

has been generalized in several ways. A function $f(t_1, \cdots, t_k)$ can be transformed with respect to each variable separately to produce a multiple Laplace transform $\bar{f}(s_1, \cdots, s_k)$ [4]. The kernel e^{-st} can be replaced by a similar but more complicated function such as a Whittaker function or Meijer's G-function with argument st; there is an extensive but scattered literature about this type of generalization, and a number of references can be found in [5]. Of course the kernel of the multiple transform also can be replaced by a more complicated function, as in [9], for example. In the present paper the function $f(t)$ to be transformed depends on only one variable, while the transform $\bar{F}(s_1, \cdots, s_k)$ depends on several variables and can be constructed from $\bar{f}(s)$ by an averaging process. Because $\bar{F}(s, \cdots, s) = \bar{f}(s)$, inversion is accomplished by inverting the ordinary Laplace transformation.

It is the averaging process [2] which provides a natural starting point and determines the kernel of the transformation. Let a function g be continuous on a domain in the z-plane containing the points $z_1, \cdots, z_k$ and their convex combinations $\mathbf{u} \cdot \mathbf{z} = \sum_{i=1}^k u_i z_i$. We define

$$G(\mathbf{b}, \mathbf{z}) = \int_E g(\mathbf{u} \cdot \mathbf{z})\, d\mu_{\mathbf{b}}(\mathbf{u}), \tag{1.2}$$

* Received by the editors August 23, 1971, and in revised form April 3, 1972.

† Ames Laboratory of the U.S. Atomic Energy Commission, Iowa State University, Ames, Iowa. Now with the Department of Mathematics, South Dakota School of Mines and Technology, Rapid City, South Dakota 57701. The paper is based in part on this author's doctoral thesis written at Iowa State University.

‡ Ames Laboratory of the U.S. Atomic Energy Commission, and Departments of Physics and Mathematics, Iowa State University, Ames, Iowa 50010.

where the region of integration is the set E of all positive weights $(u_1, \cdots, u_k)$ with $u_k = 1 - u_1 - \cdots - u_{k-1}$. The measure is

$$d\mu_{\mathbf{b}}(\mathbf{u}) = \Gamma\left(\sum_{i=1}^{k} b_i\right) \prod_{j=1}^{k} \frac{u_j^{b_j - 1}}{\Gamma(b_j)} du_1 \cdots du_{k-1}, \tag{1.3}$$

where Γ is the gamma function and the parameters $b_1, \cdots, b_k$ have positive real parts (briefly $\operatorname{Re} \mathbf{b} > 0$). Since $\int_E d\mu_{\mathbf{b}}(\mathbf{u}) = 1$, $G(\mathbf{b}, \mathbf{z})$ is an integral average of g over the convex hull of $\{z_1, \cdots, z_k\}$, denoted by con $(\mathbf{z})$. Moreover, $G(b_1, \cdots, b_k; z_1, \cdots, z_k)$ is symmetric in the indices $1, \cdots, k$.

Let the average of $\bar{f}$ in (1.1) be

$$\bar{F}(\mathbf{b}, \mathbf{s}) = \int_E \bar{f}(\mathbf{u} \cdot \mathbf{s})\, d\mu_{\mathbf{b}}(\mathbf{u}), \tag{1.4}$$

and let the average of the exponential function be

$$S(\mathbf{b}, \mathbf{z}) = \int_E e^{\mathbf{u} \cdot \mathbf{z}}\, d\mu_{\mathbf{b}}(\mathbf{u}). \tag{1.5}$$

We shall inquire under what conditions (1.1) implies

$$\bar{F}(\mathbf{b}, \mathbf{s}) = \int_0^\infty f(t) S(\mathbf{b}, -\mathbf{s}t)\, dt, \tag{1.6}$$

where $-\mathbf{s}t = (-s_1 t, \cdots, -s_k t)$. The kernel $S(\mathbf{b}, -\mathbf{s}t)$ of this averaged Laplace transformation is in many ways closely analogous to e^{-st}, and (1.6) reduces to (1.1) if $s_1 = \cdots = s_k$. The particular cases of S include some important special functions such as the confluent hypergeometric function

$$e^{-wt} {}_1F_1(b, c; zt) = S(b, c - b; -wt + zt, -wt). \tag{1.7}$$

Integrals containing the left side of (1.7) in the integrand can thus be put in the form (1.6) and in some cases evaluated most easily by way of (1.1) and (1.4).

Several other transforms will be averaged in a similar way. In the case of the Stieltjes transform the kernel is a power instead of an exponential function, and the average of a power will be denoted by

$$R_t(\mathbf{b}, \mathbf{z}) = \int_E (\mathbf{u} \cdot \mathbf{z})^t\, d\mu_{\mathbf{b}}(\mathbf{u}), \qquad 0 \notin \text{con}\,(\mathbf{z}). \tag{1.8}$$

Special cases of the R-function include Legendre functions, elliptic integrals, and the hypergeometric function ${}_2F_1$.

For each type of transformation, the same problem occurs that we confront in passing from (1.1) to (1.6). The integration which converts $\bar{f}(s)$ into $\bar{F}(\mathbf{b}, \mathbf{s})$ must be taken inside the integral in (1.1) to obtain (1.6). The change in order of integration will be justified in § 2 when the first integral is a multiple integral over E of the type occurring in (1.4). However, if $\bar{f}$ is analytic on a domain in the s-plane, the conversion to $\bar{F}$ can be accomplished by a single contour integration. This procedure is used in § 3 and leads to less restrictive conditions of validity in some cases.

We discuss in § 4 some operational properties of the averaged transforms and in § 5 some examples of the averaged Laplace and Stieltjes transforms. The last section contains four lemmas which are used frequently in § 2 and § 3. For other connections between averages and transforms, see [3] and [1].[1]

2. Averaging by multiple integrals. We shall average the Fourier, Laplace and Stieltjes transforms and the inverse Fourier, Laplace and Mellin transforms. In each case we shall assume conditions on a function which allow the order of integration to be changed. In the first three cases both the ordinary and averaged transforms are analytic on certain domains.

2.1. Fourier transform. If h is a function defined on the real line $\mathbb{R}$, the Fourier transform of h is defined for possibly complex s by

$$\bar{h}(s) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} h(t)\, e^{-ist}\, dt, \tag{2.1}$$

provided that this integral exists. We denote the average of $\bar{h}$ by $\bar{H}(\mathbf{b}, \mathbf{s})$.

THEOREM 1. *Let K_1 and K_2 be positive constants, and let a_1 and a_2 be real constants with $a_1 < a_2$. Assume that h is measurable on $\mathbb{R}$ and that $|h(t)| \leqq K_1 \exp(-a_1 t)$, $-\infty < t < 0$, and $|h(t)| \leqq K_2 \exp(-a_2 t)$, $0 < t < \infty$. Then $\bar{h}(s)$ is analytic on the open strip $D = \{s : a_1 < \operatorname{Im} s < a_2\}$. Moreover, if* $\operatorname{Re} \mathbf{b} > 0$ *and* $\mathbf{s} = (s_1, \cdots, s_k) \in D^k$, *then*

$$\bar{H}(\mathbf{b}, \mathbf{s}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} h(t) S(\mathbf{b}, -i\mathbf{s}t)\, dt, \tag{2.2}$$

where S is defined by (1.5).

Proof. We write $\bar{h}(s) = \bar{h}_1(s) + \bar{h}_2(s)$, where

$$\bar{h}_i(s) = (2\pi)^{-1/2} \int_0^{\infty} \chi_i(s, t)\, dt$$

and $\chi_1(s, t) = h(-t)\, e^{ist}$, $\chi_2(s, t) = h(t)\, e^{-ist}$ for $t \in \mathbb{R}_+$ (the positive real line). Suppose that $a_1 < \rho_1 < \rho_2 < a_2$. To apply Lemma 2 (see § 6), let $dv(t) = dt$, $X = \mathbb{R}_+$, $\Omega_1 = \{s : \operatorname{Im} s > \rho_1\}$, $\Omega_2 = \{s : \operatorname{Im} s < \rho_2\}$. Now $\chi_i(s, t)$, $i = 1, 2$, is analytic in s for each $t \in \mathbb{R}_+$ and measurable in t for each finite s, and $|\chi_i(s, t)| \leqq K_i \exp(-|\rho_i - a_i| t)$, $s \in \Omega_i$, $t \in \mathbb{R}_+$. Thus $|\chi_i(s, t)|$ is majorized by an integrable function of t and, by Lemma 2, $\bar{h}_i$ is analytic on Ω_i. Hence $\bar{h}$ is analytic on $\Omega_1 \cap \Omega_2 = \{s : \rho_1 < \operatorname{Im} s < \rho_2\}$. Given any $s \in D$ we can choose ρ_1 and ρ_2 so that $s \in \Omega_1 \cap \Omega_2$. Therefore $\bar{h}$ is analytic on D.

The integral average of $\bar{h}$ is

$$\bar{H}(\mathbf{b}, \mathbf{s}) = \sum_{i=1}^{2} (2\pi)^{-1/2} \int_E \int_0^{\infty} \chi_i(\mathbf{u} \cdot \mathbf{s}, t)\, dt\, d\mu_{\mathbf{b}}(\mathbf{u})$$

by (1.2). We now choose ρ_1 and ρ_2 so that $s_1, \cdots, s_k \in \Omega_1 \cap \Omega_2$. Then $\mathbf{u} \cdot \mathbf{s} \in \Omega_1 \cap \Omega_2$ and hence $|\chi_i(\mathbf{u} \cdot \mathbf{s}, t)|$ is majorized by an integrable function of t as before. We can therefore apply Lemma 3 with $\lambda = \mathbb{R}_+$ and $\psi\varphi = \chi_i$ to obtain (2.2). It follows from Lemma 1 that $\bar{H}(\mathbf{b}, \mathbf{s})$ is analytic in $\mathbf{b}$ and $\mathbf{s}$ on a domain in $\mathbb{C}^{2k}$ defined by the conditions $\operatorname{Re} \mathbf{b} > 0$ and $\mathbf{s} \in D^k$; similar remarks apply to the Laplace and Stieltjes transforms below.

[1] We are obliged to an Editor for these references.

The two-sided Laplace transformation is the same as (2.1) except that the kernel e^{-ist} is replaced by e^{-st}. Thus Theorem 1 still holds if the strip D is now defined to be $\{s : a_1 < -\mathrm{Re}\, s < a_2\}$ and the kernel in (2.2) is replaced by $S(\mathbf{b}, -\mathbf{s}t)$.

2.2. Laplace transform. The one-sided Laplace transform $\bar{f}$ of a function f is defined by (1.1) and its average $\bar{F}$ by (1.4). The proof of the next theorem is similar to that of Theorem 1.

THEOREM 2. *Let f be measurable on $\mathbb{R}_+$. For real constants $M > 0$ and σ, assume that $|f(t)| \leqq M e^{\sigma t}$, $t \in \mathbb{R}_+$. Then $\bar{f}$ is analytic on the half-plane* $\mathrm{Re}\, s > \sigma$. *Also, if $s_1, \cdots, s_k$ lie in this half-plane and if* $\mathrm{Re}\, \mathbf{b} > 0$, *then* (1.6) *is true.*

The averaged Laplace transform is related to other integral transforms, the most interesting case being the Hankel transform

$$\bar{h}_\nu(x) = \int_0^\infty h(t) J_\nu(xt)(xt)^{1/2}\, dt, \tag{2.3}$$

where J_ν is a Bessel function. By (5.7) we have

$$\begin{aligned} \bar{h}_\nu(x) &= \int_0^\infty h(t) \frac{(xt/2)^\nu}{\Gamma(1+\nu)} S(\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; ixt, -ixt)(xt)^{1/2}\, dt \\ &= \frac{2^{-\nu} x^{\nu + 1/2}}{\Gamma(1+\nu)} \bar{F}(\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; -ix, ix), \end{aligned} \tag{2.4}$$

where $\bar{F}$ is the averaged Laplace transform of $f(t) = t^{\nu + 1/2} h(t)$ with $b_1 = b_2 = \nu + \frac{1}{2}$, $s_1 = -ix$ and $s_2 = ix$.

Incidentally, $\bar{F}(\mathbf{b}, \mathbf{s})$ in (1.6) is not the multiple Laplace transform of $F(\mathbf{b}, \mathbf{t})$, where $F(\mathbf{b}, \mathbf{t})$ is the average of $f(t)$. This is shown by the example $f(t) = t$, $k = 2$ and $b_1 = b_2 = 1$. We have $\bar{f}(s) = s^{-2}$, $\mathrm{Re}\, s > 0$, and $\bar{F}(1, 1; x, y) = (xy)^{-1}$, $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$. On the other hand, $F(1, 1; t, v) = \frac{1}{2}(t + v)$ and its double Laplace transform is

$$\frac{1}{2} \int_0^\infty e^{-xt}\, dt \int_0^\infty e^{-yv}(t + v)\, dv = \tfrac{1}{2}(xy)^{-1}(x^{-1} + y^{-1}).$$

2.3. Stieltjes transform. If g is a function defined on $\mathbb{R}_+$ and a is any complex number, then the Stieltjes transform of g is defined by

$$\bar{g}_a(s) = \int_0^\infty g(t)(s + t)^{-a}\, dt \tag{2.5}$$

whenever this integral exists. Since the integral is not well-defined if s lies on the negative real axis, we assume $s \neq 0$ and $|\arg s| < \pi$ and we take $|\arg(s + t)| < \pi$ for $t \in \mathbb{R}_+$. The average of $\bar{g}_a$ will be denoted by $\bar{G}_a(\mathbf{b}, \mathbf{s})$.

THEOREM 3. *Let g be measurable on $\mathbb{R}_+$ and let a be a fixed complex number. Assume that $|g(t)| \leqq M t^{\zeta - 1}(1 + t)^{\mathrm{Re}\, a - 2\zeta}$, $t \in \mathbb{R}_+$, where M and ζ are positive constants. Then $\bar{g}_a$ is analytic on the s-plane cut along the nonpositive real axis. Moreover, if* $\mathrm{Re}\, \mathbf{b} > 0$ *and if the convex hull* $\mathrm{con}(\mathbf{s})$ *is contained in the cut plane, then*

$$\bar{G}_a(\mathbf{b}, \mathbf{s}) = \int_0^\infty g(t) R_{-a}(\mathbf{b}, \mathbf{s} + \mathbf{t})\, dt, \tag{2.6}$$

where R_{-a} is defined by (1.8) *and* $\mathbf{s} + \mathbf{t} = (s_1 + t, \cdots, s_k + t)$.

Proof. In Lemma 2 let $X = \mathbb{R}_+$ and let Ω be a bounded open set whose closure is contained in the cut plane. Take $d\nu(t) = dt$ and $\varphi(s, t) = g(t)(s + t)^{-a}$. Then $\varphi(s, t)$ is measurable in t for each $s \in \Omega$ and analytic on Ω for each $t \in \mathbb{R}_+$. Since $|\arg(s + t)| < \pi$ by assumption, we have

$$|\varphi(s, t)| = |g(t)(s + t)^{-a}| \leqq |g(t)| e^{\pi|\mathrm{Im}\, a|} |s + t|^{-\mathrm{Re}\, a}$$
$$\leqq e^{\pi|\mathrm{Im}\, a|} M t^{\zeta - 1}(1 + t)^{\mathrm{Re}\, a - 2\zeta} f(t),$$

where

$$f(t) = \sup_{s \in \Omega} |s + t|^{-\mathrm{Re}\, a}, \qquad t \in \mathbb{R}_+.$$

Being the supremum of a collection of measurable functions, f is measurable. Since $|s|$ is bounded away from zero on Ω and also bounded above, $f(t)$ remains bounded as $t \to 0$ and $t^{\mathrm{Re}\, a} f(t)$ remains bounded as $t \to \infty$. Hence $|\varphi(s, t)|$ is majorized by an integrable function, and it follows from Lemma 2 that $\bar{g}_a$ is analytic on Ω. Given any s in the cut plane we can choose Ω so that $s \in \Omega$. Hence $\bar{g}_a$ is analytic on the cut plane.

The average of $\bar{g}_a$ is

$$\bar{G}_a(\mathbf{b}, \mathbf{s}) = \int_E \int_0^\infty g(t)(\mathbf{u} \cdot \mathbf{s} + t)^{-a}\, dt\, d\mu_{\mathbf{b}}(\mathbf{u}), \tag{2.7}$$

the integrand being measurable on $E \times \mathbb{R}_+$. Choose a bounded open set Ω in the cut plane so that $\mathrm{con}(\mathbf{s}) \subset \Omega$. Then, since $\mathbf{u} \cdot \mathbf{s} \in \Omega$, the same argument as before with s replaced by $\mathbf{u} \cdot \mathbf{s}$ shows that $|g(t)(\mathbf{u} \cdot \mathbf{s} + t)^{-a}|$ is majorized by an integrable function of t. Applying Lemma 3 with $\lambda = \mathbb{R}_+$ and $\psi = g$ and using (1.8), we have (2.6).

The inverse Fourier, Laplace and Mellin transforms are of the form

$$f(t) = \int_\lambda \bar{f}(s) \varphi(s, t)\, ds,$$

where λ is a path in the complex plane. We are not concerned here with inversion theorems, and the notation is not intended to imply that $\bar{f}$ is necessarily the direct transform of f. In each case the conditions which $\bar{f}$ will be assumed to satisfy include analyticity on a strip containing λ. They make f continuous on either $\mathbb{R}$ or $\mathbb{R}_+$.

2.4. Inverse Fourier transform. If $\bar{h}$ is analytic on a horizontal strip in $\mathbb{C}$, its inverse Fourier transform is defined by

$$h(t) = (2\pi)^{-1/2} \int_{-\infty + i\gamma}^{\infty + i\gamma} \bar{h}(s)\, e^{ist}\, ds, \qquad t \in \mathbb{R}, \tag{2.8}$$

where the path of integration is a straight line $\mathrm{Im}\, s = \gamma$ lying in the strip. The average of h will be denoted by $H(\mathbf{b}, \mathbf{t})$.

THEOREM 4. *Let $\bar{h}(s)$ be analytic on the strip $a_1 < \mathrm{Im}\, s < a_2$ and satisfy $\bar{h}(s) \to 0$ as $|s| \to \infty$ in the strip. Assume $\bar{h}$ is Lebesgue integrable on some line $\mathrm{Im}\, s = \gamma$, $a_1 < \gamma < a_2$. Then h as defined by* (2.8) *is continuous on $\mathbb{R}$ and independent* *of* γ *for* $a_1 < \gamma < a_2$. *Also, if* $\operatorname{Re} \mathbf{b} > 0$ *and* $\mathbf{t} = (t_1, \cdots, t_k) \in \mathbb{R}^k$, *then*

$$(2.9) \qquad H(\mathbf{b}, \mathbf{t}) = (2\pi)^{-1/2} \int_{-\infty + i\gamma}^{\infty + i\gamma} \bar{h}(s) S(\mathbf{b}, is\mathbf{t})\, ds,$$

where $is\mathbf{t} = (ist_1, \cdots, ist_k)$.

Proof. Let γ, $a_1 < \gamma < a_2$, be fixed. Then

$$(2.10) \qquad h(t) = (2\pi)^{-1/2} e^{-\gamma t} \int_{-\infty}^{\infty} \bar{h}(\sigma + i\gamma)\, e^{i\sigma t}\, d\sigma, \qquad t \in \mathbb{R}.$$

Since $\bar{h}$ is Lebesgue integrable on the line $\operatorname{Im} s = \gamma$, this integral exists and defines a continuous function of t on $\mathbb{R}$, and so h is continuous on $\mathbb{R}$ for this γ. A simple application of Cauchy's theorem shows that h is independent of γ.

We write the average of (2.10) as

$$H(\mathbf{b}, \mathbf{t}) = (2\pi)^{-1/2} \int_E \int_{-\infty}^{\infty} \bar{h}(\sigma + i\gamma)\, e^{i(\sigma + i\gamma)(\mathbf{u} \cdot \mathbf{t})}\, d\sigma\, d\mu_{\mathbf{b}}(\mathbf{u}).$$

The exponential factor is measurable on $E \times \mathbb{R}$ and is bounded in modulus by $f = e^{|\gamma| T}$, where $T = \max\{|t_1|, \cdots, |t_k|\}$. Since $|\bar{h}| f$ is integrable on $\mathbb{R}$, we may apply Lemma 3 with $\lambda = \mathbb{R}$ and $\psi = \bar{h}$ to obtain (2.9).

The proofs of the next two theorems are similar to that of Theorem 4.

2.5. Inverse Laplace transform. If $\bar{f}$ is analytic on a vertical strip in $\mathbb{C}$, its inverse Laplace transform is defined by

$$(2.11) \qquad f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)\, e^{st}\, ds, \qquad t \in \mathbb{R},$$

where the path of integration is a straight line $\operatorname{Re} s = \gamma$ lying in the strip. We denote the average of f by $F(\mathbf{b}, \mathbf{t})$.

THEOREM 5. *Let* $\bar{f}$ *be analytic on the strip* $a_1 < \operatorname{Re} s < a_2$ *and satisfy* $\bar{f}(s) \to 0$ *as* $|s| \to \infty$ *in the strip. Assume* $\bar{f}$ *is Lebesgue integrable on some line* $\operatorname{Re} s = \gamma$, $a_1 < \gamma < a_2$. *Then* f *as defined by* (2.11) *is continuous on* $\mathbb{R}$ *and independent of* γ *for* $a_1 < \gamma < a_2$. *Moreover, if* $\operatorname{Re} \mathbf{b} > 0$ *and* $\mathbf{t} \in \mathbb{R}^k$, *then*

$$(2.12) \qquad F(\mathbf{b}, \mathbf{t}) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s) S(\mathbf{b}, s\mathbf{t})\, ds.$$

2.6. Inverse Mellin transform. The inverse Laplace transform is converted into the inverse Mellin transform by choosing the new variable $x = e^{-t}$, $t \in \mathbb{R}$. Thus, if $\bar{m}$ is analytic on a vertical strip in $\mathbb{C}$, its transform is defined by

$$(2.13) \qquad m(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{m}(s) x^{-s}\, ds, \qquad x \in \mathbb{R}_+.$$

We denote the average of m by $M(\mathbf{b}, \mathbf{x})$.

THEOREM 6. *Let* $\bar{m}$ *satisfy the same assumptions as* $\bar{f}$ *in Theorem* 5. *Then* m *as defined by* (2.13) *is continuous on* $\mathbb{R}_+$ *and independent of* γ *for* $a_1 < \gamma < a_2$. *If* $\operatorname{Re} \mathbf{b} > 0$ *and* $\mathbf{x} \in \mathbb{R}_+^k$, *then*

$$(2.14) \qquad M(\mathbf{b}, \mathbf{x}) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{m}(s) R_{-s}(\mathbf{b}, \mathbf{x})\, ds.$$

The function M is continuous in $\mathbf{x}$ on $\mathbb{R}_+^k$ by Lemma 1 because m is continuous on $\mathbb{R}_+$. Similar remarks apply to H and F above. All three functions are holomorphic in $\mathbf{b}$ for $\operatorname{Re} \mathbf{b} > 0$.

3. Averaging by contour integrals. If a point s of the complex plane is encircled by a positively oriented contour γ, and if g is holomorphic on γ and its inner region, the Cauchy integral formula states that

$$g(s) = \frac{1}{2\pi i} \int_\gamma g(z)(z - s)^{-1}\, dz. \tag{3.1}$$

Since the average of s^{-1} is $R_{-1}(\mathbf{b}, \mathbf{s})$ by (1.8), the analogue of (3.1) is

$$G(\mathbf{b}, \mathbf{s}) = \frac{1}{2\pi i} \int_\gamma g(z) R_{-1}(\mathbf{b}, \mathbf{z} - \mathbf{s})\, dz, \tag{3.2}$$

where $\mathbf{z} - \mathbf{s} = (z - s_1, \cdots, z - s_k)$ and γ encircles $s_1, \cdots, s_k$ but need not encircle con $(\mathbf{s})$. It is shown in [2] that $G(\mathbf{b}, \mathbf{s})$ is a holomorphic function of $\mathbf{b}$ and $\mathbf{s}$ if $\sum_{i=1}^k b_i \neq 0, -1, -2, \cdots$ and $s_1, \cdots, s_k$ lie in the inner region of γ. The function defined by (3.2) is an analytic continuation of the function defined by (1.2).

To avoid dependence on the rather lengthy proof of these statements [2], we preferred in § 2 to use only the representation (1.2). However, we now show briefly how to use (3.2) to relax the assumption made previously for the three direct transforms that $\operatorname{Re} \mathbf{b} > 0$. For the Stieltjes transform one can also allow the variables $s_1, \cdots, s_k$ to lie on both sides of the negative real axis instead of requiring their convex hull to lie in the cut plane. However, the contour-integral method is not useful for the inverse transforms, which are not in general restrictions of analytic functions to the real axis.

Taking $g(z) = e^{-izt}$ in (3.2) and defining $\mathbf{z} = (z, \cdots, z)$ and $\mathbf{t} = (t, \cdots, t)$, we find

$$S(\mathbf{b}, -i\mathbf{s}t) = \frac{1}{2\pi i} \int_\gamma e^{-izt} R_{-1}(\mathbf{b}, \mathbf{z} - \mathbf{s})\, dz, \tag{3.3}$$

and similarly, with $g(z) = (t + z)^{-a}$ where $t \in \mathbb{R}_+$,

$$R_{-a}(\mathbf{b}, \mathbf{t} + \mathbf{s}) = \frac{1}{2\pi i} \int_\gamma (t + z)^{-a} R_{-1}(\mathbf{b}, \mathbf{z} - \mathbf{s})\, dz. \tag{3.4}$$

Throughout this section γ denotes a positively oriented rectifiable Jordan curve which encircles $s_1, \cdots, s_k$, and in (3.4) γ lies in the z-plane cut along the nonpositive real axis. We define $c = \sum_{i=1}^k b_i$ and assume $c \neq 0, -1, -2, \cdots$. For the following theorems the proofs of analyticity are the same as for the corresponding theorems of § 2.

3.1. Fourier and Laplace transforms.

THEOREM 7. *Theorems* 1 *and* 2 *remain true if the assumption* $\operatorname{Re} \mathbf{b} > 0$ *is replaced by* $c \neq 0, -1, -2, \cdots$, *with* S *now given by* (3.3).

Proof. Referring to the proof of Theorem 1, we choose γ so that $\gamma \subset \Omega_1 \cap \Omega_2$. By (2.1) and (3.2) we have

$$\bar{H}(\mathbf{b}, \mathbf{s}) = \frac{(2\pi)^{-1/2}}{2\pi i} \int_\gamma \int_0^\infty \{h(-t)\, e^{itz} + h(t)\, e^{-itz}\} R_{-1}(\mathbf{b}, \mathbf{z} - \mathbf{s})\, dt\, dz.$$

Now h is measurable on $\mathbb{R}$ and, by the same argument as before, $h(\mp t)\, e^{\pm itz}$ is majorized on γ by an integrable function of t. By Lemma 4 and (3.3) we obtain (2.2). The proof for the Laplace transform is similar.

3.2 Stieltjes transform.

THEOREM 8. *Theorem* 3 *remains true if* $s_1, \cdots, s_k$ *lie in the cut plane and the assumption* Re $\mathbf{b} > 0$ *is replaced by* $c \neq 0, -1, -2, \cdots$, *with* R_{-a} *now given by* (3.4).

Proof. Let γ lie in the cut z-plane. By (2.5) and (3.2) we have

$$\bar{G}_a(\mathbf{b}, \mathbf{s}) = \frac{1}{2\pi i} \int_\gamma \int_0^\infty g(t)(z + t)^{-a} R_{-1}(\mathbf{b}, \mathbf{z} - \mathbf{s})\, dt\, dz.$$

Now g is measurable on $\mathbb{R}_+$ and, since $t \in \mathbb{R}_+$ and γ lies in the cut plane, $(z + t)^{-a}$ is continuous on $\gamma \times \mathbb{R}_+$. As in the proof of Theorem 3, $|g(t)(z + t)^{-a}|$ is majorized on γ by an integrable function of t. By Lemma 4 and (3.4) we obtain (2.6). The function $\bar{G}_a(\mathbf{b}, \mathbf{s})$ is holomorphic in $\mathbf{b}$ and $\mathbf{s}$ if $c \neq 0, -1, -2, \cdots$ and $s_1, \cdots, s_k$ lie in the cut s-plane; similar remarks apply to the Fourier or Laplace transform with $s_1, \cdots, s_k$ in a strip or half-plane.

4. Operational properties and convolutions. Some of the operational properties of ordinary Fourier, Laplace, and Stieltjes transforms carry over at once to the averaged transforms while other properties do not. If $G(\mathbf{b}, \mathbf{z})$ is defined by (1.2) and if $H(\mathbf{b}, \mathbf{z})$ is similarly the average of $h(z)$, then $\alpha G(\mathbf{b}, \mathbf{z}) + \beta H(\mathbf{b}, \mathbf{z})$ is the average of $\alpha g(z) + \beta h(z)$, where α and β are constants, but the average of $g(z)\, h(z)$ has in general no simple relation to G and H. Also, the average of $g(\alpha z + \beta)$ is $G(\alpha \mathbf{z} + \boldsymbol{\beta})$, where $\alpha \mathbf{z} + \boldsymbol{\beta} = (\alpha z_1 + \beta, \cdots, \alpha z_k + \beta)$. We assume here that g is continuous on con $(\alpha \mathbf{z} + \boldsymbol{\beta})$.

As an example consider the function $f(t)$ with Laplace transform $\bar{f}(s)$ and averaged transform $\bar{F}(\mathbf{b}, \mathbf{s})$. The function $f(t/\alpha)$ has ordinary Laplace transform $\alpha \bar{f}(\alpha s)$, and it follows from the last paragraph that the averaged transform is $\alpha \bar{F}(\mathbf{b}, \alpha \mathbf{s})$. The function $e^{\beta t} f(t)$ has ordinary transform $\bar{f}(s - \beta)$ and averaged transform $\bar{F}(\mathbf{b}, \mathbf{s} - \boldsymbol{\beta})$. Similar statements hold for the Fourier transform and, in the first case, for the Stieltjes transform.

If n is a positive integer the function $(-t)^n f(t)$ has the ordinary Laplace transform $\bar{f}^{(n)}(s)$, and its averaged transform is

$$\bar{F}^{(n)}(\mathbf{b}, \mathbf{s}) = \left(\sum_{i=1}^{k} \frac{\partial}{\partial s_i} \right)^n \bar{F}(\mathbf{b}, \mathbf{s}) \tag{4.1}$$

(see [2, (5.16)]). A similar statement holds for the Fourier transform. The averaged Stieltjes transform (2.6) satisfies

$$\bar{G}_a^{(n)}(\mathbf{b}, \mathbf{s}) = (-1)^n (a)_n \bar{G}_{a+n}(\mathbf{b}, \mathbf{s}), \tag{4.2}$$

which reduces if $s_1 = \cdots = s_k$ to the corresponding known property of the ordinary transform. Here $(a)_n = a(a + 1) \cdots (a + n - 1)$.

The Laplace transforms of $f'(t) = df/dt$ and $\int_0^t f(x)\,dx$ involve $s\bar{f}(s)$ and $s^{-1}\bar{f}(s)$, respectively, and the averages of these functions do not have a simple relation to $\bar{F}(\mathbf{b}, \mathbf{s})$. Thus the property of Laplace (and Fourier) transforms which makes them useful in solving differential and integral equations does not carry over to the averaged transforms. If $g(t)$ has the ordinary Stieltjes transform $\bar{g}_a(s)$, as in (2.5), the averaged Stieltjes transform of $g'(t)$ is easily found to be

$$a\bar{G}_{a+1}(\mathbf{b}, \mathbf{s}) - R_{-a}(\mathbf{b}, \mathbf{s})g(0), \qquad \operatorname{Re} a > 0, \tag{4.3}$$

while that of $\int_0^t g(x)\,dx$ is

$$(a - 1)^{-1}\bar{G}_{a-1}(\mathbf{b}, \mathbf{s}), \qquad \operatorname{Re} a > 1. \tag{4.4}$$

Both statements are generalizations of well-known properties of the ordinary Stieltjes transform.

The ordinary Fourier or Laplace transform of a convolution of two functions is the product of their transforms. This property also fails to hold for averaged transforms because the average of a product is not the product of the averages. However, it is useful to regard the convolution

$$h(x) = f * g(x) = \int_{-\infty}^{\infty} f(x - t)g(t)\,dt, \qquad x \in \mathbb{R}, \tag{4.5}$$

as an integral transformation with $f(x - t)$ as kernel. We can then express the averaged transform,

$$H(\mathbf{b}, \mathbf{x}) = \int_E h(\mathbf{u} \cdot \mathbf{x})\,d\mu_{\mathbf{b}}(\mathbf{u}), \tag{4.6}$$

as a convolution transform of g with an averaged kernel.

THEOREM 9. *Let f and g be Lebesgue integrable on $\mathbb{R}$, and assume that f is finite and essentially bounded on $\mathbb{R}$. If $h = f * g$ and* $\operatorname{Re} \mathbf{b} > 0$, *then*

$$H(\mathbf{b}, \mathbf{x}) = \int_{-\infty}^{\infty} F(\mathbf{b}, \mathbf{x} - \mathbf{t})g(t)\,dt, \qquad \mathbf{x} \in \mathbb{R}^k, \tag{4.7}$$

where $\mathbf{x} - \mathbf{t} = (x_1 - t, \cdots, x_k - t)$.

Proof. Since f is essentially bounded on $\mathbb{R}$, there exists a finite constant $M > 0$ such that $|f| \leqq M$ almost everywhere on $\mathbb{R}$. Thus

$$\int_{-\infty}^{\infty} |f(x - t)g(t)|\,dt \leqq M \int_{-\infty}^{\infty} |g(t)|\,dt = MN,$$

where N is finite. It follows by (4.5) that $|h(x)| \leqq MN$. Since h is bounded and Lebesgue integrable (see [8, Theorem 7.14]), and since f is finite, essentially bounded and Lebesgue integrable, the averages $H(\mathbf{b}, \mathbf{x})$ and $F(\mathbf{b}, \mathbf{x} - \mathbf{t})$ exist and are finite on $\mathbb{R}^k$ by [7, Theorem 2.3]. Moreover, by (4.5) and (4.6),

$$H(\mathbf{b}, \mathbf{x}) = \int_E \int_{-\infty}^{\infty} f(\mathbf{u} \cdot \mathbf{x} - t)g(t)\,dt\,d\mu_{\mathbf{b}}(\mathbf{u}). \tag{4.8}$$

The integrand is measurable on $E \times \mathbb{R}$ and, if $|\mu_{\mathbf{b}}|$ is the total variation of $\mu_{\mathbf{b}}$,

$$\int_E \int_{-\infty}^{\infty} |f(\mathbf{u} \cdot \mathbf{x} - t)g(t)|\, dt\, d|\mu_{\mathbf{b}}(\mathbf{u})| \leqq MN \int_E d|\mu_{\mathbf{b}}(\mathbf{u})| < \infty .$$

By Fubini's theorem we may change the order of integration in (4.8) and thus prove (4.7).

5. Examples. Besides establishing some interesting integral representations of certain special functions, Theorems 1 to 8 provide a new way of evaluating the ordinary transforms of some rather complicated functions. As examples we shall consider the Laplace transforms of some products of confluent hypergeometric functions. The averaged Stieltjes transform will be used to evaluate some integrals of elliptic integrals and of other hypergeometric functions.

We shall need the following list of properties of the R and S functions and their relations to other special functions. Their homogeneity is expressed by

$$R_t(\mathbf{b}, \lambda \mathbf{z}) = \lambda^t R_t(\mathbf{b}, \mathbf{z}), \qquad S(\mathbf{b}, \mathbf{z} + \boldsymbol{\lambda}) = e^{\lambda} S(\mathbf{b}, \mathbf{z}), \tag{5.1}$$

where $\lambda \mathbf{z} = (\lambda z_1, \cdots, \lambda z_k)$ and $\mathbf{z} + \boldsymbol{\lambda} = (z_1 + \lambda, \cdots, z_k + \lambda)$. The R-function has the properties

$$R_t(0, b, b'; x, y, z) = R_t(b, b'; y, z), \tag{5.2}$$

$$y^a R_{-a}(b, c - b; x, y) = y^b R_{-b}(a, c - a; x, y), \tag{5.3}$$

$$R_{-c}(\mathbf{b}, \mathbf{z}) = \prod_{i=1}^{k} z_i^{-b_i}, \qquad c = \sum_{i=1}^{k} b_i. \tag{5.4}$$

The inverse circular, error, Bessel, and second Legendre functions are related to R and S by

$$R_{-1/2}(\tfrac{1}{2}, 1; x, y) = (y - x)^{-1/2} \arccos (x/y)^{1/2}, \qquad 0 \leqq x < y, \tag{5.5}$$

$$\operatorname{erf}(x) = 2\pi^{-1/2} x S(\tfrac{1}{2}, 1; -x^2, 0), \tag{5.6}$$

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(1 + \nu)} S(\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; ix, -ix), \tag{5.7}$$

$$Q_\nu(x) = \frac{\pi^{1/2}\Gamma(\nu + 1)}{2^{\nu+1}\Gamma(\nu + 3/2)} R_{-\nu-1}(\nu + 1, \nu + 1; x + 1, x - 1). \tag{5.8}$$

Finally, R has the integral representation [10, (1.5)]

$$R_{-a}(\mathbf{b}, \mathbf{z}) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(a')} \int_0^\infty t^{a'-1} \prod_{i=1}^{k} (t + z_i)^{-b_i}\, dt, \tag{5.9}$$

$$a + a' = c = \sum b_i,$$

where a and a' have positive real parts and $|\arg z_i| < \pi$ for $i = 1, \cdots, k$. The integral is elliptic if four of the parameters $a, a', b_1, \cdots, b_k$ are half-odd-integers and the rest are integers. For example, the special case denoted by R_K is related to Legendre's integral $K(k)$ by

$$R_K(x, y) = R_{-1/2}(\tfrac{1}{2}, \tfrac{1}{2}; x, y) = \frac{2}{\pi} y^{-1/2} K[(1 - x/y)^{1/2}]. \tag{5.10}$$

By applying Theorem 7 to the Laplace transformation

$$\int_0^\infty t^{a-1}\, e^{-st}\, dt = \Gamma(a) s^{-a}, \tag{5.11}$$

where a and s have positive real parts, we find

$$\int_0^\infty t^{a-1} S(\mathbf{b}, -\mathbf{s}t)\, dt = \Gamma(a) R_{-a}(\mathbf{b}, \mathbf{s}), \qquad c \neq 0, -1, -2, \cdots. \tag{5.12}$$

By (5.1),

$$\int_0^\infty t^{a-1}\, e^{-pt} S(\mathbf{b}, -\mathbf{s}t)\, dt = \Gamma(a) R_{-a}(\mathbf{b}, \mathbf{p} + \mathbf{s}), \tag{5.13}$$

where $\operatorname{Re} a > 0$, $\mathbf{p} = (p, \cdots, p)$, $\operatorname{Re}(\mathbf{p} + \mathbf{s}) > 0$, and $c \neq 0, -1, -2, \cdots$. In particular, if $k = 2$ and $a = c = b + b'$, (5.4) shows that

$$\int_0^\infty t^{c-1}\, e^{-pt} S(b, b'; -zt, -wt)\, dt = \Gamma(c)(p + z)^{-b}(p + w)^{-b'}. \tag{5.14}$$

Regarding this as a Laplace transformation, we apply Theorem 2 and [2, (4.22)] to obtain

$$\begin{aligned} &\int_0^\infty t^{c-1} S(\beta, \beta'; -qt, -rt) S(b, b'; -zt, -wt)\, dt \\ &\quad = \Gamma(c) \int_0^1 [uq + (1-u)r + z]^{-b} [uq + (1-u)r + w]^{-b'}\, d\mu_{(\beta,\beta')}(u) \\ &\quad = \Gamma(c)(r + z)^{-b}(r + w)^{-b'} R_{-\beta}\left(\gamma - c, b, b'; 1, \frac{q + z}{r + z}, \frac{q + w}{r + w}\right), \end{aligned} \tag{5.15}$$

where $c = b + b'$, $\gamma = \beta + \beta'$, $\operatorname{Re} c > 0$, $\operatorname{Re} \beta > 0$, $\operatorname{Re} \beta' > 0$, and $q + z, q + w$, $r + z, r + w$ are nonzero and have arguments less than $\pi/2$ in magnitude.

We illustrate the use of (5.15) by finding two Laplace transforms, the first one previously known and the second apparently not. With the help of (5.1), (5.2), (5.7) and (5.8), we have

$$\begin{aligned} &\int_0^\infty e^{-st} J_\nu(at) J_\nu(bt)\, dt \\ &\quad = \frac{(ab/4)^\nu}{[\Gamma(\nu + 1)]^2} \int_0^\infty t^{2\nu} S(\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; -st + iat, -st - iat) \\ &\qquad \cdot S(\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; ibt, -ibt)\, dt \\ &\quad = \frac{a^\nu b^\nu \Gamma(\nu + \frac{1}{2})}{\pi^{1/2}\Gamma(\nu + 1)} R_{-\nu-1/2}[\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; s^2 + (a + b)^2, s^2 + (a - b)^2] \\ &\quad = \frac{1}{\pi (ab)^{1/2}} Q_{\nu - 1/2}\left(\frac{s^2 + a^2 + b^2}{2ab}\right), \end{aligned} \tag{5.16}$$

where $\operatorname{Re} \nu > -\frac{1}{2}$ and all four of the numbers $s \pm ia \pm ib$ have positive real parts. The second illustration, which makes use of (5.1), (5.2), (5.5) and (5.6), is

$$\begin{aligned}
&\int_0^\infty e^{-st}t^{-1/2} \operatorname{erf}[(zt)^{1/2}] \operatorname{erf}[(wt)^{1/2}]\, dt \\
&= \frac{4}{\pi}(zw)^{1/2} \int_0^\infty t^{1/2} S(\tfrac{1}{2}, 1; -st - zt, -st) S(\tfrac{1}{2}, 1; -wt, 0)\, dt \\
&= 2\left(\frac{zw}{\pi s}\right)^{1/2} R_{-1/2}[\tfrac{1}{2}, 1; s(s + z + w), (s + z)(s + w)] \\
&= 2(\pi s)^{-1/2} \arcsin\left[\frac{zw}{(s + z)(s + w)}\right]^{1/2},
\end{aligned} \tag{5.17}$$

where s, $s + z$, $s + w$, and $s + z + w$ are nonzero and have arguments less than $\pi/2$ in magnitude. This integral is not listed in the compilation by Ng and Geller [6], nor is a similar integral [7] involving the product of an error function and a Bessel function.

By applying Theorem 8 to the Stieltjes transformation

$$\int_0^\infty t^{\nu-1}(s + t)^{-a}\, dt = B(\nu, a - \nu)s^{\nu - a}, \tag{5.18}$$

where $\operatorname{Re} a > \operatorname{Re} \nu > 0$ and B is the beta function, we find

$$\int_0^\infty t^{\nu-1} R_{-a}(\boldsymbol{\beta}, \mathbf{s} + \mathbf{t})\, dt = B(\nu, a - \nu) R_{\nu-a}(\boldsymbol{\beta}, \mathbf{s}), \tag{5.19}$$

where $s_1, \cdots, s_\varkappa$ are points in the cut s-plane, $\mathbf{t} = (t, \cdots, t)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_\varkappa)$, $\gamma = \sum \beta_j$, and $\gamma \neq 0, -1, -2, \cdots$. If $a = \gamma$, (5.19) reduces to (5.9) by way of (5.4). Few other special cases of (5.19) are listed in present tables, even for $\varkappa = 2$. Choosing $\varkappa = 2$ and $a = \gamma = \beta + \beta'$, we use (5.3) to obtain

$$\begin{aligned}
\int_0^\infty t^{\nu-1}(z + t)^{-\beta}(w + t)^{-\beta'}\, dt &= B(\nu, \gamma - \nu) R_{\nu-\gamma}(\beta, \beta'; z, w) \\
&= B(\nu, \gamma - \nu) w^{\beta + \nu - \gamma} R_{-\beta}(\gamma - \nu, \nu; z, w).
\end{aligned} \tag{5.20}$$

Regarding this as a Stieltjes transformation, we apply Theorem 3 and [2, (4.21)] to prove

$$\begin{aligned}
&\int_0^\infty t^{\nu-1} R_{-\beta}(\mathbf{b}, \mathbf{z} + \mathbf{t})(w + t)^{-\beta'}\, dt \\
&= B(\nu, \gamma - \nu) w^{\nu - \beta'} \int_E R_{-\beta}(\gamma - \nu, \nu; \mathbf{u} \cdot \mathbf{z}, w)\, d\mu_{\mathbf{b}}(\mathbf{u}) \\
&= B(\nu, c) w^{\nu - \beta'} R_{-\beta}(b_1, \cdots, b_k, \nu; z_1, \cdots, z_k, w),
\end{aligned} \tag{5.21}$$

where we require $\gamma - \nu = c$ in the last step. In summary, the first and third members of (5.21) are equal if $\nu + c = \beta + \beta'$, $c = \sum_{i=1}^k b_i$, $\operatorname{Re} \mathbf{b} > 0$, $\operatorname{Re} \nu > 0$, and $z_1, \cdots, z_k, w$ are points in the plane cut along the nonpositive real axis. The last condition is permitted by analytic continuation.

We illustrate the use of (5.21) by evaluating two integrals which do not seem to be listed in present tables. The first, with an elementary integrand, is

$$\int_0^\infty t^{a-2}(z+t)^{-a}\arccos\left(\frac{x+t}{y+t}\right)^{1/2}dt$$

$$(5.22)\qquad = (y-x)^{1/2}\int_0^\infty t^{a-2}(z+t)^{-a}R_{-1/2}(\tfrac{1}{2},1;x+t,y+t)\,dt$$

$$= B(a-1,3/2)(y-x)^{1/2}z^{-1}R_{-1/2}(\tfrac{1}{2},1,a-1;x,y,z),$$

where $\operatorname{Re} a > 1, 0 \leqq x \leqq y$, and z is in the cut plane. If a is half an odd integer, the last R-function is a complete elliptic integral of the third kind [10].

The second example, with the elliptic integral (5.10) in the integrand, is

$$(5.23)\qquad \int_0^\infty t^{s-1}R_K(x+t,y+t)(z+t)^{-s-1/2}\,dt = B(s,1)z^{-1/2}R_{-1/2}(\tfrac{1}{2},\tfrac{1}{2},s;x,y,z),$$

where $\operatorname{Re} s > 0$ and x, y, z are points in the cut plane. If s is half an odd integer, the R-function on the right side is an incomplete elliptic integral of the first (if $s=\frac{1}{2}$) or second kind. If s is an integer, it is a complete elliptic integral of the third kind.

6. Four lemmas. The first two lemmas were used in showing that certain transforms are analytic or continuous on specified domains. Lemmas 3 and 4 were used in changing the order of integration in § 2 and § 3, respectively. Although Lemma 2 is useful in other contexts, we have not seen it stated elsewhere in print except in a weaker form [8, p. 220, Problem 15] in which $|\varphi|$ is required to be bounded.

LEMMA 1. *If g is holomorphic on a domain D in the complex plane $\mathbb{C}$, then $G(\mathbf{b},\mathbf{z}) = \int_E g(\mathbf{u}\cdot\mathbf{z})\,d\mu_{\mathbf{b}}(\mathbf{u})$ is a holomorphic function of $\mathbf{b}$ and $\mathbf{z}$ on a domain $Q \subset \mathbb{C}^{2k}$ defined by the conditions* $\operatorname{Re}\mathbf{b} > 0$ *and* $\operatorname{con}(\mathbf{z}) \subset D$. *If g is continuous on an interval I of the real line, then $G(\mathbf{b},\mathbf{t})$ is holomorphic in $\mathbf{b}$ for* $\operatorname{Re}\mathbf{b} > 0$ *and continuous in* $\mathbf{t}$ on I^k.

Proof. Lemma 1 is a modification of [2, Theorem 1].

LEMMA 2. *Let v be a real or complex measure on a measure space X, Ω an open set in $\mathbb{C}$, and φ a function defined on $\Omega \times X$. Assume that $\varphi(s,t)$ is a measurable function of t for each $s\in\Omega$, $\varphi(s,t)$ is analytic on Ω for each $t \in X$, and $|\varphi(s,t)|$ is majorized on Ω by a function $\psi(t)$ which is Lebesgue integrable with respect to v on X. For each $s\in\Omega$ define*

$$(6.1)\qquad \bar{\varphi}(s) = \int_X \varphi(s,t)\,dv(t).$$

Then $\bar{\varphi}$ is analytic on Ω.

Proof. From the definition of $\bar{\varphi}$ we have, for $s, s_0\in\Omega$ and $s \neq s_0$,

$$(6.2)\qquad \frac{\bar{\varphi}(s)-\bar{\varphi}(s_0)}{s-s_0} = \int_X \frac{\varphi(s,t)-\varphi(s_0,t)}{s-s_0}\,dv(t).$$

Let $K \subset \Omega$ be a compact set and let $\gamma \subset \Omega$ be a positively oriented rectifiable Jordan curve of length L such that $\gamma \cup I(\gamma) \subset \Omega$ and $K \subset I(\gamma)$, where $I(\gamma)$ denotes the interior of γ. Let δ be the distance between K and γ. Then, since φ is analytic on $\gamma \cup I(\gamma)$, we apply Cauchy's integral formula and obtain

$$\begin{aligned}\left|\frac{\varphi(s,t)-\varphi(s_0,t)}{s-s_0}\right| &= \frac{1}{2\pi|s-s_0|}\left|\int_\gamma \frac{\varphi(z,t)}{z-s}\,dz - \int_\gamma \frac{\varphi(z,t)}{z-s_0}\,dz\right| \\ &\leqq \frac{1}{2\pi}\int_\gamma \frac{|\varphi(z,t)|}{|z-s|\,|z-s_0|}\,|dz| \\ &\leqq L\psi(t)/2\pi\delta^2,\end{aligned}$$

where $s, s_0 \in \boldsymbol{K}$, $s \neq s_0$, and $t \in X$. Thus

$$\frac{\varphi(s,t)-\varphi(s_0,t)}{s-s_0}$$

is measurable in t and is majorized by an integrable function. We choose s_0 to be an interior point of K and let s tend to s_0 through a sequence of points in K. By the Lebesgue dominated convergence theorem we may take the limit under the integral sign in (6.2). Since $\varphi(s, t)$ is analytic in s, the right side of that equation tends to the finite limit

$$\int_X \frac{\partial\varphi(s_0,t)}{\partial s_0}\,d\nu(t),$$

and hence $\bar{\varphi}$ has a derivative at s_0. For every $s_0 \in \Omega$ we can find a compact set $K \subset \Omega$ containing s_0 in its interior, and thus $\bar{\varphi}$ is analytic on Ω.

LEMMA 3. *Let $z_1, \cdots, z_k$ be fixed complex numbers, let λ be a connected subset of a straight line in $\mathbb{C}$, and let $\mu_{\mathbf{b}}$ be the measure defined by* (1.3) *on the set E of all positive weights $(u_1, \cdots, u_k)$ whose sum is unity. Assume that $\psi(t)$ is measurable on λ, $\varphi(\mathbf{u}\cdot\mathbf{z}, t)$ is measurable on $E \times \lambda$, and $|\varphi(\mathbf{u}\cdot\mathbf{z}, t)| \leqq f(t)$ on $E \times \lambda$, where $|\psi|\, f$ is Lebesgue integrable on λ. Then, for* Re $\mathbf{b} > 0$,

$$\int_E\int_\lambda \psi(t)\varphi(\mathbf{u}\cdot\mathbf{z},t)\,dt\,d\mu_{\mathbf{b}}(\mathbf{u}) = \int_\lambda\int_E \psi(t)\varphi(\mathbf{u}\cdot\mathbf{z},t)\,d\mu_{\mathbf{b}}(\mathbf{u})\,dt. \tag{6.3}$$

Proof. The integrand on the left of (6.3) is a measurable function on $E \times \lambda$. We have also

$$\int_E\int_\lambda |\psi(t)\varphi(\mathbf{u}\cdot\mathbf{z},t)|\,|dt|d|\mu_{\mathbf{b}}(\mathbf{u})| \leqq |\mu_{\mathbf{b}}|(E)\int_\lambda |\psi(t)|f(t)\,|dt|,$$

where $|\mu_{\mathbf{b}}|$ denotes the total variation of $\mu_{\mathbf{b}}$. It is easy to verify that $|\mu_{\mathbf{b}}|(E) = \int_E d|\mu_{\mathbf{b}}(\mathbf{u})| < \infty$. Since the second factor on the right also is finite, Fubini's theorem can be used to change the order of integration on the left side of (6.3).

LEMMA 4. *Let γ be a positively oriented rectifiable Jordan curve in $\mathbb{C}$ and let $s_1, \cdots, s_k$ be fixed points in the inner region of γ. Let λ be a connected subset of a straight line in $\mathbb{C}$ and assume that ψ is measurable on λ and that $\varphi(z, t)$ is continuous on $\gamma \times \lambda$. Define $\mathbf{z} = (z, \cdots, z)$. If $\int_\lambda |\psi(t)\varphi(z, t)|\,dt$ is bounded on γ and if $\sum_{i=1}^k b_i \neq 0, -1, -2, \cdots$, then*

$$\int_\gamma\int_\lambda \psi(t)\varphi(z,t)R_{-1}(\mathbf{b},\mathbf{z}-\mathbf{s})\,dt\,dz = \int_\lambda\int_\gamma \psi(t)\varphi(z,t)R_{-1}(\mathbf{b},\mathbf{z}-\mathbf{s})\,dz\,dt. \tag{6.4}$$

Proof. Since $R_{-1}(\mathbf{b}, \mathbf{z} - \mathbf{s})$ is continuous in z on γ [2, Theorem 4], the integrand on the left of (6.4) is measurable on $\gamma \times \lambda$. Even if the integrand and differentials are replaced by their absolute values, the integral is finite because of the assumption of the lemma and because R_{-1} is bounded on γ. We may therefore apply Fubini's theorem again.

REFERENCES

[1] R. Askey, *Orthogonal expansions of averaged functions*, J. Math. Anal. Appl., 14 (1965), pp. 326–331.

[2] B. C. Carlson, *A connection between elementary functions and higher transcendental functions*, SIAM J. Appl. Math., 17 (1969), pp. 116–148.

[3] R. R. Goldberg, *Convolutions and general transforms on L^p*, Duke Math. J., 27 (1960), pp. 251–260.

[4] J. Hladik, *La transformation de Laplace à plusieurs variables*, Masson et Cie., Paris, 1969.

[5] V. K. Kapoor and S. Masood, *On a generalized L-H transform*, Proc. Cambridge Philos. Soc., 64 (1968), pp. 399–406.

[6] E. W. Ng and M. Geller, *A table of integrals of the error functions*, J. Res. Nat. Bur. Standards Sec. B, 73B (1969), pp. 1–20.

[7] D. M. Rognlie, *Generalized integral transforms*, Doctoral thesis, Iowa State University, Ames, 1969.

[8] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.

[9] H. M. Srivastava, *On a generalized integral transform*, Math. Z., 108 (1969), pp. 197–201.

[10] D. G. Zill and B. C. Carlson, *Symmetric elliptic integrals of the third kind*, Math. Comp., 24 (1970), pp. 199–214.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

SPHERICAL CONVERGENCE AND CERTAIN SINGULAR INTEGRALS*

RICHARD ESCOBEDO†

Abstract. This paper establishes a partial generalization to Euclidean N-space of a one-variable result which relates convergence of the conjugate Fourier series of a function to the existence of the conjugate function. Conjugates are taken with respect to a certain class of Calderón–Zygmund kernels.

1. Introduction. Let f be a function in $L^1(-\pi, \pi]$ and periodic of period 2π. For m an integer we set

$$\hat{f}(m) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(y)\, e^{-imy}\, dy.$$

If $K(x) = 2x^{-1}$, then $\hat{K}(x)$ denotes its principal-valued Fourier transform:

$$\hat{K}(x) = \lim_{r \to 0} \lim_{R \to \infty} (2\pi)^{-1} \int_{(-R,R)-(-r,r)} K(y)\, e^{-ixy}\, dy.$$

In particular, $\hat{K}(0) = 0$ and $\hat{K}(m) = -i \operatorname{sgn} m$.

Let x^0 be a fixed point. Suppose that $\hat{f}(m) = O(1/m)$. Hardy and Littlewood [2] showed that a necessary and sufficient condition that

$$\lim_{R \to \infty} \sum_{|m| < R} \hat{f}(m)\hat{K}(m)\, e^{-imx^0} = \alpha \text{ (finite)}$$

is that

$$\lim_{r \to 0} \lim_{R \to \infty} (2\pi)^{-1} \int_{r \leqq |y| \leqq R} f(x^0 - y)K(y)\, dy = \alpha.$$

We intend to establish here a (partial) generalization of this theorem to Euclidean N-space, E_N, $N \geqq 2$, when $K(x)$ belongs to a certain class of Calderón–Zygmund kernels. We shall use some techniques from a recent paper by Professor Victor L. Shapiro [5].

2. Notation. We shall use vector notation:

$$x = (x_1, \cdots, x_N), \qquad y = (y_1, \cdots, y_N),$$

$$(x, y) = x_1 y_1 + \cdots + x_N y_N \quad \text{and} \quad |x| = (x, x)^{1/2}.$$

We let $T_N = \{x \mid -\pi < x_j \leqq \pi, j = 1, \cdots, N\}$. Ω will stand for the N-dimensional unit sphere centered at 0 and $d\Omega$ for the natural $(N-1)$-dimensional volume element of Ω.

If f is a function in $L^1(T_N)$ and periodic of period 2π in each variable, and m is an integral lattice point in E_N, then

$$\hat{f}(m) = (2\pi)^{-N} \int_{T_N} f(y)\, e^{-i(m,y)}\, dy.$$

---

* Received by the editors June 9, 1972.

† Department of Mathematics, University of California at San Diego, La Jolla, California 92037. This work was supported in part by the Economic Opportunity Program and by the United States Air Force Office of Scientific Research of the Air Research and Development Command under Contract AFOSR 69-1689 A.

Let $P_n(x)$ be an homogeneous polynomial of degree n, $n \geqq 1$, which is harmonic in E_N, and let $x' = x/|x|$. Then $P_n(x')$ is called a surface spherical harmonic of order n and $P_n(x')/|x|^N = P_n(x)/|x|^{n+N}$ is called a spherical harmonic kernel of Calderón–Zygmund type. Let $K(x) = P_n(x)/|x|^{n+N}$. Then $\hat{K}(x)$ designates the principal-valued Fourier transform of K:

$$\hat{K}(x) = \lim_{r \to 0} \lim_{R \to \infty} (2\pi)^{-N} \int_{B(0,R)-B(0,r)} K(y)\, e^{-i(x,y)}\, dy,$$

where $B(x, R)$ denotes the N-dimensional closed ball with center x and radius R. From [4, p. 69] we see that $\hat{K}(0) = 0$ and for $m \neq 0$,

$$\hat{K}(m) = \left\{ (-i)^n \frac{\Gamma(n/2)}{2^N \Gamma((n+N)/2)\pi^{N/2}} \right\} \cdot P_n\left(\frac{m}{|m|}\right).$$

If Y_n is a surface spherical harmonic of order n, then $|||Y_n|||$ will denote the sup norm of Y_n on Ω.

For future reference, we list here two useful formulas (viz., (2.1) and (2.2)), the first of which can be found in [7, p. 368] and the second in [4, p. 5].

$$e^{iz\cos\varphi} = 2^{\nu}\Gamma(\nu) \sum_{j=0}^{\infty} (\nu + j) i^j \frac{J_{\nu+j}(z)}{z^{\nu}} P_j^{\nu}(\cos\varphi), \tag{2.1}$$

where $\nu = (k-2)/2$, J_β is the Bessel function of the first kind of order β and the P_j^ν are the Gegenbauer (ultraspherical) polynomials defined by the equation

$$(1 - 2r\cos\theta + r^2)^{-\nu} = \sum_{j=1}^{\infty} r^j P_j^{\nu}(\cos\theta).$$

For the second formula, assume that W is a function in L^1 on Ω with respect to $d\Omega$. Then W can be expanded in a series of surface spherical harmonics [1, Chap. 11], $W \sim \sum_{j=0}^{\infty} Y_j(x)$, where

$$Y_j(x) = \frac{\Gamma(\nu)(j+\nu)}{2\pi^{\nu+1}} \int_{\Omega} P_j^{\nu}[(x, y)] W(y)\, d\Omega(y). \tag{2.2}$$

(Y_j is a surface spherical harmonic of order j.)

3. Statement of Theorem. For our theorem we shall take a more general Calderón–Zygmund kernel.

THEOREM. *Let $K(x) = W(x)/|x|^N$, where $W(x) = \sum_{n=1}^{\infty} Y_n(x/|x|)$, Y_n is a surface spherical harmonic of order n and*

$$\sum_{n=1}^{\infty} n|||Y_n||| < \infty. \tag{3.1}$$

Let $f \in L^1(T_N)$ be periodic of period 2π in each variable. Assume

$$\sum_{R \leqq |m| \leqq R+1} |\hat{f}(m)| = o(1/R) \tag{3.2}$$

and for the fixed point $x^0 \in E_N$,

$$\lim_{R \to \infty} \sum_{1 \leqq |m| \leqq R} \hat{f}(m)\hat{K}(m)\, e^{i(m,x^0)} = \alpha \text{ (finite)}.$$

Then

$$\lim_{r\to 0}\lim_{R\to\infty}(2\pi)^{-N}\int_{B(0,R)-B(0,r)} f(x^0-y)K(y)\,dy=\alpha. \tag{3.3}$$

Proof of Theorem. We first establish a lemma which will be useful in the proof of our theorem.

LEMMA. *Assume the hypotheses of the above theorem and also assume that* $\hat{f}(0)=0$. *Then*

$$\lim_{R\to\infty}\int_{B(0,R)-B(0,r)} f(x-y)K(y)\,dy=(2\pi)^{\nu+1}\lim_{R'\to\infty}\sum_{1\le|m|\le R'} e^{i(m,x)}\hat{f}(m)\cdot\sum_{n=1}^{\infty}(-i)^n Y_n\left(\frac{m}{|m|}\right)x_{n,\nu}(|m|r) \tag{3.4}$$

for every x in E^N, *where* $\nu=(k-2)/2$ *and* $x_{n,\nu}(s)=\int_s^\infty (J_{n+\nu}(u)/u^{1+\nu})\,du$.

Proof of Lemma. Define $G(x)=\int_{B(0,R)-B(0,r)} f(x-y)K(y)\,dy$. Then

$$\begin{aligned}\hat{G}(m)&=\frac{1}{(2\pi)^N}\int_{T_N} e^{-i(m,x)}\,dx\int_{B(0,R)-B(0,r)} f(x-y)K(y)\,dy\\ &=\frac{1}{(2\pi)^N}\int_{B(0,R)-B(0,r)} K(y)\,dy\int_{T_N} e^{-i(m,x)}f(x-y)\,dx\\ &=\frac{1}{(2\pi)^N}\int_{B(0,R)-B(0,r)} e^{-i(m,y)}K(y)\,dy\int_{T_N} e^{-i(m,x)}f(x)\,dx\\ &=\hat{f}(m)\int_{B(0,R)-B(0,r)} e^{-i(m,y)}K(y)\,dy.\end{aligned}$$

Since $G(x)$ is continuous we have by [3, p. 55] that the Fourier series of $G(x)$ is Abel summable to $G(x)$ everywhere, that is, for every $x\in E_N$,

$$G(x)=\lim_{t\to 0}\sum_{m\neq 0} e^{i(m,x)-|m|t}\hat{f}(m)\int_{B(0,R)-B(0,r)} e^{-i(m,y)}K(y)\,dy. \tag{3.5}$$

We now calculate $\int_{B(0,R)-B(0,r)} e^{-i(m,y)}K(y)\,dy$. From hypothesis (3.1) and the Lebesgue dominated convergence theorem we have

$$\begin{aligned}\int_{B(0,R)-B(0,r)} e^{-i(m,y)}K(y)\,dy&=\int_r^R\frac{1}{\rho}\,d\rho\int_\Omega e^{-i|m|\rho\cos\varphi}\left(\sum_{n=1}^{\infty}Y_n(y)\right)d\Omega(y)\\ &=\int_r^R\frac{1}{\rho}\,d\rho\sum_{n=1}^{\infty}\int_\Omega e^{-i|m|\rho\cos\varphi}Y_n(y)\,d\Omega(y)\\ &=\sum_{n=1}^{\infty}\int_r^R\frac{1}{\rho}\,d\rho\int_\Omega e^{-i|m|\rho\cos\varphi}Y_n(y)\,d\Omega(y),\end{aligned} \tag{3.6}$$

where φ is the angle between y and m. By using (2.1) and the orthogonality of the surface spherical harmonics we see that (3.6) is equal to

$$2^{\nu}\Gamma(\nu)\sum_{n=1}^{\infty}(\nu+n)i^{n}\int_{r}^{R}\frac{J_{n+\nu}(-|m|\rho)}{(-|m|\rho)^{\nu}\rho}\,d\rho\int_{\Omega}P_{n}^{\nu}(\cos\varphi)Y_{n}(y)\,d\Omega(y),$$

which in turn, by (2.2), is equal to

$$\begin{aligned}&2^{\nu}\Gamma(\nu)\sum_{n=1}^{\infty}(\nu+n)i^{n}\frac{2\pi^{\nu+1}}{(n+\nu)\Gamma(\nu)}Y\left(\frac{m}{|m|}\right)\int_{r}^{R}\frac{J_{n+\nu}(-|m|\rho)}{\rho(-|m|\rho)^{\nu}}\,d\rho\\&=2^{\nu}\cdot 2\pi^{\nu+1}\sum_{n=1}^{\infty}i^{n}(-1)^{n}Y_{n}\left(\frac{m}{|m|}\right)\int_{r}^{R}\frac{J_{n+\nu}(|m|\rho)}{\rho(|m|\rho)^{\nu}}\,d\rho\\&=(2\pi)^{\nu+1}\sum_{n=1}^{\infty}(-i)^{n}Y_{n}\left(\frac{m}{|m|}\right)\int_{|m|r}^{|m|R}\frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{B(0,R)-B(0,r)}e^{-i(m,y)}K(y)\,dy\\&\quad=(2\pi)^{\nu+1}\sum_{n=1}^{\infty}(-i)^{n}Y_{n}\left(\frac{m}{|m|}\right)\int_{|m|r}^{|m|R}\frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du,\end{aligned}$$

and so from (3.5) we obtain

$$\begin{aligned}G(x)=(2\pi)^{\nu+1}\lim_{t\to 0}\sum_{m\neq 0}e^{i(m,x)-|m|t}\hat{f}(m)\sum_{n=1}^{\infty}(-i)^{n}Y_{n}\left(\frac{m}{|m|}\right)\\\cdot\int_{|m|r}^{|m|R}\frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du.\end{aligned}\tag{3.7}$$

Now if we can show that

$$\lim_{R'\to\infty}\sum_{1\leq|m|\leq R'}e^{i(m,x)}\hat{f}(m)\sum_{n=1}^{\infty}(-i)^{n}Y_{n}\left(\frac{m}{|m|}\right)\int_{|m|r}^{|m|R}\frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du$$

exists (finite), then we shall have

$$\begin{aligned}G(x)=(2\pi)^{\nu+1}\lim_{R'\to\infty}\sum_{1\leq|m|\leq R'}e^{i(m,x)}\hat{f}(m)\sum_{n=1}^{\infty}(-i)^{n}Y_{n}\left(\frac{m}{|m|}\right)\\\cdot\int_{|m|r}^{|m|R}\frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du,\end{aligned}\tag{3.8}$$

since spherical summability implies Abel summability to the same sum.

We shall now show that the series is indeed spherically summable. We first estimate $\int_{|m|r}^{|m|R}(J_{n+\nu}(u)/u^{1+\nu})\,du$. Assuming for the moment that $n\geqq 3$, we integrate by parts and obtain

$$\int_{|m|r}^{|m|R} \frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du = \int_{|m|r}^{|m|R} \frac{J_{n+\nu}(u)}{u^{n+\nu-1}}\,u^{n-2}\,du$$

$$(3.9)\qquad = -\frac{J_{n+\nu-1}(u)}{u^{n+\nu-1}}u^{n-2}\Bigg|_{|m|r}^{|m|R} + (n-2)\int_{|m|r}^{|m|R} \frac{J_{n+\nu-1}(u)}{u^{n+\nu-1}}u^{n-3}\,du$$

$$= -\frac{J_{n+\nu-1}(u)}{u^{\nu+1}}\Bigg|_{|m|r}^{|m|R} + (n-2)\int_{|m|r}^{|m|R} \frac{J_{n+\nu-1}(u)}{u^{\nu+2}}\,du,$$

where we have used the fact that $(d/dx)[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$. For the second term in (3.9) we get

$$(3.10)\qquad \left|(n-2)\int_{|m|r}^{|m|R} \frac{J_{n+\nu-1}(u)}{u^{\nu+2}}\,du\right| \leqq (n-2)\int_{|m|r}^{|m|R} \frac{|J_{n+\nu-1}(u)|}{u^{\nu+2}}\,du.$$

Using the fact that $|J_{\nu+n}(u)| \leqq u^{\nu}$, n a positive integer, we see that

$$(n-2)\int_{|m|r}^{|m|R} \frac{|J_{n+\nu-1}(u)|}{u^{\nu+2}}\,du$$

$$\leqq (n-2)\int_{|m|r}^{|m|R} \frac{u^{\nu}}{u^{\nu+2}}\,du = (n-2)\int_{|m|r}^{|m|R} \frac{1}{u^2}\,du$$

$$= (n-2)\left(\frac{1}{|m|r} - \frac{1}{|m|R}\right) \leqq (n-1)\left(\frac{1}{|m|r} + \frac{1}{|m|R}\right)$$

$$\leqq \frac{2(n-1)}{|m|r}.$$

Similarly, the first term in (3.9) is, in absolute value, bounded by $2/|m|r$. Therefore $|\int_{|m|r}^{|m|R} (J_{n+\nu}(u)/u^{1+\nu})\,du| \leqq 2n/|m|r$, $n \geqq 3$. Hence there is a constant A such that

$$(3.11)\qquad \left|\int_{|m|r}^{|m|R} \frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du\right| \leqq \frac{A\cdot n}{|m|r}, \qquad n \geqq 1.$$

Returning to the series in (3.8) we get

$$\lim_{R'\to\infty} \sum_{1\leqq|m|\leqq R'} |\hat{f}(m)| \left|\sum_{n=1}^{\infty} (-i)^n Y_n\!\left(\frac{m}{|m|}\right)\int_{|m|r}^{|m|R} \frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du\right|$$

$$\leqq \lim_{R'\to\infty} \sum_{1\leqq|m|\leqq R'} |\hat{f}(m)| \sum_{n=1}^{\infty} \left|Y_n\!\left(\frac{m}{|m|}\right)\right| \left|\int_{|m|r}^{|m|R} \frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du\right|$$

$$\leqq \lim_{R'\to\infty} A \sum_{1\leqq|m|\leqq R'} |\hat{f}(m)| \sum_{n=1}^{\infty} \frac{n\|| Y_n \||}{|m|r},$$

by (3.11). This last expression is equal to

$$(3.12)\qquad A\sum_{j=1}^{\infty} \sum_{j\leqq|m|<j+1} |\hat{f}(m)| \sum_{n=1}^{\infty} \frac{n\||Y_n\||}{|m|r} \leqq \frac{A\cdot A'}{r}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{j\leqq|m|<j+1} |\hat{f}(m)|,$$

where $A' = \sum_{n=1}^{\infty} n|||Y_n|||$ ($A' < \infty$, by hypothesis). By hypothesis (3.2),

$$\sum_{j \leqq |m| \leqq j+1} |\hat{f}(m)| = o(1/j),$$

so that the expression in (3.12) is less than or equal to $(\text{const.}/r)\sum_{j=1}^{\infty} 1/j^2 < \infty$, thus establishing (3.8).

Write (3.8) as

$$(3.13)\qquad \begin{aligned} G(x) = (2\pi)^{\nu+1} \lim_{R' \to \infty} \sum_{1 \leqq |m| \leqq R'} e^{i(m,x)}\hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right) \\ \cdot \{\chi_{n,\nu}(|m|r) - \chi_{n,\nu}(|m|R)\}. \end{aligned}$$

The same method that was used to obtain inequality (3.11) can be used to obtain

$$(3.14)\qquad \left|\int_{|m|R}^{\infty} \frac{J_{n+\nu}(u)}{u^{1+\nu}}\,du\right| \leqq \frac{n \cdot B}{|m|R}, \qquad B \text{ a constant}, \quad n \geqq 1.$$

We then can rewrite (3.13) as

$$(3.15)\qquad \begin{aligned} G(x) &= (2\pi)^{\nu+1} \lim_{R' \to \infty} \sum_{1 \leqq |m| \leqq R'} e^{i(m,x)}\hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right)\chi_{n,\nu}(|m|r) \\ &\quad - (2\pi)^{\nu+1} \lim_{R' \to \infty} \sum_{1 \leqq |m| \leqq R'} e^{i(m,x)}\hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right)\chi_{n,\nu}(|m|R) \\ &= S_r - S_R, \end{aligned}$$

each of the two series being convergent by virtue of (3.14). For S_R we have

$$(3.16)\qquad |S_R| \leqq (2\pi)^{\nu+1} \frac{\text{const.}}{R} \sum_{j=1}^{\infty} \frac{1}{j^2},$$

using the facts that $\sum_{n=1}^{\infty} n|||Y_n||| < \infty$ and $\sum_{R \leqq |m| \leqq R+1} |\hat{f}(m)| = o(1/R)$, along with inequality (3.14). Since the constant in (3.16) is independent of R, we see that $S_R \to 0$ as $R \to \infty$. Hence from (3.15) we get

$$\begin{aligned} \lim_{R \to \infty} G(x) &= \lim_{R \to \infty} \int_{B(0,R) - B(0,r)} f(x - y)K(y)\,dy \\ &= (2\pi)^{\nu+1} \lim_{R' \to \infty} \sum_{1 \leqq |m| \leqq R'} e^{i(m,x)}\hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right)\chi_{n,\nu}(|m|r), \end{aligned}$$

which is the conclusion (3.4).

4. Proof of Theorem. Without loss of generality, we can assume that $x^0 = 0$, $\hat{f}(0) = 0$ and $\alpha = 0$.

We first write (3.4) of the lemma (at $x = 0$) as

$$\begin{aligned} &\lim_{R \to \infty} \frac{1}{(2\pi)^N} \int_{B(0,R) - B(0,r)} f(-y)K(y)\,dy \\ &\qquad = (2\pi)^{-N/2}\left\{\sum_{j=1}^{[1/r]} \sum_{j \leqq |m| < j+1} \hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right)\chi_{n,\nu}(|m|r)\right. \end{aligned}$$

$$+ \sum_{[1/r]+1}^{\infty} \sum_{j \leqq |m| < j+1} \hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right) \chi_{n,\nu}(|m|r)\Bigg\}$$

$$= S_1(r) + S_2(r),$$

where $[1/r]$ is the greatest integer in $1/r$. The task is to show that

$$\lim_{r \to 0} S_1(r) = 0, \tag{4.1}$$

$$\lim_{r \to 0} S_2(r) = 0. \tag{4.2}$$

We first establish (4.2).

$$\begin{aligned} |S_2(r)| &\leqq \sum_{[1/r]+1}^{\infty} \sum_{j \leqq |m| < j+1} |\hat{f}(m)| \sum_{n=1}^{\infty} |||Y_n||| \, |\chi_{n,\nu}(|m|r)| \\ &\leqq \text{const.} \sum_{[1/r]+1}^{\infty} \sum_{j \leqq |m| < j+1} |\hat{f}(m)| \sum_{n=1}^{\infty} \frac{n|||Y_n|||}{|m|r}, \end{aligned} \tag{4.3}$$

by (3.14). By the hypotheses, the expression in (4.3) is bounded by

$$\frac{\text{const.}}{r} \sum_{[1/r]+1}^{\infty} o(1/j^2) = \frac{o(1)}{r} \sum_{[1/r]+1}^{\infty} 1/j^2 = \frac{o(1)}{r} O(r) = o(1),$$

and (4.2) is disposed of.

From [4, p. 69] we have for $m \neq 0$,

$$\hat{K}(m) = \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right) \frac{\Gamma(n/2)}{2^N \Gamma((n+N)/2) \pi^{N/2}},$$

and from [7, p. 391] we have

$$\frac{1}{(2\pi)^{N/2}} \int_0^{\infty} \frac{J_{n+\nu}(u)}{u^{1+\nu}} du = \frac{\Gamma(n/2)}{2^N \Gamma((n+N)/2) \pi^{N/2}}.$$

Using these two equations we conclude that

$$\begin{aligned} 0 &= \lim_{R \to \infty} \sum_{1 \leqq |m| \leqq R} \hat{f}(m) \hat{K}(m) \\ &= \lim_{R \to \infty} \frac{1}{(2\pi)^{N/2}} \sum_{1 \leqq |m| \leqq R} \hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right) \int_0^{\infty} \frac{J_{n+\nu}(u)}{u^{1+\nu}} du. \end{aligned}$$

Let

$$S(r) = \frac{1}{(2\pi)^{N/2}} \sum_{j=1}^{[1/r]} \sum_{j \leqq |m| < j+1} \hat{f}(m) \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right) \int_0^{\infty} \frac{J_{n+\nu}(u)}{u^{1+\nu}} du.$$

Then $\lim_{r \to 0} S(r) = 0$. Thus if it can be shown that

$$\lim_{r \to 0} (S(r) - S_1(r)) = 0, \tag{4.4}$$

then (4.1) will be established and the theorem will be proved. We now establish (4.4).

From the definitions of $S(r)$ and $S_1(r)$ we can write

$$S(r) - S_1(r) = \frac{1}{(2\pi)^{N/2}} \sum_{j=1}^{[1/r]} \sum_{j \leqq |m| < j+1} \hat{f}(m) \cdot \sum_{n=1}^{\infty} (-i)^n Y_n\left(\frac{m}{|m|}\right) \int_0^{|m|r} \frac{J_{n+\nu}(u)}{u^{1+\nu}} du. \tag{4.5}$$

From [6, p. 61] we see that $|J_{n+\nu}(u)| \leqq \text{const.} \cdot u^{1+\nu}$, where the constant depends only on ν. With this inequality we obtain

$$\left| \int_0^{|m|r} \frac{J_{n+\nu}(u)}{u^{1+\nu}} \right| \leqq \text{const.}\, |m|r. \tag{4.6}$$

Using (4.6) in (4.5) we have

$$|S(r) - S_1(r)| \leqq \frac{1}{(2\pi)^{N/2}} \sum_{j=1}^{[1/r]} \sum_{j \leqq |m| < j+1} |\hat{f}(m)| \sum_{n=1}^{\infty} |||Y_n||| \,\text{const.}\, |m|r$$
$$\leqq \frac{\text{const.}\, r}{(2\pi)^{N/2}} \sum_{j=1}^{[1/r]} o(1/j)j.$$

But

$$r \sum_{j=1}^{[1/r]} o(1/j)j = r \sum_{j=1}^{[1/r]} o(1) = o(1),$$

which is the required result.

REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. 2, McGraw-Hill, New York, 1953.

[2] G. H. HARDY AND J. E. LITTLEWOOD, *Two theorems concerning Fourier series*, J. London Math. Soc., 1 (1926), pp. 19–25.

[3] V. L. SHAPIRO, *Fourier series in several variables*, Bull. Amer. Math. Soc., 70 (1964), pp. 48–93.

[4] ———, *Topics in Fourier and geometric analysis*, Mem. Amer. Math. Soc., no. 39, 1961, 100 pp.

[5] ———, *Singular integrals and spherical convergence*, to appear.

[6] E. C. TITCHMARSH, *Theory of Functions*, 2nd ed., Oxford University Press, Oxford, 1939.

[7] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1966.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

LYAPUNOV THEORY AND PERTURBATIONS OF DIFFERENTIAL EQUATIONS*

JOHN V. ERHART†

Abstract. In this paper we discuss qualitative properties of the solution of systems of ordinary differential equations and perturbations of such systems in the event a Lyapunov function is known whose derivative along solutions of the system satisfies a strong negative definite condition. Boundedness and stability of sets are discussed along with the observation that a Lyapunov function with a strongly negative definite derivative must be positive definite and radially unbounded. These results are used to discuss certain types of perturbations of systems of differential equations. Several examples are given to illustrate the main results.

1. Introduction. The qualitative behavior of the solutions of perturbed nonlinear systems of differential equations is often studied by obtaining a Lyapunov function for the unperturbed system and using it as a Lyapunov function for the perturbed system. In this paper, we wish to investigate the properties of solutions of a system of differential equations when a Lyapunov function is known whose derivative along solutions of the system satisfies a strong negative definitive condition, and to discuss several types of perturbations of such systems. The theorems of § 1 discuss stability and boundedness of solutions of systems when such a Lyapunov function is known. In § 2 we show that, under certain conditions, this type of Lyapunov function must necessarily be radially unbounded and positive definite. The results of § 1 and § 2 are used in § 3 and § 4 to obtain similar results for perturbed nonlinear systems. In particular, Theorem 2.2 enables us to apply standard stability theorems to perturbed systems.

Consider the system of differential equations

$$x' = f(t, x) \tag{N}$$

and its perturbed system

$$y' = f(t, y) + g(t, y), \tag{P}$$

where x and y are elements of R^n, the set of n-dimensional column vectors, and $f, g: [0, \infty) \times R^n \to R^n$ are continuous. We denote by $p(t, t_0, x_0)$ a solution of (N) such that $p(t_0, t_0, x_0) = x_0$ and by $y(t, t_0, y_0)$ a solution of (P) such that $y(t_0, t_0, y_0) = y_0$. Let D be an open and connected subset of R^n containing the origin and let $I = [0, \infty)$.

Throughout this paper, $\|x\|$ will denote the Euclidean length of a vector $x \in R^n$ since a differentiable norm is desirable. For $\varepsilon > 0$ and $H \subset R^n$, let

$$S(\varepsilon) = \{x \in R^n : \|x\| < \varepsilon\}$$

and

$$S(H, \varepsilon) = \{x \in R^n : d(H, x) < \varepsilon\},$$

* Received by the editors November 23, 1971, and in revised form May 15, 1972.

† Division of Mathematics, Northeast Missouri State College, Kirksville, Missouri 63501. This paper is part of the author's Ph.D. dissertation at Southern Illinois University.

where $d(H,x) = \inf\{\|x - y\| : y \in H\}$. Also let

$$\bar{H} = \{x \in R^n : d(H,x) = 0\},$$
$$H^c = \{x \in R^n : x \notin H\},$$

and

$$\Delta H = \{x \in R^n : x \in \bar{H} \text{ and } x \in \overline{H^c}\}.$$

The family of functions $F: I \times D \to R^n$, $D \subset R^m$ that are locally Lipschitzian with respect to $x \in D$ will be denoted by $C_0(x)$. If, for every compact set $K \subset R^n$, there exists a constant $L(K)$ such that $\|F(t,x_1) - F(t,x_2)\| \leqq L(K)\|x_1 - x_2\|$ for x_1 and x_2 in $K \subset D$ and $t \in I$, we shall say $F(t,x)$ belongs to the class $\overline{C_0(x)}$. C' will denote the family of functions $F(t,x)$ defined on $I \times D$, $D \subset R^n$, whose components have continuous first partial derivatives with respect to $t, x_1, x_2, \cdots, x_n$.

DEFINITION. A continuous function $V: I \times D \to R^1$, $D \subset R^n$, is called a *Lyapunov function* if $V \in C_0(x)$ and V is bounded from below.

Let V be a Lyapunov function. For $(t,x) \in I \times D$ we define

$$V'_{(N)}(t,x) = \limsup_{h \to 0^+} (1/h)\{V(t+h, x + hf(t,x)) - V(t,x)\}.$$

If $V'_{(N)}(t,x) \leqq 0$ for $(t,x) \in [t_0, t_1] \times D$ and $p(t)$ is a solution of (N) that remains in D for $t \in [t_0, t_1]$, then (using Lebesgue integration)

$$V(t, p(t)) - V(t_0, p(t_0)) \leqq \int_{t_0}^{t} V'_{(N)}(t, p(s))\, ds \quad \text{for } t \in [t_0, t_1].$$

DEFINITION. $V'_{(N)}(t,x)$ is *strongly negative definite* relative to a compact set $H \subset R^n$ if for every $\varepsilon > 0$, there exists a constant $d = d(\varepsilon) > 0$ such that $(t,x) \in I \times S^c(H,\varepsilon)$ implies that $V'_{(N)}(t,x) \leqq -d\|f(t,x)\|/(1 + \|x\|)$.

The above definition can be extended to any closed set $H \subset R^n$, but this will not be necessary for the results of this paper. We will often make use of the following:

Hypothesis A. The function $f(t,x)$ of equation (N) is said to satisfy Hypothesis A for $x_0 \in R^n$ if there exists $\eta > 0$ and a continuous function $v: I \to I$ such that $\int_0^\infty v(t)\,dt = \infty$ and $\|f(t,x)\| \geqq v(t)$ for all $(t,x) \in I \times S(x_0, \eta)$.

The types of stability and boundedness discussed in this paper are as defined in Yoshizawa [4].

1. Stability and boundedness.

THEOREM 1.1. *Let H be a compact subset of R^n. Assume there exists a Lyapunov function $V: I \times \overline{H^c} \to R^1$ such that:*

(i) *$V(t,x)$ is bounded for $(t,x) \in I \times \Delta H$;*

(ii) *there exists a continuous function $q: I \to I$ such that $\int_0^\infty q(t)\,dt < \infty$, and for every $\varepsilon > 0$ there exists $d = d(\varepsilon) > 0$ such that for (t,x) in $I \times S^c(H,\varepsilon)$, we have*

$$V'_{(N)}(t,x) \leqq -d\|f(t,x)\|/(1 + \|x\|) + q(t).$$

Then solutions of (N) *are equibounded.*

Proof. H compact in R^n implies that there exists $\zeta > 0$ such that $S(H, 1) \subset S(\zeta)$. Let $t_0 \in I$ and $\alpha \geqq \zeta$ be given and define the following constants:

d: By (ii) there exists $d > 0$ such that

$$V'_{(N)}(t, x) \leqq -d\|f(t, x)\|/(1 + \|x\|) + q(t)$$

for $(t, x) \in I \times S^c(H, 1)$.

L_1: Set $L_1 = \max\{V(t_0, x): x \in \overline{S(\alpha) \cap H^c}\}$.

L_2: Hypothesis (i) implies there exists $L_2 \in R^1$ such that $V(t, x) \leqq L_2$ for all $(t, x) \in I \times \Delta H$.

L: Let $L = \max\{L_1, L_2\} + \int_0^\infty q(t)\, dt$.

M: Let M be such that $V(t, x) \geqq M$ for all $(t, x) \in I \times \overline{H^c}$.

β: Set $\beta = \beta(t_0, \alpha) = \exp(1/d)[|M| + L + \ln(1 + \alpha)]$.

Notice that $\beta > 1 + \alpha$, so that $S(H, 1) \subset S(\alpha) \subset S(\beta)$.

We now show that for $x \in S(\alpha)$, $\|p(t, t_0, x)\| < \beta$ for all $t \geqq t_0$. Assume there exists a solution $p(t) = p(t, t_0, x_0)$ of (N) with right interval of definition $[t_0, T)$, $T \leqq \infty$, where $\|x\| < \alpha$, such that for some $t^* > t_0$, $\|p(t^*, t_0, x_0)\| \geqq \beta$.

Since $S(H, 1) \subset S(\alpha)$, either:

(a) $p(t) \in H^c$ for $t \in [t_0, t^*]$; or

(b) there exists $\bar{t} \geqq t_0$ such that $p(\bar{t}) \in \Delta H$ and $p(t) \in \overline{H^c}$ for $t \in [\bar{t}, t^*]$.

Assume (a) holds. There exists $t_1 > t_0$ such that $p(t_1) \in \Delta S(\alpha)$ and $p(t) \in S^c(\alpha)$ for $t \in [t_1, t^*]$. Consider $U(t, x) = V(t, x) + \int_t^\infty q(s)\, ds$. For all $(t, x) \in I \times S^c(H, 1)$, we have $U'_{(N)}(t, x) \leqq -d\|f(t, x)\|/(1 + \|x\|)$. Therefore,

$$U(t^*, p(t^*)) - U(t_0, p(t_0)) \leqq \int_t^{t^*} -d|\,\|f(s, p(s))\|/(1 + \|p(s)\|)|\, ds.$$

Using the fact that $|\,\|p(t)\|'| \leqq \|p'(s)\|$, we obtain

$$\begin{aligned} U(t^*, p(t^*)) - U(t_0, p(t_0)) &\leqq -d\left|\int_{t_1}^{t^*} \|p(s)\|'/(1 + \|p(s)\|)\, ds\right| \\ &\leqq -d|\ln(1 + \|p(t^*)\|) - \ln(1 + \|p(t_1)\|)| \\ &< -d[(1/d)(|M| + L + \ln(1 + \alpha))] + d\ln(1 + \alpha) \\ &= -|M| - L. \end{aligned}$$

Therefore,

$$\begin{aligned} V(t^*, p(t^*)) &= U(t^*, p(t^*)) - \int_{t^*}^\infty q(s)\, ds \\ &< V(t_0, p(t_0)) + \int_{t_0}^{t^*} q(s)\, ds - |M| - L \leqq -|M|, \end{aligned}$$

which contradicts the fact that M is a lower bound for V in $I \times \overline{H^c}$.

Assuming (b) holds, we obtain as above, $U(t^*, p(t^*)) - U(\bar{t}, p(\bar{t})) \leqq -|M| - L$ which again contradicts the fact that M is a lower bound for V in $I \times \overline{H^c}$. Therefore, for $x_0 \in S(\alpha)$, $\|p(t, t_0, x_0)\| < \beta(t_0, \alpha)$ for all $t \geqq t_0$. For $\alpha < \zeta$ let $\beta(t_0, \alpha) = \beta(t_0, \zeta)$, proving that solutions of (N) are equibounded.

THEOREM 1.2. *Let H be a compact subset of R^n. Assume there exists a Lyapunov function $V: I \times \overline{H^c} \to I$ such that $V(t, x) = 0$ for all $(t, x) \in I \times \Delta H$ and hypothesis* (ii) *of Theorem* 1.1 *holds. Then, if $f(t, x)$ satisfies Hypothesis* A *for all $x \in H^c$, all solutions of* (N) *approach the set H as t tends to infinity.*

Proof. By Theorem 1.1, all solutions of (N) are bounded. Therefore, given any $(t_0, x_0) \in I \times R^n$, the solution $p(t, t_0, x_0)$ of (N) is defined for all $t \geqq t_0$. Let $p(t) = p(t, t_0, x_0)$ be any solution of (N). We first show that there does not exist a sequence $\{t_n\}$ tending monotonically to $+\infty$ (the notation we shall use hereafter is $\{t_n\} \uparrow +\infty$) such that the sequence $\{p(t_n)\}$ has a cluster point in H^c.

Assume the contrary. Then there exists an $\bar{x} \in H^c$ and a sequence $\{t_n\} \uparrow +\infty$ such that $p(t_n) \to \bar{x}$ as $n \to \infty$. Let $\varepsilon = d(\bar{x}, H)/3$ and let the constant d of hypothesis (ii) be found. Let the $\zeta > 0$ and the $v(t)$ of Hypothesis A be found for $\bar{x}$ and $f(t, x)$, and let $\alpha = \min\{\varepsilon, \zeta\}$. Since $p(t)$ is bounded, there exists $M > 0$ such that $1 + \|p(t)\| < M$ for all $t \geqq t_0$. Let $\bar{d} = d/M$.

If $p(t) \to \bar{x}$ as $t \to \infty$, there exists $\bar{t} \geqq t_0$ such that $p(t) \in S(\bar{x}, \alpha)$ for all $t \geqq \bar{t}$. Therefore, for all $t \geqq \bar{t}$, $\|f(t, p(t))\| > v(t)$ and $V'_{(\mathrm{N})}(t, p(t)) \leqq -\bar{d}\|f(t, p(t))\| + q(t)$. Letting $U(t, x) = V(t, x) + \int_t^\infty q(s)\,ds$, so that $U'_{(\mathrm{N})}(t, p(t)) \leqq -\bar{d}v(t)$, we have $V(t, p(t)) \leqq U(t, p(t)) \leqq U(\bar{t}, p(\bar{t})) - \bar{d}\int_{\bar{t}}^t v(s)\,ds$ which tends to $-\infty$ as t approaches $+\infty$, contradicting the fact that $V(t, x)$ is bounded from below. Therefore, there does not exist a $\bar{t} \geqq t$ such that $p(t) \in S(\bar{x}, \alpha)$ for all $t \geqq \bar{t}$, and so $p(t) \not\to \bar{x}$ as $t \to \infty$.

Since $\bar{x}$ is a cluster point of the sequence $\{p(t_n)\}$ and $p(t) \not\to \bar{x}$ as $t \to \infty$, there exist sequences $\{t'_n\} \uparrow +\infty$ and $\{t''_n\} \uparrow +\infty$ such that $p(t) \in S(\bar{x}, \alpha)$ for $t \in [t'_n, t''_n]$ and $\|p(t''_n) - p(t'_n)\| > \alpha/2$ for all n. We now consider two possibilities: either

(a) there exists t^* such that $p(t) \in H^c$ for all $t > t^*$; or

(b) there exist sequences $\{\bar{t}_n\} \uparrow +\infty$ and $\{\bar{\bar{t}}_n\} \uparrow +\infty$ such that $p(t) \in H$ for $t \in [\bar{t}_n, \bar{\bar{t}}_n]$ and $p(t) \in H^c$ for $t \in (\bar{\bar{t}}_n, \bar{t}_{n+1})$.

Assume (a) holds. Choose k large enough so that $t'_k > t^*$. Letting $U(t, x) = V(t, x) + \int_t^\infty q(s)\,ds$, we have $U'_{(\mathrm{N})}(t, x) \leqq 0$ for all (t, x) in $I \times H^c$ and $U'_{(\mathrm{N})}(t, x) \leqq -\bar{d}\|f(t, x)\|$ for $(t, x) \in I \times S(\bar{x}, \alpha)$. Therefore, for all $t \geqq t'_k$,

$$\begin{aligned} U(t, p(t)) - U(t'_k, p(t'_k)) &\leqq \int_{t'_k}^t U'_{(\mathrm{N})}(s, p(s))\,ds \\ &\leqq \sum_{j=k}^n \int_{t'_j}^{t''_j} -\bar{d}\|f(s, p(s))\|\,ds \\ &\leqq \sum_{j=k}^n -\bar{d}\|p(t''_j) - p(t'_j)\|. \end{aligned}$$

As $t \to \infty$ (and therefore, as $n \to \infty$) this expression approaches $-\infty$, contradicting the fact that V is bounded from below.

Now assume (b) holds. Let $m \geqq 1$ be such that $t'_m > \bar{t}_1$. Pick $r \geqq m$ large enough that $\bar{d}\alpha(r - m + 1)/2 > \int_0^\infty q(t)\,dt$ and let k be large enough that $\bar{t}_k > t''_r$. Since $p(\bar{t}_n) \in \Delta H$ and $p(\bar{\bar{t}}_n) \in H$ for every n, we have

$$\begin{aligned} 0 &= V(\bar{t}_k, p(\bar{t}_k)) - V(\bar{\bar{t}}_1, p(\bar{\bar{t}}_1)) \\ &\leqq \sum_{j=1}^{k-1} [U(\bar{t}_{j+1}, p(\bar{t}_{j+1})) - U(\bar{\bar{t}}_j, p(\bar{\bar{t}}_j))] + \int_0^\infty q(t)\,dt \qquad \text{(cont.)} \end{aligned}$$

$$\leqq \sum_{j=1}^{k} \int_{\bar{t}_j}^{\bar{t}_{j+1}} U'_{(N)}(s, p(s))\, ds + \int_0^\infty q(t)\, dt$$

$$\leqq \sum_{j=m}^{r} \int_{t'_j}^{t''_j} -\bar{d}\|f(s, p(s))\|\, ds + \int_0^\infty q(t)\, dt$$

$$\leqq \sum_{j=m}^{r} -\bar{d}\|p(t''_j) - p(t'_j)\| + \int_0^\infty q(t)\, dt$$

$$\leqq -\bar{d}(r - m + 1)\alpha/2 + \int_0^\infty q(t)\, dt < 0.$$

This contradiction establishes the fact that no sequence $\{t_n\}\uparrow +\infty$ exists such that $\{p(t_n)\}$ has a cluster point in H^c. Therefore, if $p(t)$ does not approach H as $t \to \infty$, there exists a sequence $\{t_n\}\uparrow +\infty$ and $\beta > 0$ such that $\beta > d(H, p(t_n))$ for $n = 1, 2, \cdots$. But $p(t)$ is bounded, implying that the sequence $\{p(t_n)\}$ has a cluster point in H^c, which was just shown to be impossible. Therefore, $p(t)$ approaches H as $t \to \infty$ and the theorem is proved.

COROLLARY 1.1. *If, in Theorem* 1.2, $H = \{0\}$, *then every solution of* (N) *tends to zero as t approaches* ∞.

Theorem 1.2 is similar to the following theorem of Burton [1].

THEOREM. *If there exists a Lyapunov function* $V: I \times R^n \to R^1$ *such that* $V'_{(N)}(t, x) \leqq 0$ *for all* $(t, x) \in I \times R^n$, $V'_{(N)}(t, x)$ *is strongly negative definite relative to a compact set* $H \subset R^n$, *and* $f(t, x)$ *satisfies Hypothesis* A *for all* $x \in H^c$, *then each solution* $p(t, t_0, x_0)$ *of* (N) *approaches H as t approaches T, where* (t_0, T) *is the maximal right interval of definition of* $p(t, t_0, x_0)$.

Theorem 1.2 does not require that V be defined on all of $I \times R^n$, but only on $I \times \overline{H^c}$; and the condition that $V'_{(N)}(t, x)$ be strongly negative definite relative to H is modified somewhat by hypothesis (ii), making Theorem 1.2 a very useful tool in the perturbation theorems in § 2. Burton's theorem, however, does not require knowledge of V on $I \times \Delta H$ as does Theorem 1.2.

THEOREM 1.3. *Let H be a compact subset of* R^n. *Assume there exists a Lyapunov function* $V: I \times \overline{H^c} \cap S(H, B) \to I$, $B > 0$, *such that* $V(t, x) = 0$ *for all* $(t, x) \in I \times \Delta H$ *and* $V'_{(N)}(t, x)$ *is strongly negative definite relative to H for all* $(t, x) \in I \times H^c \cap S(H, B)$. *Then H is stable, that is, for every* $\varepsilon > 0$ *and* $t_0 \in I$, *there exists* $\delta > 0$ *such that* $x_0 \in S(H, \delta)$ *implies* $p(t, t_0, x_0) \in S(H, \varepsilon)$ *for all* $t \geqq t_0$.

Proof. Let $0 < \varepsilon < B$ and $t_0 \geqq 0$ be given. There exists $d = d(\varepsilon/2) > 0$ such that for all (t, x) in $I \times S(H, B) \cap S^c(H, \varepsilon/2)$ we have $V'_{(N)}(t, x) \leqq -d\|f(t, x)\|$. V continuous implies that there exists α, $0 < \alpha < \varepsilon/2$, such that $x_0 \in \overline{H^c} \cap S(H, \alpha)$ implies $V(t_0, x_0) < d\varepsilon/4$.

Let $x_0 \in H^c \cap S(H, \alpha)$ and consider $p(t) = p(t, t_0, x_0)$. If $p(t) \notin S(H, \varepsilon)$ for all $t \geqq t_0$, there exist $t_1, t_2 > t_0$ such that $p(t_1) \in \Delta S(H, \varepsilon/2)$, $p(t_2) \in \Delta S(H, \varepsilon)$, $p(t) \in \overline{S(H, \varepsilon)}$ for all $t \in [t_1, t_2]$, and $p(t) \in \overline{S(H, \varepsilon)} \cap S^c(H, \varepsilon/2)$ for all $t \in [t_1, t_2]$. Since $V(t, x) = 0$ for all $(t, x) \in I \times \Delta H$ and $V'_{(N)}(t, x) \leqq 0$ for all $(t, x) \in I \times H^c \cap$

$S(H, B)$ we have $V(t_1, p(t_1)) \leqq V(t_0, p(t_0))$. The condition on $V'_{(N)}(t, x)$ implies

$$\begin{aligned} V(t_2, p(t_2)) &\leqq V(t_1, p(t_1)) + \int_{t_1}^{t_2} -d\|f(s, p(s))\| \, ds \\ &\leqq V(t_1, p(t_1)) - d\|p(t_2) - p(t_1)\| \\ &\leqq V(t_0, x_0) - d\varepsilon/2 < -d\varepsilon/4 \end{aligned}$$

contradicting the fact that $V(t, x) \geqq 0$ for all (t, x).

Therefore, $p(t) \in S(H, \varepsilon)$ for all $t \geqq t_0$, proving the theorem.

THEOREM 1.4. *If, in addition to the hypotheses of Theorem* 1.3, $f(t, x)$ *satisfies Hypothesis* A *for all* x *in* $H^c \cap S(H, B)$, *then* H *is asymptotically stable. In particular, if* $H = \{0\}$ *and* $f(t, 0) = 0$ *for all* $t \in I$, *the zero solution of* (N) *is asymptotically stable.*

Proof. By Theorem 1.3, H is stable. Therefore, given $t_0 \geqq 0$, there exists $\alpha > 0$ such that $x_0 \in S(H, \alpha)$ implies that $p(t, t_0, x_0) \in S(H, B)$ for all $t \geqq t_0$. Therefore, we can show, as in the proof of Theorem 1.2, that $p(t, t_0, x_0)$ tends to H as t approaches infinity, proving the theorem.

The following example shows that the conditions of Theorem 1.4 do not necessarily imply uniform or exponential asymptotic stability.

Example 1.1. Consider the scalar equation

$$x' = -g(x)/(t + a), \tag{1.1}$$

where $a > 0$, $g: R^1 \to R^1$ is continuous, and $xg(x) > 0$ for $x \neq 0$. The function $f(t, x) = -g(x)/(t + a)$ satisfies Hypothesis A for all $x \in R^1 - \{0\}$. To show this we choose $\eta = |x|/2$ and let $k = \inf\{g(\bar{x}) : \bar{x} \in S(x, \eta)\}$. Then $|f(t, x)| \geqq k/(t + a)$ and $\int_0^\infty k/(t + a)\, dt = +\infty$.

Consider the following Lyapunov function:

$$V(x) = |x|.$$

We have that $V'_{(1.1)}(t, x) = -g(x)/(t + a) = -|f(t, x)|$. The conditions of Theorem 1.4 are satisfied and so the zero solution of (1.1) is asymptotically stable.

In particular, for $g(x) = x$, we can easily see that the zero solution of (1.1) is asymptotically stable but not uniformly asymptotically stable: given $\varepsilon > 0$ and $(t_0, x_0) \in I \times R^1$, the solution $x(t) = x_0(t_0 + a)/(t + a) < \varepsilon$ if and only if

$$t > [x_0(t_0 + a)/\varepsilon] - a.$$

2. Properties of $V(t, x)$**.**

THEOREM 2.1. *Let* H *be a compact subset of* R^n. *Assume there exists a continuous function* $f: I \times R^n \to R^n$ *that satisfies Hypothesis* A *for all* $x \in H^c$ *and a Lyapunov function* V *such that the conditions of Theorem* 1.1 *hold. Then* $V(t, x) \to +\infty$ *as* $\|x\| \to \infty$ *uniformly in* t.

Proof. Let $\varepsilon > 0$ be fixed. There exists $d > 0$ such that for all $(t, x) \in I \times S^c(H, \varepsilon/2)$,

$$V'_{(N)}(t, x) \leqq -d\|f(t, x)\|/(1 + \|x\|) + q(t).$$

Let $J = \max\{\|x\| : x \in \Delta S(H,\varepsilon)\}$, $K = \ln(1 + J) + (1/d)\int_0^\infty q(t)\,dt$, and let M be a constant such that $V(t,x) \geqq M$ for all $(t,x) \in I \times H^c$.

If $V(t,x) \not\to \infty$ as $\|x\| \to \infty$, then there exists $L > 0$ such that for every $N > 0$, there exist an x_N and $t_0 \geqq 0$ such that $\|x_N\| > N$ and $V(t_0, x_N) < L$. Let N be so large that $\|x_N\| \geqq \exp[\{(L + |M|)/d\} + K]$ so that $\ln(1 + \|x_N\|) > [(L + |M|)/d] + K$ and $V(t_0, x_N) < L$.

Let $p(t) = p(t, t_0, x_N)$ be a solution of (N). *Claim*: There exists $t_1 > t_0$ such that $p(t_1) \in \Delta S(H,\varepsilon)$. Suppose the assertion is false. Then $p(t) \in S^c(H,\varepsilon)$ for all $t \geqq t_0$. Since by Theorem 1.1, $p(t)$ is bounded, the sequence $\{p(n)\}$ must have a cluster point in $S^c(H,\varepsilon)$ which we can show is not possible as was done in the proof of Theorem 1.2. The only alternative is to have $p(t_1) \in \Delta H$ for some $t_1 > t_0$, proving the claim.

We may choose t_1 so that $p(t) \in S^c(H,\varepsilon)$ for all $t \in [t_0, t_1]$. Now, as in the proof of Theorem 1.1, we obtain

$$V(t_1, p(t_1)) - V(t_0, p(t_0)) \leqq -d|\ln(1 + \|p(t_1)\|) - \ln(1 + \|p(t_0)\|)| + \int_0^\infty q(t)\,dt \leqq -|M| - L,$$

which contradicts the fact that V is bounded below by M.

Therefore, $V(t,x) \to +\infty$ as $\|x\| \to \infty$, proving the theorem.

THEOREM 2.2. *Assume there exists a Lyapunov function V satisfying the conditions of Theorem* 1.3 *for $H = \{0\}$, and some $B > 0$. Then, if $f(t,0) = 0$ and $f(t,x)$ satisfies Hypothesis* A *for $x \in S(B) - \{0\}$, then V is positive definite for $(t,x) \in I \times S(B/2)$. More precisely, $V(t,x) \geqq k(\|x\|)\|x\|$ where $k: I \to I$ is a continuous nondecreasing function such that $k(0) = 0$ and $k(r) > 0$ for $r > 0$.*

Proof. Let $\bar{x} \in S(B/2)$. If $\|\bar{x}\| = 0$, then $V(t,\bar{x}) = 0$ for all $t \in I$ by hypothesis. If $\|\bar{x}\| \neq 0$, let $\varepsilon = \|\bar{x}\|/2$ and let the $d = d(\varepsilon) > 0$ be found such that for $(t,x) \in I \times [S(B) \cap S^c(\varepsilon)]$, we have $V'_{(\mathrm{N})}(t,x) \leqq -d\|f(t,x)\|$. Let $p(t)$ be a solution of (N) passing through $\bar{x}$. If $p(t) \in S^c(\varepsilon) \cap \overline{S(3\varepsilon)}$ for all t sufficiently large, the sequence $\{p(n)\}$ has a cluster point in $S^c(\varepsilon) \cap \overline{S(3\varepsilon)}$. Since $f(t,x)$ satisfies Hypothesis A for all x in this annulus, we can show as in the proof of Theorem 1.2 that the existence of such a cluster point is impossible.

Therefore, given any $t_0 \in I$, there exists $t_1 > t_0$ such that $\|p(t_1, t_0, \bar{x}) - \bar{x}\| = \varepsilon$ and $p(t, t_0, \bar{x})$ lies in $S^c(\varepsilon) \cap \overline{S(3\varepsilon)}$ for $t \in [t_0, t_1]$. Therefore, we have

$$\begin{aligned} V(t_1, p(t_1, t_0, \bar{x})) - V(t_0, \bar{x}) &\leqq -d\int_{t_0}^{t_1} \|f(s, p(s, t_0, \bar{x}))\|\,ds \\ &\leqq -d\|p(t_1, t_0, \bar{x}) - \bar{x}\| = -d\varepsilon. \end{aligned}$$

Since $V(t_1, p(t_1, t_0, \bar{x})) \geqq 0$, we have $V(t_0, \bar{x}) \geqq d\varepsilon = d\|\bar{x}\|/2$. Since $d(\varepsilon)$ is a nondecreasing function of ε, we take $k(\|x\|) = d(\|x\|/2)/2$, so $V(t,x) \leqq k(\|x\|)\|x\|$ for all $(t,x) \in I \times S(B/2)$. The theorem is proved.

3. Perturbation theorems. In this and the following section we discuss perturbations of systems (N) for which is known a Lyapunov function whose derivative along solutions of (N) satisfies a strong negative definite condition.

Consider the following example:

(S) $$x' = -x/(t + a) \qquad \text{for } a > 0 \text{ and } x \text{ a scalar.}$$

The zero solution of (S) is asymptotically stable but neither uniformly asymptotically stable nor exponentially asymptotically stable. The function $g(t, x) = x/(t + a)$ satisfies:

(i) $\|g(t, x)\| = o(\|x\|)$ (as $\|x\| \to 0$);

(ii) $\|g(t, x)\| \leqq \gamma(t)$, where $\int_t^{t+1} \gamma(s)\, ds \to 0$ as $t \to \infty$;

but does not satisfy:

(iii) $\|g(t, x)\| \leqq \lambda(t)\|x\|$, where $\int_0^\infty \lambda(t)\, dt < \infty$.

The zero solution of $x' = -x/(t + a) + g(t, x) = 0$ is stable but not asymptotically stable. Also notice that the function $V(t, x) = |x|/(t + a)$ is such that $V'_{(S)}(t, x) = -|x|/(t + a)$ is strongly negative definite relative to any compact set $H \subset R^n$. Therefore, we cannot expect to discuss perturbation terms that only satisfy conditions (i) or (ii) above. In this section we discuss perturbation terms which satisfy condition (iii) above. Section 4 deals with perturbation terms that are dependent upon the function $f(t, x)$ of (N).

THEOREM 3.1. *Assume there exists a Lyapunov function*

$$V: I \times S(B) \to I, \qquad 0 < B \leqq \infty,$$

such that:

(i) $V(t, 0) = 0$ *for all* $t \in I$;

(ii) $|V(t, x_1) - V(t, x_2)| \leqq L\|x_1 - x_2\|$ *for some* $L > 0$, *for all* $t \in I$, *and all* $x_1, x_2 \in S(B)$;

(iii) $V'_{(N)}(t, x) \leqq -\|f(t, x)\|$ *for* $(t, x) \in I \times S(B) - \{0\}$.

Assume $f(t, 0) = 0$ *for all* $t \in I$ *and that* f *satisfies Hypothesis* A *for all* $x \in S(B) - \{0\}$. *Then, if for all* $x \in S(B)$, $\|g(t, x)\| \leqq \lambda(t)\|x\|$, *where* $\lambda: I \to I$ *and* $\int_0^\infty \lambda(t)\, ds < \infty$, *the zero solution of* (P) *is asymptotically stable. In particular, if* $B = \infty$, *the zero solution is asymptotically stable in the large.*

Proof. Let $y(t) = y(t, t_0, x_0)$ be any solution of (P) for $(t_0, x_0) \in I \times S(B)$. Hypotheses (ii) and (iii) imply that for $(t, y(t)) \in I \times S(B) - \{0\}$, we have

$$\begin{aligned} V'_{(P)}(t, y(t)) &\leqq V'_{(N)}(t, y(t)) + L\|g(t, y(t))\| \\ &\leqq -\|f(t, y(t))\| + L\|g(t, y(t))\| \\ &= -\|y'(t) - g(t, y(t))\| + L\|g(t, y(t))\| \\ &\leqq -\|y'(t)\| + \|g(t, y(t))\| + L\|g(t, y(t))\|. \end{aligned} \tag{3.1}$$

Letting $U(t, x) = V(t, x) + \|x\|$, we have for $(t, y(t)) \in I \times S(B)$, $U'_{(P)}(t, y(t)) \leqq (1 + L)\|g(t, y(t))\| \leqq (1 + L)\lambda(t)U(t, y(t))$. Therefore,

$$\|y(t)\| \leqq U(t_0, y_0) \exp (1 + L) \int_0^\infty \lambda(s)\, ds \quad \text{for } (t, y(t)) \in I \times S(B).$$

Since $V(t, 0) = 0$, given $t_0 \in I$ and $\varepsilon > 0$, there exists $\eta = \eta(t_0, \varepsilon) > 0$ such that $\|y\| < \eta$ implies $\|y(t)\| < \varepsilon$ for all $t \geqq t_0$. Therefore, the zero solution of (P) is stable.

Now, for given $t_0 \in I$ and $\|y_0\| < \eta(t_0)$, since $\|y(t, t_0, y_0)\|$ is bounded, there exists $M > 0$ such that

$$\int_t^\infty \|g(s, y(s))\| \, ds \leqq \int_t^\infty \lambda(t)\|y(t)\| \, dt \leqq M < \infty .$$

Therefore, using (3.1) we have, as in the proof of Theorem 1.2, $\|y(t, t_0, y_0)\| \to 0$ as $t \to \infty$. This proves that the zero solution of (P) is asymptotically stable.

It is interesting to compare this theorem with a result of Strauss and Yorke [3, Theorem 4.1] which requires uniform asymptotic stability of the zero solution of (N). The following example illustrates our result in the case that the zero solution of (N) is asymptotically stable but not uniformly so.

Example 3.1. Consider the following system of differential equations:

$$\text{(Q)} \qquad x' = Ax/(t + a), \qquad a > 0,$$

where $x \in R^n$ and A is a constant $n \times n$ matrix whose characteristic roots have negative real parts.

We make use of the following theorem (cf. LaSalle and Lefschetz [2, p. 83]).

THEOREM. *Let A be an $n \times n$ matrix whose characteristic roots have negative real parts. Let C be a positive matrix (that is, the quadratic form x^TCx is positive for all nonzero vectors in R^n). Then, if C is a symmetric matrix, the matrix equation $A^TB + BA = -C$ has a unique solution B, and B is a positive symmetric matrix.*

We also note that for a given positive symmetric matrix B, there exists a nonsingular matrix D such that $B = D^TD$ (cf. LaSalle and Lefschetz [2, p. 16]).

Let E be the identity matrix, and let B be the symmetric positive matrix such that $A^TB + BA = -E$. Consider the Lyapunov function $V(x) = (x^TBx)^{1/2}$. Since B is a positive matrix, $V(x) > 0$ for all $x \neq 0$. Let D as above be such that $B = D^TD$, so that $(x^TBx)^{1/2} = (x^TD^TDx)^{1/2} = \|Dx\| \leqq \|D\| \cdot \|x\|$ where

$$\|D\| = \left[\sum_{i,j=1}^{n} |d_{ij}|^2 \right]^{1/2} .$$

Then we have

$$\begin{aligned} V'_{(Q)}(x) &= [(x^T)'Bx + x^TBx']/2(x^TBx)^{1/2} \\ &= [x^TA^TBx/(t + a) + x^TBAx/(t + a)]/2(x^TBx)^{1/2} \\ &= [(-1/(t + a))x^Tx]/2(x^TBx)^{1/2} \\ &\leqq -(1/(t + a))\|x\|^2/2\|D\| \cdot \|x\| \\ &\leqq -d\|Ax/(t + a)\|, \end{aligned}$$

where $d = 1/(2\|A\| \cdot \|D\|)$. Therefore, V satisfies hypothesis (iii) of Theorem 3.1.

Claim. There exists $L > 0$ such that for all $x_1, x_2 \in R^n$ we have

$$|V(t, x_1) - V(t, x_2)| \leqq L\|x_1 - x_2\| .$$

We note that for a fixed nonsingular matrix P, the function $v(x) = \|Px\|$ defines a norm on R^n. Also, if $\| \cdot \|_1$ is any norm on R^n, we have

$$| \|x_1\|_1 - \|x_2\|_1 | \leqq \|x_1 - x_2\|_1 \qquad \text{for all } x_1, x_2 \in R^n .$$

We noted above that there exists a nonsingular matrix D such that $D^TD = B$, and that $V(x) = \|Dx\|$ for all $x \in R^n$. Therefore, since V is actually a norm on R^n, V satisfies

$$|V(x_1) - V(x_2)| \leqq V(x_1 - x_2) = \|D(x_1 - x_2)\| \leqq \|D\| \cdot \|x_1 - x_2\|$$

proving the claim for $L = \|D\|$.

Since $Ax/(t + a)$ satisfies Hypothesis A for all nonzero x in R^n, if $\|g(t, x)\| \leqq \lambda(t)\|x\|$, where $\int_0^\infty \lambda(t)\,dt < \infty$, then the zero solution of $x' = Ax/(t + a) + g(t, x)$, $a > 0$ is asymptotically stable in the large.

THEOREM 3.2. *Let H be a compact subset of R^n. Assume there exists a Lyapunov function $V: I \times \overline{H^c} \to I$, $V \in C'$, such that:*

(i) *$V(t, x)$ is bounded for all $(t, x) \in I \times \Delta H$;*

(ii) *there exists $k > 0$ such that for $(t, x) \in I \times \overline{H^c}$*

$$\|\text{grad } V(t, x)\| \cdot \|x\| \leqq kV(t, x);$$

(iii) *$V'_{(N)}(t, x) \leqq -\|f(t, x)\|$ for all $(t, x) \in I \times \overline{H^c}$.*

Then if $\|g(t, x)\| \leqq \lambda(t)\|x\|$, where $\lambda: I \to I$ and $\int_0^\infty \lambda(t)\,dt < \infty$, solutions of (P) *are equibounded.*

Proof. H compact implies there exists $\zeta > 0$ such that $H \subset S(\zeta)$. Let $\alpha \geqq \zeta$ and $t_0 \geqq 0$ be given and define the following constants:

K: Since V is continuous, there exists a constant $K = K(t_0, \alpha)$ such that $V(t_0, x_0) \leqq K$ for all $x \in \overline{S(\alpha)}$.

L: By (i) there exists a constant $L > 0$ such that $V(t, x) < L$ for all $(t, x) \in I \times \Delta H$.

M: Let $M = \max\{K, L\}$.

β: Let $\beta = \beta(t_0, \alpha) = (M + \alpha) \exp k \int_t^\infty \lambda(t)\,dt$.

Note that $\beta \geqq \alpha$. We now show that for all $y_0 \in S(\alpha)$, the solution $y(t) = y(t, t_0, y_0)$ of (P) remains in $S(\beta)$.

Assume there exists a solution $y(t) = y(t, t_0, x_0)$ of (P) with $\|y_0\| < \alpha$ such that for some $t^* > t_0$, $\|y(t^*)\| \geqq \beta$. Either:

(a) $y(t) \in H^c$ for all $t \in [t_0, t^*]$; or

(b) there exists $\bar{t} \geqq t_0$ such that $y(\bar{t}) \in \Delta H$ and $y(t) \in \overline{H^c}$ for $t \in [\bar{t}, t^*]$.

Assume (a) holds. For $t \in [t_0, t^*]$, we have (assuming, without loss of generality, that $k \geqq 1$)

$$\begin{aligned} V'_{(P)}(t, y(t)) &\leqq V'_{(N)}(t, y(t)) + \|\text{grad } V(t, y(t))\| \cdot \|g(t, y(t))\| \\ &\leqq -\|f(t, y(t))\| + \|\text{grad } V(t, y(t))\| \cdot \|y(t)\|\lambda(t) \\ &\leqq -\|y'(t)\| + \|g(t, y(t))\| + kV(t, y(t))\lambda(t) \\ &\leqq -\|y'(t)\| + k\lambda(t)[\|y(t)\| + V(t, y(t))]. \end{aligned}$$

Letting $U(t, y) = \|y\| + V(t, y)$, we have $U'_{(P)}(t, y(t)) \leqq k\lambda(t)U(t, y(t))$. Therefore, for $t \in [t_0, t^*]$,

$$\|y(t)\| \leqq U(t, y(t)) \leqq U(t_0, y(t_0)) \exp k \int_{t_0}^\infty \lambda(t)\,dt < \beta,$$

contradicting the fact that $\|y(t^*)\| \geqq \beta$. Assuming (b) holds, the argument is exactly the same for $t \in [\bar{t}, t^*]$.

Therefore, $\|y(t, t_0, y_0)\| < \beta(t_0, \alpha)$ for all $t \geqq t_0$ and $y_0 \in S(\alpha)$. For $\alpha \leqq \zeta$, let $\beta(t_0, \alpha) = \beta(t_0, \zeta)$ so that $\|y_0\| < \alpha$ implies $\|y_0\| < \zeta$ and $\|y(t, t_0, y_0)\| < \beta(t_0, \alpha)$ for all $y_0 \in S(\alpha)$ and $t \geqq t_0$, proving the theorem.

THEOREM 3.3. *If, in addition to the hypotheses of Theorem* 3.2, $V(t, x) = 0$ *for* $(t, x) \in I \times \Delta H$ *and* f *satisfies Hypothesis* A *for all* $x \in H^c$, *then every solution* $y(t)$ *of* (P) *approaches* H *as* t *tends to infinity.*

Proof. By Theorem 3.2, every solution $y(t, t_0, y_0)$ of (P) is bounded and therefore defined for all $t \geqq t_0$.

Let $y(t) = y(t, t_0, y_0)$ be any solution of (P) and fix $\alpha > \|y_0\|$. We have shown in the proof of Theorem 3.2 that there exists $\beta = \beta(t_0, \alpha)$ such that for $t \geqq t_0$, $V(t, y(t)) + \|y(t)\| < \beta$. Therefore, for any t such that $(t, y(t)) \in I \times \overline{H^c}$, we have

$$\begin{aligned} V'_{(P)}(t, y(t)) &\leqq -\|f(t, y(t))\| + \|\text{grad } V(t, y(t))\| \cdot \|g(t, y(t))\| \\ &\leqq -\|f(t, y(t))\| + kV(t, y(t))\lambda(t) \\ &\leqq -\|f(t, y(t))\| + k\beta\lambda(t). \end{aligned}$$

Therefore, by Theorem 1.2, we have that $y(t)$ approaches H as t approaches infinity.

COROLLARY 3.1. *If, under the hypotheses of Theorem* 3.3, $H = \{0\}$ *and* $f(t, 0) = 0$ *for all* $t \in I$, *then the zero solution of* (P) *is asymptotically stable in the large.*

Proof. Let $y(t) = y(t, t_0, y_0)$ be any solution of (P). For all $t \geqq t_0$, we have that (as in the proof of Theorem 3.2)

$$\|y(t)\| \leqq [V(t_0, y_0) + \|y_0\|] \exp k \int_{t_0}^{\infty} \lambda(t)\, dt.$$

Stability of the zero solution of (P) follows from this expression as in the proof of Theorem 3.1. By Theorem 3.3, $\|y(t)\| \to 0$ as $t \to \infty$, proving the corollary.

4. Perturbations dependent on $f(t, x)$.

THEOREM 4.1. *Let* H *be a compact set of* R^n. *Assume there exists a Lyapunov function* $V: I \times \overline{H^c} \to R$ *such that*:

(i) $V(t, x)$ *is bounded for all* $(t, x) \in I \times \Delta H$;

(ii) $|V(t, x_1) - V(t, x_2)| \leqq L\|x_1 - x_2\|$ *for some* $L > 0$, *and all* $t \in I, x_1, x_2 \in \overline{H^c}$;

(iii) $V'_{(N)}(t, x) \leqq -\|f(t, x)\|$ *for all* $(t, x) \in I \times \overline{H^c}$.

Assume that f *satisfies Hypothesis* A *for all* $x \in H^c$. *Then, if* $\|g(t, x)\| \leqq (1/L)\|f(t, x)\|$ *for all* $(t, x) \in I \times S^c(\eta)$ *for some* $\eta > 0$, *the solutions of* (P) *are equibounded.*

Proof. There exists $\zeta > 0$ such that $H \subset S(\zeta)$. Let $\gamma = \max\{\zeta, \eta\}$. Hypotheses (ii) and (iii) imply that for (t, y) in $I \times S^c(\gamma)$, $V'_{(P)}(t, y) \leqq -\|f(t, y)\| + L\|g(t, y)\| \leqq 0$. Now let $\alpha \geqq \gamma$ and $t_0 \in I$ be given and define the following constants:

K: Let $K = K(t_0, \alpha)$ be such that $V(t_0, y) \leqq K$ for all $y \in \overline{S(\alpha)}$.

$\bar{L}$: Hypothesis (i) implies that there exists $\bar{L} > 0$ such that $V(t, y) \leqq \bar{L}$ for $(t, y) \in I \times \Delta H$.

M: Set $M = \max\{K, \bar{L}\}$,

β: By Theorem 2.1 there exists $\beta = \beta(t_0, \alpha) > 0$ such that $\|y\| > \beta$ implies that $V(t, y) > M$ for all $t \geqq t_0$.

Assume there exists a solution $y(t) = y(t, t_0, y_0)$ of (P) with $y_0 \in S(\alpha)$ such that for some $t^* > t_0$, $\|y(t^*)\| \geqq \beta$. Either:

(a) $y(t) \in H^c$ for $t \in [t_0, t^*]$; or

(b) there exists $\bar{t} \geqq t_0$ such that $y(\bar{t}) \in \Delta H$ and $y(t) \in \overline{H^c}$ for $t \in [\bar{t}, t^*]$.

If (a) holds, integrate $V'_{(P)}(t, y(t)) \leqq 0$ from t_0 to t^*, obtaining

$$K \leqq M < V(t^*, y(t^*)) \leqq V(t_0, y_0) \leqq K,$$

a contradiction.

Assuming (b) holds, integrate the same expression from $\bar{t}$ to t^*, obtaining $L \leqq M < V(t^*, y(t^*)) \leqq V(\bar{t}, y(\bar{t})) \leqq L$, again a contradiction. Therefore, $y(t) \in S(\beta)$ for all $t \geqq t_0$.

For $\alpha \leqq \zeta$, let $\beta(t_0, \alpha) = \beta(t_0, \zeta)$ so that $\|y_0\| < \alpha$ implies $\|y_0\| < \zeta$ and so $\|y(t, t_0, y_0)\| < \beta(t_0, \alpha)$ for all $t \geqq t_0$, proving the theorem.

THEOREM 4.2. *Assume there exists a Lyapunov function* $V: I \times S(B) \to I$, $B > 0$, *such that*:

(i) $V(t, 0) = 0$ *for all* $t \in I$;

(ii) $|V(t, x_1) - V(t, x_2)| \leqq L\|x_1 - x_2\|$ *for some* $L > 0$, *and all* $t \in I$, $x_1, x_2 \in S(B)$;

(iii) $V'_{(N)}(t, x) \leqq -\|f(t, x)\|$ *for all* $(t, x) \in I \times S(B)$.

Assume that $f(t, 0) = 0$ *for all* $t \in I$ *and that* f *satisfies Hypothesis* A *for all* $x \in S(B) - \{0\}$. *Then, if there exist* $T > 0$ *and* $\varepsilon > 0$ *such that for some* $c > 0$ *and all* (t, x) *in* $[T, \infty) \times S(\varepsilon)$, $\|g(t, x)\| \leqq [(1/L) - c]\|f(t, x)\|$, *the zero solution of* (P) *is asymptotically stable for all* $t_0 \geqq T$.

Proof. For $(t, x) \in [T, \infty) \times S(\varepsilon)$, we have

$$V'_{(P)}(t, x) \leqq -\|f(t, x)\| + L\|g(t, x)\| \leqq -Lc\|f(t, x)\|.$$

By Theorem 2.2, V is positive definite for (t, x) in $[T, \infty) \times S(\varepsilon/2)$. Therefore, the zero solution of (P) is stable for $t_0 \geqq T$.

For $t_0 \in [T, \infty)$, let the $\delta = \delta(\varepsilon/2)$ of stability be found. Let $y(t) = y(t, t_0, y_0)$ be a solution of (P) with $y_0 \in S(\delta)$. Since $V'_{(P)}(t, y(t)) \leqq -Lc\|f(t, y(t))\|$ for $(t, y(t)) \in [T, \infty) \times S(\varepsilon/2)$, we have, as in the proof of Theorem 1.2, that

$$\|y(t, t_0, y_0)\| \to 0$$

as t tends to infinity, proving the theorem.

Example 4.1. Consider the scalar differential equation:

$$x' = -w(x)/(t + a), \qquad a > 0,$$

where $w: R^1 \to R^1$ is continuous and $xw(x) > 0$ for $x > 0$. Also, $-w(x)/(t + a)$ satisfies Hypothesis A for all x in $R^1 - \{0\}$, since for $\bar{x} \neq 0$ we can choose $\eta = |\bar{x}|/2$. Letting

$$k = \inf\{|w(x)| : x \in S(\bar{x}, \eta)\} > 0,$$

we have

$$|w(x)/(t + a)| \geqq k/(t + a)$$

and

$$\int_{t_0}^{\infty} \frac{k}{t + a}\, dt = +\infty \quad \text{for all } t_0 \geqq 0.$$

For the Lyapunov function $V(x) = |x|$, the hypotheses of Theorem 4.2 are satisfied with $L = 1$. Therefore, by Theorem 4.2, $x' = -w(x)/(t + a) + g(t, x)$, $a > 0$ is asymptotically stable for any $g(t, x)$ satisfying the condition stated in Theorem 4.2.

In particular, the zero solution of each of the following equations is asymptotically stable for all $t_0 \geqq 0$:

(a) $x' = -x^{1/3}/(t + a) + x^r/(t + b)$, for $a, b > 0$ and $r > 1/3$;

(b) $x' = -x \exp(-x^2)/(t + a) + x^r/(t + b)$, for $a, b > 0$ and $r > 1$.

THEOREM 4.3. *Assume there exists a Lyapunov function* $V: I \times R^n \to R^1$ *such that*:

(i) $|V(t, x_1) - V(t, x_2)| \leqq \gamma(t)\|x_1 - x_2\|$ *for all* $t \in I, x_1, x_2 \in R^n$, *where* $\gamma: I \to I$ *is continuous*;

(ii) $V'_{(N)}(t, x) \leqq -\|f(t, x)\|/(1 + \|x\|)$ *for all* $(t, x) \in I \times R^n$.

Assume that f satisfies Hypothesis A *for all* $x \in R^n - \{0\}$. *Then if*

$$\|g(t, x)\| \leqq \lambda(t)\|f(t, x)\|/(1 + \|x\|)$$

where $\lambda: I \to I$ *is bounded and continuous and there exists* $T \in I$ *and a constant c*, $0 < c < 1$, *such that* $\lambda(t)\gamma(t) < c$ *for all* $t \geqq T$, *then for all* $t_0 \in [T, \infty)$, *every solution* $y(t) = y(t, t_0, y_0)$ *of* (P) *is bounded and satisfies* $\|y(t)\| \to 0$ *as t tends to infinity.*

Proof. Let $y(t) = y(t, t_0, y_0)$ be a solution of (P) with $t_0 \geqq T$. Hypotheses (i) and (ii) imply that for $t \geqq T$,

$$\begin{aligned} V'_{(P)}(t, y(t)) &\leqq -\|f(t, y(t))\|/(1 + \|y(t)\|) + \gamma(t)\|g(t, y(t))\| \\ &\leqq -k\|f(t, y(t))\|/(1 + \|y(t)\|), \end{aligned}$$

where $k = 1 - c > 0$. From equation (P) we see that

$$\|y'(t)\| \leqq \|f(t, y(t))\| + \|g(t, y(t))\| \leqq \|f(t, y(t))\|[1 + M/(1 + \|y(t)\|)],$$

where M is such that $\lambda(t) \leqq M$ for all $t \leqq 0$. Therefore,

$$\|f(t, y(t))\| \geqq \|y'(t)\|(1 + \|y(t)\|)/(1 + M + \|y(t)\|).$$

From the above, therefore, we have

$$V'_{(P)}(t, y(t)) \leqq -k\|y'(t)\|/(1 + M + \|y(t)\|). \tag{4.1}$$

Using the fact that $|d\|y(t)\|/dt| \leqq \|y'(t)\|$, we obtain by integration

$$V(t, y(t)) - V(t_0, y_0) \leqq -k|\ln[(1 + M + \|y(t)\|)/(1 + M + \|y_0\|)]|,$$

which approaches $-\infty$ as $\|y(t)\| \to \infty$. Since $V(t, x)$ is bounded from below, we have that $\|y(t)\|$ is bounded.

Now, for $t_0 \geqq T$, $\|y(t, t_0, y_0)\|$ is bounded, so there exists $K > 0$ such that $1 + \|y(t, t_0, y_0)\| \leqq K$ for all $t \geqq t_0$. Therefore, for all $t \geqq t_0$,

$$V'_{(P)}(t, y(t, t_0, y_0)) \leqq -(k/K)\|f(t, y(t, t_0, y_0))\|. \tag{4.2}$$

As in the proof of Theorem 1.2, $\|y(t, t_0, y_0)\| \to 0$ as $t \to \infty$, using (4.1) and (4.2), proving the theorem.

COROLLARY 4.1. *If, in Theorem* 4.3, *hypothesis* (ii) *is replaced by*

(ii)′ $V'_{(N)}(t, x) \leqq -\|f(t, x)\|$ *for all* $(t, x) \in I \times R^n$, *then the conclusion of the theorem holds for any* $g(t, x)$ *such that* $\|g(t, x)\| \leqq \lambda(t)\|f(t, x)\|$, *where* $\lambda(t)$ *is as in Theorem* 4.3.

Proof. The proof of the corollary follows exactly the same as the proof of the theorem.

COROLLARY 4.2. *If, in addition to the hypotheses of Theorem* 4.3 *or Corollary* 4.1, $V(t, 0) = 0 = f(t, 0)$ *for all* $t \in I$ *and* $V(t, x) \geqq 0$ *for all* $(t, x) \in I \times R^n$, *then the zero solution of* (P) *is asymptotically stable for* $t_0 \geqq T$ *in the large; that is, the zero solution of* (P) *is stable for* $t_0 \geqq T$, *and for all* $(t_0, y_0) \in [T, \infty) \times R^n$,

$$\|y(t, t_0, y_0)\| \to 0 \quad \text{as } t \text{ approaches infinity.}$$

Proof. The conditions on g assure us that $g(t, 0) = 0$ for all $t \in I$, so that (P) has the zero solution.

By Theorem 2.2, V is positive definite and the condition $\lambda(t)\gamma(t) < c < 1$ for $t \leqq T$ implies, as in Theorem 4.3, that $V'_{(P)}(t, y(t)) \leqq 0$ for all $t \leqq T$, where $y(t) = y(t, t_0, y_0)$ is any solution of (P) with $t_0 \geqq T$. Therefore, the zero solution of (P) is stable for all $t_0 \geqq T$.

By Theorem 4.3, we have that $\|y(t, t_0, y_0)\| \to 0$ as t approaches infinity, proving the corollary.

Under the hypotheses of Theorem 4.3 or Corollary 4.1, it is possible that there exist solutions of the perturbed equation that have finite escape time.

Example 4.2. Consider the scalar differential equation

$$x' = -x^3 \tag{4.3}$$

and its perturbed equation

$$y' = -y^3 + g(t, y), \tag{4.4}$$

where

$$g(t, y) = \begin{cases} 2y^3, & \text{for } 0 \leqq t \leqq 10, \\ 200y^3/t^2, & \text{for } t \geqq 10. \end{cases}$$

For $t \leqq 10$, solutions of (4.4) are of the form $y(t, t_0, y_0) = [2(t_0 - t) + y_0^{-2}]^{-1/2}$ which increases without bound as $t \to (t_0 + y_0^{-2}/2)^+$. Note that the Lyapunov function $V(x) = |x|$ satisfies the conditions of Corollary 4.1 for this example.

Under the conditions of the following theorem solutions of the perturbed equation cannot have finite escape time.

THEOREM 4.4. *Assume there exists a Lyapunov function* $V: I \times R^n \to R^1$ *such that*:

(i) $|V(t, x_1) - V(t, x_2)| \leqq \gamma(t)\|x_1 - x_2\|$ *for all* $t \in I, x_1, x_2 \in R^n$, *where* $\gamma: I \to I$ *is continuous*;

(ii) $V'_{(N)}(t, x) \leqq -\|f(t, x)\|/(1 + \|x\|)$ *for all* $(t, x) \in I \times R^n$.

Assume that f *satisfies Hypothesis* A *for all* $x \in R^n - \{0\}$. *Then if*

$$\|g(t, x)\| \leqq \lambda(t)\|f(t, x)\|/(1 + \|x\|^{1+\varepsilon})$$

for $(t, x) \in I \times R^n$, *where* $\varepsilon > 0$, $\lambda: I \to I$ *is bounded and continuous and there exist* $T \in I$ *and a constant* c, $0 < c < 1$, *such that* $\lambda(t)\gamma(t) < c$ *for all* $t \geqq T$, *every solution* $y(t) = y(t, t_0, y_0)$ *of* (P) *is bounded and satisfies* $\|y(t)\| \to 0$ *as* t *tends to infinity.*

Proof. The theorem follows from the previous theorem if we show that no solution of (P) can have finite escape time.

Assume there exists a solution $y(t) = y(t, t_0, y_0)$ of (P) with finite escape time. Since f and g are continuous, $\text{limit}_{t \to \bar{t}} \|y(t)\| = +\infty$ for some $\bar{t} > t_0$, $\bar{t} < \infty$. Let

$$M = \max_{t \in [t_0, \bar{t}]} \{\lambda(t)\gamma(t)\}.$$

Hypotheses (i) and (ii) imply that

$$V'_{(\mathrm{P})}(t, y(t)) \leqq -\|f(t, y(t))\|/(1 + \|y(t)\|) + \gamma(t)\|g(t, y(t))\|,$$

so, for $t_0 \leqq t \leqq \bar{t}$,

$$V'_{(\mathrm{P})}(t, y(t)) \leqq \|f(t, y(t))\|[-1/(1 + \|y(t)\|) + M/(1 + \|y(t)\|^{1+\varepsilon})].$$

Let $K > 1$ be such that $\|y\| \geqq K$ implies $\|y\|^\varepsilon > 4M$. Since $\|y(t)\| \to +\infty$ as $t \to \bar{t}$ from the left, there exists $t^* \geqq t_0$ such that $\|y(t)\| \geqq K$ for $t \in [t^*, \bar{t})$. Therefore, for $t \in [t^*, \bar{t})$,

$$\begin{aligned} \|y(t)\|^{1+\varepsilon} &= \|y(t)\| \cdot \|y(t)\|^\varepsilon/2 + \|y(t)\|^{1+\varepsilon}/2 \\ &> \|y(t)\|2M + (2M - 1) \\ &= 2M(1 + \|y(t)\|) - 1, \end{aligned}$$

which implies that $(1 + \|y(t)\|^{1+\varepsilon})/M > 2(1 + \|y(t)\|)$. Therefore, for $t \in [t^*, \bar{t})$,

$$V'_{(\mathrm{P})}(t, y(t)) \leqq -\|f(t, y(t))\|/2(1 + \|y(t)\|).$$

Now, from the differential equation (P), we have

$$\begin{aligned} \|y'(t)\| &\leqq \|f(t, y(t))\| + \|g(t, y(t))\| \\ &\leqq \|f(t, y(t))\|(1 + k + \|y(t)\|)/(1 + \|y(t)\|), \end{aligned}$$

where k is the bound on $\lambda(t)$ for $t \in [t_0, \bar{t}]$. So we have

$$\|f(t, y(t))\| \geqq \|y'(t)\|(1 + \|y(t)\|)/(1 + k + \|y(t)\|).$$

Therefore, for $t \in [t^*, \bar{t})$,

$$V'_{(\mathrm{P})}(t, y(t)) \leqq -\|y'(t)\|/(1 + k + \|y(t)\|)2.$$

Integrating from t^* to $t \in [t^*, \bar{t})$, we obtain

$$V(t, y(t)) - V(t^*, y(t^*)) \leqq -\tfrac{1}{2}|\ln(1 + k + \|y(t)\|)/(1 + k + \|y(t^*)\|)|,$$

which approaches $-\infty$ as $\|y(t)\| \to +\infty$, contradicting the fact that $V(t, x)$ is bounded from below for all $(t, x) \in I \times R^n$.

Therefore, no solution of (P) can have finite escape time, proving the theorem.

We state the following without proof.

COROLLARY 4.3. *If, in Theorem* 4.4, *the hypothesis* (ii) *is replaced by*

(ii)′ $V'_{(\mathrm{N})}(t, x) \leqq -\|f(t, x)\|$ *for all* $(t, x) \in I \times R^n$,

then the conclusion of Theorem 4.4 *holds for any* $g(t, x)$ *such that*

$$\|g(t, x)\| \leqq \lambda(t)\|f(t, x)\|/(1 + \|x\|^\varepsilon)$$

for $\varepsilon > 0$ *and* $\lambda(t)$ *as in the hypotheses of Theorem* 4.4.

Acknowledgment. The author wishes to thank Professor T. A. Burton for his constant advice and encouragement.

REFERENCES

[1] T. A. BURTON, *An extension of Liapunov's direct method*, J. Math. Anal. Appl., 28 (1969), pp. 545–552.

[2] P. LASALLE AND S. LEFSCHETZ, *Stability of Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.

[3] A. STRAUSS AND J. A. YORKE, *Perturbation theorems for ordinary differential equations*, J. Differential Equations, 3 (1967), pp. 15–30.

[4] T. YOSHIZAWA, *Stability Theory by Liapunov's Second Method*, Math. Soc. of Japan, Tokyo, 1966.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

SOME q-ANALOGUES OF CERTAIN COMBINATORIAL NUMBERS*

L. CARLITZ†

Abstract. Stanton and Cowan have discussed the recurrence

$$g(n + 1, r + 1) = g(n, r + 1) + g(n + 1, r) + g(n, r)$$

subject to the initial conditions

$$g(n, 0) = g(0, r) = 1, \qquad n \geqq 0, \quad r \geqq 0.$$

The present paper is concerned with two generalizations of this recurrence. In particular the recurrence

$$A(n, r) = A(n - 1, r - 1) + q^n A(n, r - 1) + q^r A(n - 1, r)$$

is treated. The more general recurrence

$$A(n, r) = A(n - 1, r - 1) + p^n A(n, r - 1) + q^r A(n - 1, r)$$

is also discussed; the special case $p = 1$ leads to simpler results.

1. Stanton and Cowan [3] have discussed the recurrence

$$g(n + 1, r + 1) = g(n, r + 1) + g(n + 1, r) + g(n, r) \tag{1.1}$$

subject to the initial conditions

$$g(n, 0) = g(0, r) = 1, \qquad n \geqq 0, \quad r \geqq 0. \tag{1.2}$$

They show for example that

$$g(n, r) = \sum_{k=0}^{n} \binom{n}{k} \binom{r + k}{n} \tag{1.3}$$

and

$$g(n, r) = \sum_{k=0}^{\min(n,r)} 2^k \binom{n}{k} \binom{r}{k} = g(r, n). \tag{1.4}$$

Since, by (1.4),

$$\sum_{n,r=0}^{\infty} g(n, r) x^n y^r = \frac{1}{1 - x - y - xy},$$

it follows also that

$$\begin{aligned} g(n, r) &= \sum_{k=0}^{\min(n,r)} \frac{(n + r - k)!}{k!(n - k)!(r - k)!} = \sum_{k=0}^{n} \binom{n}{k} \binom{n + r - k}{n} \\ &= \sum_{k=0}^{r} \binom{r}{k} \binom{n + r - k}{r} = \sum_{k=0}^{n} \binom{n}{k} \binom{r + k}{n} = \sum_{k=0}^{r} \binom{r}{k} \binom{n + k}{r}. \end{aligned}$$

* Received by the editors November 11, 1971, and in revised form May 22, 1972.

† Department of Mathematics, Duke University, Durham, North Carolina 27706. This research was supported in part by the National Science Foundation under Grant GP-17031.

Moreover

$$g(n,r)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}\binom{n+r-j}{n}2^{r-j}=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{n+r-j}{r}2^{n-j}.$$

See also [1].

The referee (of the present paper) has pointed out that the $g(n,r)$ have the following combinatorial meaning arising in Kaplansky's lemma [2]:

The number of ways of selecting r of the $2n-1$ letters $A_1, A_2, \cdots, A_{n-1}$; $B_1, B_2, \cdots, B_n$ such that no selection contains A_i and B_j with $j=i-1, i, i+1$ is equal to $g(n-r,r)$ ($g(n,0)=1$, by convention).

The recurrence (1.1) suggests various generalizations. We consider first the numbers $A(n,r)=A(n,r;q)$ defined by

$$A(n,r)=A(n-1,r-1)+q^nA(n,r-1)+q^rA(n-1,r) \tag{1.5}$$

and

$$A(n,0)=A(0,r)=1, \qquad n\geqq 0, \quad r\geqq 0. \tag{1.6}$$

Clearly

$$A(n,r;1)=g(n,r).$$

Put

$$F(x,y)=F(x,y;q)=\sum_{n,r=0}^{\infty}A(n,r)x^ny^r.$$

Then, by (1.5) and (1.6),

$$\begin{aligned}F(x,y)&=1+\frac{x}{1-x}+\frac{y}{1-y}+\sum_{n,r=1}^{\infty}\{A(n-1,r-1)+q^nA(n,r-1)\\&\qquad+q^rA(n-1,r)\}x^ny^n\\&=1+\frac{x}{1-x}+\frac{y}{1-y}+xyF(x,y)+y\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}A(n,r)(qx)^ny^r\\&\quad+x\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}A(n,r)x^n(qy)^r\\&=1+xyF(x,y)+y\sum_{n,r=0}^{\infty}A(n,r)(qx)^ny^r+x\sum_{n,r=0}^{\infty}A(n,r)x^n(qy)^r.\end{aligned}$$

Thus

$$F(x,y)=\frac{1}{1-xy}+\frac{y}{1-xy}F(qx,y)+\frac{x}{1-xy}F(x,qy). \tag{1.7}$$

Iteration of (1.7) gives

$$\begin{aligned} F(x,y) &= \frac{1}{1-xy} + \frac{y}{1-xy}\left\{\frac{1}{1-qxy} + \frac{y}{1-qxy}F(q^2x,y) + \frac{qx}{1-qxy}F(qx,qy)\right\} \\ &\quad + \frac{x}{1-xy}\left\{\frac{1}{1-qxy} + \frac{qy}{1-qxy}F(qx,qy) + \frac{x}{1-qxy}F(x,q^2y)\right\} \\ &= \frac{1}{1-xy} + \frac{x+y}{(1-xy)(1-qxy)} \\ &\quad + \frac{1}{(1-xy)(1-qxy)}\left\{y^2F(q^2x,xy) + 2qxyF(qx,qy) + x^2F(x,q^2y)\right\}. \end{aligned}$$

This suggests the possibility of an expansion of the form

$$F(x,y) = \sum_{n=0}^{\infty} \frac{P_n(x,y)}{(xy)_{n+1}}, \tag{1.8}$$

where

$$(xy)_{n+1} = (1-xy)(1-qxy)\cdots(1-q^nxy).$$

Substituting from (1.8) in (1.7) we are led to

$$P_{n+1}(x,y) = yP_n(qx,y) + xP_n(x,qy), \tag{1.9}$$

with

$$P_0(x,y) = 1, \qquad P_1(x,y) = x + y. \tag{1.10}$$

More precisely, if $P_n(x, y)$ satisfies (1.9) and (1.10), then $F(x, y)$ as defined by (1.8) will satisfy (1.7).

It is convenient to write

$$P_{n+1}(x,y) = (yE_x + xE_y)P_n(x,y), \tag{1.11}$$

where

$$E_xf(x,y) = f(qx,y), \qquad E_yf(x,y) = f(x,qy)$$

and $f(x, y)$ is an arbitrary function of x and y. Clearly (1.11) implies

$$P_n(x,y) = (yE_x + xE_y)^n \cdot 1. \tag{1.12}$$

We have

$$(yE_x + xE_y)^n = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)}x^{n-k}y^kE_x^kE_y^{n-k}. \tag{1.13}$$

Indeed, assuming that (1.13) holds up to n, then

$$\begin{aligned} (yE_x + xE_y)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} q^{(k+1)(n-k)}x^{n-k}y^{k+1}E_x^{k+1}E_y^{n-k} \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k+1)}x^{n-k+1}y^kE_x^kE_y^{n-k+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} q^{k(n-k+1)}x^{n-k+1}y^kE_x^kE_y^{n-k+1}. \end{aligned}$$

It follows from (1.12) and (1.13) that

$$P_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} x^{n-k} y^k. \tag{1.14}$$

Since (1.11) implies

$$P_{m+n}(x, y) = (yE_x + xE_y)^n P_m(x, y),$$

we have the more general result

$$P_{m+n}(x, y) = \sum_{k=0}^{n} \binom{n}{k} q^{k(n-k)} x^{n-k} y^k P_m(q^k x, q^{n-k} y). \tag{1.15}$$

In the next place, since

$$\frac{1}{(z)_{n+1}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} z^k,$$

where

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \frac{(1-q^{n+k})(1-q^{n-k-1})\cdots(1-q^{n+1})}{(1-q)(1-q^2)\cdots(1-q^k)},$$

it follows from (1.8) and (1.14) that

$$\begin{aligned} F(x, y) &= \sum_{k=0}^{\infty} \frac{P_k(x, y)}{(xy)_{k+1}} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k}{j} q^{j(k-j)} x^j y^{k-j} \sum_{s=0}^{\infty} \begin{bmatrix} k+s \\ s \end{bmatrix} x^s y^s \\ &= \sum_{n,r=0}^{\infty} x^n y^r \sum_{\substack{j+s=n \\ k-j+s=r}} \binom{k}{j} \begin{bmatrix} k+s \\ s \end{bmatrix} q^{j(k-j)} \\ &= \sum_{n,r=0}^{\infty} x^n y^r \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} \begin{bmatrix} n+r-s \\ s \end{bmatrix} q^{(n-s)(r-s)}. \end{aligned}$$

Therefore,

$$A(n, r) = \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} \begin{bmatrix} n+r-s \\ s \end{bmatrix} q^{(n-s)(r-s)}. \tag{1.16}$$

It is clear from (1.16) that $A(n, r)$ is a polynomial in q with nonnegative coefficients. Moreover

$$\deg A(n, r) = nr. \tag{1.17}$$

If we put

$$A(n, r) = \sum_{k=0}^{nr} a(n, r, k) q^k$$

then (1.5) implies

$$a(n, r, k) = a(n-1, r-1, k) + a(n, r-1, k-n) + a(n-1, r, k-r). \tag{1.18}$$

Corresponding to (1.6) we have the conditions

$$a(n,0,k) = a(0,r,k) = \delta_{k,0}, \qquad n \geqq 0, \quad r \geqq 0. \tag{1.19}$$

If we put

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_j p(n,k,j)q^j,$$

where $p(n,k,j)$ is the number of partitions of n with at most k parts and no part greater than j, then (1.19) implies

$$a(n,r,k) = \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} p(n+r-s, s, k-(n-s)(r-s)). \tag{1.20}$$

If we define

$$A^*(n,r) = q^{nr}A(n,r,q^{-1}),$$

where $A(n,r,q^{-1})$ is the result of replacing q by q^{-1} in $A(n,r)$, then (1.16) becomes

$$A^*(n,r) = \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} \begin{bmatrix} n+r-s \\ s \end{bmatrix} q^{s^2}. \tag{1.21}$$

It follows that

$$\begin{aligned} F^*(x,y) &= \sum_{n,r=0}^{\infty} A^*(n,r)x^n y^r \\ &= \sum_{s=0}^{\infty} (xy)^s q^{s^2} \sum_{n,r=0}^{\infty} \binom{n+r}{n} \begin{bmatrix} n+r+s \\ s \end{bmatrix} x^n y^r \\ &= \sum_{s=0}^{\infty} (xy)^s q^{s^2} \sum_{k=0}^{\infty} \begin{bmatrix} k+s \\ s \end{bmatrix} \sum_{n=0}^{k} \binom{k}{n} x^n y^{k-n} \\ &= \sum_{s=0}^{\infty} (xy)^s q^{s^2} \sum_{k=0}^{\infty} \begin{bmatrix} k+s \\ s \end{bmatrix} (x+y)^k. \end{aligned}$$

Therefore,

$$F^*(x,y) = \sum_{s=0}^{\infty} \frac{(xy)^s q^{s^2}}{(x+y)_{s+1}}. \tag{1.22}$$

The numbers $A^*(n,r)$ satisfy the recurrence

$$A^*(n,r) = q^{n+r-1}A^*(n-1,r-1) + A^*(n,r-1) + A^*(n-1,r). \tag{1.23}$$

Alternatively, we could have taken (1.23) as definition and derived all the above results.

We remark that, when $q = 1$, the right member of (1.16) reduces to

$$\sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} \binom{n+r-s}{s} = \sum_{s=0}^{n} \binom{n}{s} \binom{r+s}{n}$$

in agreement with (1.3).

2. We consider next the more general recurrence,

$$A(n,r) = A(n-1,r-1) + p^n A(n,r-1) + q^r A(n-1,r), \tag{2.1}$$

subject to the conditions

$$A(n,0) = A(0,r) = 1, \qquad n \geqq 0, \quad r \geqq 0. \tag{2.2}$$

To avoid confusion, we shall occasionally employ the fuller notation $A(n,r;p,q)$.

If again we put

$$F(x,y) = F(x,y;p,q) = \sum_{n,r=0}^{\infty} A(n,r;p,q)x^n y^r, \tag{2.3}$$

it follows that

$$F(x,y) = \frac{1}{1-xy} + \frac{y}{1-xy}F(px,y) + \frac{x}{1-xy}F(x,qy). \tag{2.4}$$

This may be written in the form

$$F(x,y) = \frac{1}{1-xy} + \frac{1}{1-xy}(yE_x + xE_y)F(x,y), \tag{2.5}$$

where now

$$E_x f(x,y) = f(px,y), \qquad E_y f(x,y) = f(x,qy).$$

It can be verified that

$$(yE_x + xE_y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k y^{n-k} E_x^{n-k} E_y^k, \tag{2.6}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{(p^n - q^n)(p^{n-1} - q^{n-1}) \cdots (p^{n-k+1} - q^{n-k+1})}{(p-q)(p^2-q^2)\cdots(p^k - q^k)}.$$

Note that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,q} = \binom{n}{k} q^{k(n-k)},$$

so that (2.5) reduces to (1.13). However (2.6) does not seem to be of much use in evaluating $F(x,y)$.

Iteration of (2.6) gives

$$F(x,y) = \frac{1}{1-xy} + \frac{1}{1-xy}(yE_x + xE_y)\frac{1}{1-xy}$$
$$+ \frac{1}{1-xy}(yE_x + xE_y)\frac{1}{1-xy}(yE_x + xE_y)F(x,y).$$

The general formula is

$$F(x,y) = \sum_{n=0}^{\infty} \left\{ \frac{1}{1-xy}(yE_x + xE_y) \right\}^n \frac{1}{1-xy} = \sum_{n=0}^{\infty} F_n. \tag{2.7}$$

To evaluate F_n, consider the sequence

$$\sigma = (\lambda_1\lambda_2\cdots\lambda_n), \tag{2.8}$$

where each λ_j is either a p or a q. Then

$$F_n = \sum_\sigma \frac{p^u q^v x^{n-k} y^k}{(1-xy)(1-\lambda_1 xy)(1-\lambda_1\lambda_2 xy)\cdots(1-\lambda_1\cdots\lambda_n xy)}, \tag{2.9}$$

where k is the number of p's in σ and the summation is over all sequences (2.8). To evaluate u, v we use (2.5). This implies the recurrences

$$\begin{aligned} u(\sigma p) &= u(\sigma) + n - k + 1, \qquad v(\sigma q) = v(\sigma), \\ u(\sigma q) &= u(\sigma), \qquad v(\sigma q) = v(\sigma) + k, \end{aligned} \tag{2.10}$$

where

$$(\sigma p) = (\lambda_1\cdots\lambda_n p), \qquad (\sigma q) = (\lambda_1\cdots\lambda_n q).$$

If we put

$$\sigma = (\underbrace{p\cdots p}_{s_1}\underbrace{q\cdots q}_{s_2}\underbrace{p\cdots p}_{s_3}\cdots\underbrace{q\cdots q}_{s_{2t}}),$$

where

$$s_1 \geqq 0,\ s_2 > 0, \cdots, s_{2t-1} > 0,\ s_{2t} \geqq 0, \tag{2.11}$$

it follows from (2.10) that

$$u(\sigma) = \sum_{i\leqq j} s_{2i}s_{2j+1}, \qquad v(\sigma) = \sum_{i\leqq j} s_{2i-1}s_{2j}. \tag{2.12}$$

Note that

$$k = s_1 + s_3 + s_k + \cdots, \qquad n - k = s_2 + s_4 + s_6 + \cdots \tag{2.13}$$

and

$$u(\sigma) + v(\sigma) = k(n-k). \tag{2.14}$$

Therefore, for $p = q$, (2.9) reduces to

$$F_n = \frac{1}{(xy)_{n+1}} \sum q^{k(n-k)} x^{n-k} y^k, \tag{2.15}$$

where the summation is over all s_i satisfying (2.11) and (2.13). This amounts to choosing k positions out of n in all possible ways, namely $\binom{n}{k}$. Thus (2.15) is in agreement with (1.15).

It is evident from (2.1) that $A(n, r; p, q)$ is a polynomial in p, q with nonnegative integral coefficients. It is of degree nr in each of p, q separately and also of total degree nr. It is also clear that

$$A(n, r; p, q) = A(r, n; q, p). \tag{2.16}$$

The first few values are easily computed by means of (2.1).

1	1	1	1
1	$1 + p + q$	$1 + p + p^2 + pq + q^2$	$1 + p + p^2 + p^3$ $+ p^2q + pq^2 + q^3$
1	$1 + q + q^2 + pq + p^2$		

$$A(2,2) = 1 + p + q + p^2 + q^2 + pq(p + q) + (p^2 + q^2)(p^2 + pq + q^2).$$

3. If $p = 0$, (2.1) reduces to

$$A_0(n,r) = A_0(n-1,r-1) + q^r A(n-1,r), \tag{3.1}$$

with

$$A_0(n,0) = A_0(0,r) = 1, \qquad n \geqq 0, \quad r \geqq 0, \tag{3.2}$$

where for brevity we put

$$A_0(n,r) = A(n,r;0,q).$$

Put

$$F_0(x,y) = \sum_{n,r=0}^{\infty} A_0(n,r)x^n y^r.$$

Then we have

$$\begin{aligned} F_0(x,y) &= \frac{1}{1-y} + x \sum_{n,r=0}^{\infty} \{A_0(n,r-1) + q^r A_0(n,r)\} x^n y^r \\ &= \frac{1}{1-y} + xyF_0(x,y) + xF_0(x,q,y), \end{aligned}$$

so that

$$F_0(x,y) = \frac{1}{(1-y)(1-xy)} + \frac{x}{1-xy} F_0(x,qy). \tag{3.3}$$

Iteration of (3.3) leads to

$$F_0(x,y) = \sum_{k=0}^{\infty} \frac{x^k}{(1-q^k y)(xy)_{k+1}}. \tag{3.4}$$

It follows that

$$\begin{aligned} F_0(x,y) &= \sum_{k=0}^{\infty} \frac{x^k}{1-q^k y} \sum_{j=0}^{\infty} \begin{bmatrix} j+k \\ j \end{bmatrix} (xy)^j \\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{y^{n-k}}{1-q^k y}. \end{aligned}$$

Thus

$$\sum_{r=0}^{\infty} A_0(n,r) y^r = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{y^{n-k}}{1-q^k y}. \tag{3.5}$$

Since

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{y^{n-k}}{1 - q^k y} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k \sum_{j=0}^{\infty} q^{j(n-k)} y^j$$
$$= \sum_{r=0}^{\infty} y^r \sum_{k=0}^{\min(n,r)} \begin{bmatrix} n \\ k \end{bmatrix} q^{(n-k)(r-k)},$$

it follows from (3.5) that

$$A_0(n, r) = \sum_{k=0}^{\min(n,r)} \begin{bmatrix} n \\ k \end{bmatrix} q^{(n-k)(r-k)}. \tag{3.6}$$

For example,

$$A_0(1, r) = 1 + q^r, \qquad r \geqq 1,$$
$$A_0(2, r) = 1 + (1 + q)q^{r-1}, \qquad r \geqq 2,$$
$$A_0(3, r) = 1 + (1 + q + q^2)q^{r-2} + (1 + q + q^2)q^{2r-2} + q^{3r}, \qquad r \geqq 3.$$

Note also that

$$A_0(n, r; 0, 1) = 2^n, \qquad r \geqq n.$$

Returning to the general case of (2.1), it is easily verified that

$$A(1, r) = \frac{1 - p^r}{1 - p} + \frac{p^{r+1} - q^{r+1}}{p - q}, \qquad r \geqq 0.$$

It follows that

$$\sum_{r=0}^{\infty} A(1, r) y^r = \frac{y}{(1 - y)(1 - py)} + \frac{1}{(1 - py)(1 - qy)}. \tag{3.7}$$

Put

$$F_n(y) = \sum_{r=0}^{\infty} A(n, r) y^r.$$

Then by (2.1),

$$(1 - p^{n+1} y) F_{n+1}(y) = y F_n(y) + F_n(qy)$$
$$= (y + E_y) F_n(y). \tag{3.8}$$

In particular, since

$$F_0(y) = (1 - y)^{-1},$$

it follows from (3.8) that

$$(1 - py) F_1(y) = \frac{y}{1 - y} + \frac{1}{1 - qy},$$

in agreement with (3.8). Next, for $n = 1$, (3.8) becomes

$$(1 - p^2 y) F_2(y) = y F_1(y) + F_1(qy),$$

which yields

$$(3.9)\qquad \begin{aligned} F_2(y) = {} & \frac{1}{(1-p^2y)(1-pqy)(1-q^2y)} + \frac{y}{(1-py)(1-p^2y)(1-qy)} \\ & + \frac{qy}{(1-p^2y)(1-pqy)(1-qy)} + \frac{y^2}{(1-y)(1-py)(1-p^2y)}. \end{aligned}$$

4. We shall now examine the case $p = 1$ of (2.1) in more detail. For brevity we put

$$A_1(n, r) = A(n, r; 1, q), \qquad F_{1,n}(y) = \sum_{r=0}^{\infty} A_1(n, r)y^r.$$

Then (3.8) becomes

$$(4.1)\qquad (1 - y)F_{1,n+1}(y) = yF_{1,n}(y) + F_{1,n}(qy).$$

We may write this in the form

$$F_{1,n+1}(y) = \frac{1}{1-y}(y + E_y)F_{1,n}(y).$$

Thus it is clear that

$$(4.2)\qquad F_{1,n}(y) = \left[\frac{1}{1-y}(y + E_y)\right]^n \frac{1}{1-y}.$$

It is therefore necessary to expand the operator

$$\Omega^n \equiv \left[\frac{1}{1-y}(y + E_y)\right]^n.$$

We may put

$$(4.3)\qquad \Omega^n = \sum_{k=0}^{n} C_{n,k}(y)E_y^k.$$

Then

$$\begin{aligned} \Omega^{n+1} &= \frac{1}{1-y}(y + E_y)\sum_{k=0}^{n} C_{n,k}(y)E_y^k \\ &= \frac{1}{1-y}\sum_{k=0}^{n+1} \{yC_{n,k}(y) + C_{n,k-1}(qy)\}E_y^k, \end{aligned}$$

so that

$$(4.4)\qquad C_{n+1,k}(y) = \frac{1}{1-y}\{yC_{n,k}(y) + C_{n,k-1}(qy)\}.$$

We have

$$C_{0,0}(y) = 1, \qquad C_{1,0}(y) = \frac{y}{1-y}, \qquad C_{1,1}(y) = \frac{1}{1-y}.$$

Also it follows from (4.4) that

$$C_{n,0}(y) = \frac{y^n}{(1-y)^n} \tag{4.5}$$

and

$$C_{n,n}(y) = \frac{1}{(y)_n} = \frac{1}{(1-y)(1-qy)\cdots(1-q^{n-1}y)}. \tag{4.6}$$

Using the recurrence (4.4), we get

$$C_{2,1}(y) = \frac{1}{1-y}\left(\frac{y}{1-y} + \frac{qy}{1-qy}\right),$$

$$C_{3,1}(y) = \frac{1}{1-y}\left(\frac{y^2}{(1-y)^2} + \frac{qy^2}{(1-y)(1-qy)} + \frac{q^2y^2}{(1-qy)^2}\right),$$

$$C_{3,2}(y) = \frac{1}{(1-y)(1-qy)}\left(\frac{y}{1-y} + \frac{qy}{1-qy} + \frac{q^2y}{1-q^2y^2}\right),$$

$$C_{4,1}(y) = \frac{1}{1-y}\left(\frac{y^3}{(1-y)^3} + \frac{qy^3}{(1-y)^2(1-qy)} + \frac{q^2y^3}{(1-y)(1-qy)^2} + \frac{q^3y^3}{(1-qy)^3}\right),$$

$$\begin{aligned} C_{4,2}(y) = \frac{1}{(1-y)(1-qy)}\Bigg(&\frac{y^2}{(1-y)^2} + \frac{qy^2}{(1-y)(1-qy)} + \frac{q^2y^2}{(1-qy)^2} \\ &+ \frac{q^2y^2}{(1-y)(1-q^2y)} + \frac{q^3y^2}{(1-qy)(1-q^2y)} \\ &+ \frac{q^4y^2}{(1-q^2y)^2}\Bigg), \end{aligned}$$

$$C_{4,3}(y) = \frac{1}{(1-y)(1-qy)(1-q^2y)}\left(\frac{y}{1-y} + \frac{qy}{1-qy} + \frac{q^2y}{1-q^2y} + \frac{q^3y}{1-q^3y}\right).$$

These formulas suggest the general result:

$$C_{n,k}(y) = \frac{1}{(y)_k} S_{n-k,k}(y), \tag{4.7}$$

where $S_{n,k}(y)$ denotes the complete symmetric function of degree n in the $k+1$ quantities

$$\frac{y}{1-y}, \frac{qy}{1-qy}, \ldots, \frac{q^ky}{1-q^ky}.$$

It is evident from the definition of $S_{n,k}$ that

$$S_{n+1,k}(y) = \frac{y}{1-y} S_{n,k}(y) + S_{n+1,k-1}(qy).$$

Hence, by comparison with (4.4), it is evident that (4.7) holds.

We recall that if

$$[(1 - \alpha_0 x)(1 - \alpha_1 x) \cdots (1 - \alpha_k x)]^{-1} = \sum_{n=0}^{\infty} h_n x^n,$$

then h_n is the complete symmetric function of degree n of the $k + 1$ quantities

$$\alpha_n, \alpha_1, \cdots, \alpha_k.$$

We have therefore

$$\prod_{j=0}^{k} \left(1 - \frac{q^j y}{1 - q^j y} x\right)^{-1} = \sum_{n=0}^{\infty} S_{n,k}(y) x^n \tag{4.8}$$

or, what is the same thing,

$$\frac{1}{((1 + x)y)_{k+1}} = \frac{1}{(y)_{k+1}} \sum_{n=0}^{\infty} S_{n,k}(y) x^n. \tag{4.9}$$

Since

$$\begin{aligned} \frac{1}{((1 - x)y)_{k+1}} &= \sum_{j=0}^{\infty} \begin{bmatrix} j + k \\ j \end{bmatrix} (1 + x)^j y^j \\ &= \sum_{j=0}^{\infty} \begin{bmatrix} j + k \\ j \end{bmatrix} y^j \sum_{n=0}^{j} \binom{j}{n} x^n, \end{aligned}$$

it follows that

$$\frac{1}{(y)_{k+1}} S_{n,k}(y) = \sum_{j=n}^{\infty} \binom{j}{n} \begin{bmatrix} j + k \\ j \end{bmatrix} y^j.$$

Therefore, by (4.7),

$$\frac{1}{1 - q^k y} C_{n+k,k}(y) = \sum_{j=n}^{\infty} \binom{j}{n} \begin{bmatrix} j + k \\ j \end{bmatrix} y^j. \tag{4.10}$$

Returning to (4.2) and (4.3), it is clear that

$$F_{1,n}(y) = \sum_{k=0}^{n} \frac{1}{1 - q^k y} C_{n,k}(y) = \sum_{k=0}^{\infty} \frac{1}{(y)_{k+1}} S_{n-k,k}(y).$$

It follows that

$$F_1(x, y) = \sum_{n=0}^{\infty} F_{1,n}(y) x^n = \sum_{k=0}^{\infty} \frac{x^k}{(y)_{k+1}} \sum_{n=0}^{\infty} S_{n,k}(y) x^n.$$

Therefore, by (4.9) we have

$$F_1(x, y) = \sum_{k=0}^{\infty} \frac{x^k}{((1 + x)y)_{k+1}}. \tag{4.11}$$

By means of (4.11) we can obtain an explicit formula for $A_1(n, r)$, namely,

$$A_1(n, r) = \sum_{k=0}^{n} \binom{r}{n - k} \begin{bmatrix} r + k \\ k \end{bmatrix}. \tag{4.12}$$

For $q = 1$, (4.12) reduces to

$$A(n,r;1,1) = \sum_{k=0}^{n} \binom{r}{n-k}\binom{r+k}{k}$$

in agreement with (1.3).

5. Another generalization of (1.5) that may be mentioned is

$$A_\lambda(n,r) = \lambda A_\lambda(n-1,r-1) + q^n A_\lambda(n,r-1) + q^r A_\lambda(n-1,r) \tag{5.1}$$

together with

$$A_\lambda(n,0) = A_\lambda(0,r) = 1, \qquad n \geqq 0, \quad r \geqq 0.$$

Exactly as in § 1 we find that

$$F_\lambda(x,y) = \sum A_\lambda(n,r)x^n y^r$$

satisfies

$$F_\lambda(x,y) = \frac{1}{1-\lambda xy} + \frac{1}{1-\lambda xy}(yE_x + xE_y)F_\lambda(x,y). \tag{5.2}$$

This implies

$$F_\lambda(x,y) = \sum_{n=0}^{\infty} \frac{P_n(x,y)}{(\lambda xy)_{n+1}}, \tag{5.3}$$

where $P_n(x, y)$ has the same meaning as in (1.8). Therefore, finally,

$$A_\lambda(n,r) = \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s}\begin{bmatrix} n+r-s \\ s \end{bmatrix} q^{(n-s)(r-s)}\lambda^s. \tag{5.4}$$

In particular, for $\lambda = 0$, (5.4) reduces to

$$A_0(n,r) = \binom{n+r}{n} q^{nr}. \tag{5.5}$$

It is of course easy to verify (5.5) directly.

The more general recurrence

$$B(n,r) = \lambda B(n-1,r-1) + p^r q^n B(n,r-1) + p^n q^r B(n-1,r), \tag{5.6}$$

where now

$$B(n,0) = p^{(1/2)n(n+1)}, \qquad B(0,r) = p^{(1/2)r(r+1)}, \tag{5.7}$$

can be reduced to (5.1).

Put

$$B(n,r) = p^{(1/2)n(n+1)+(1/2)r(r+1)-nr}C(n,r).$$

Then (5.6) becomes

$$C(n,r) = \lambda p^{-1}C(n-1,r-1) + (pq)^n C(n,r-1) + (pq)^r C(n-1,r), \tag{5.8}$$

where

$$C(n,0) = C(0,r) = 1.$$

Comparison of (5.8) with (5.1) leads to an explicit formula for $C(n, r)$, and therefore for $B(n, r)$. The final result is, by (5.4),

$$B(n,r) = p^{(1/2)n(n+1)+(1/2)r(r+1)-nr} \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} \begin{bmatrix} n+r-s \\ s \end{bmatrix}_{pq} \cdot (pq)^{(n-s)(r-s)} \lambda^s p^{-s}, \tag{5.9}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \frac{(1-p^n q^n)(1-p^{n-1}q^{n-1})\cdots(1-p^{n-k+1}q^{n-k+1})}{(1-pq)(1-p^2q^2)\cdots(1-p^kq^k)}.$$

In particular, when $p = q^{-1}$, (5.9) reduces to

$$B(n,r) = q^{-(1/2)n(n+1)-(1/2)r(r+1)+nr} \sum_{s=0}^{\min(n,r)} \binom{n+r-2s}{n-s} \binom{n+r-s}{s} \lambda^s q^s. \tag{5.10}$$

REFERENCES

[1] L. Carlitz, *A special functional equation*, Riv. Mat. Univ. Parma, 7 (1956), pp. 211–233.

[2] I. Kaplansky, *The asymptotic distribution of runs and consecutive elements*, Ann. Math. Statist., 16 (1945), pp. 200–203.

[3] R. G. Stanton and D. D. Cowan, *Note on a "square" functional equation*, SIAM Rev., 12 (1970), pp. 277–279.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

IMPROVABLE ESTIMATES IN SOME NON-WELL-POSED PROBLEMS FOR A SYSTEM OF ELLIPTIC EQUATIONS*

PHILIP W. SCHAEFER†

Abstract. By means of the logarithmic convexity of a suitable functional, an a priori inequality is developed which, when used with a previously derived pointwise inequality, provides new bounds for the solution functions for the Cauchy problem for the elliptic systems considered by Conlan and Trytten. In these improperly posed problems, the solution functions are assumed to be uniformly bounded. The new estimates may be improved by the Ritz method.

1. Introduction. It is well known that the Cauchy problem for elliptic equations is not well-posed in that a slight variation in the Cauchy data may result in a large variation in the solution (see [3]). However, some physical problems do give rise to Cauchy problems for elliptic equations (see [5] and [8]) and it has been shown by John [4] and Pucci [9] that under suitable conditions such problems do become stable. Consequently, assuming the solution to be uniformly bounded, Payne [6] presented a method for computing error bounds in the Cauchy problem for the Laplace equation in n dimensions.

In [2], Conlan and Trytten extended the results of Payne in obtaining pointwise bounds for solutions to the Cauchy problem for elliptic partial differential equations and elliptic systems. They treated systems of the form

$$\begin{aligned} \mathscr{L}_1 u^1 &= h_1(x, u, u_{,i}), \\ &\vdots \\ \mathscr{L}_m u^m &= h_m(x, u, u_{,i}), \end{aligned} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)$, $u = (u^1, \cdots, u^m)$, the comma i notation indicates partial differentiation with respect to x_i, and the $\mathscr{L}_j$ are uniformly elliptic operators. An example of such a system and its application is also given in [2]. The results contained therein apply (when $m = 1$) to the problem mentioned in [10] but not treated there (see p. 233 where, in fact, the right side of the single equation is void of first order derivative terms). In both of these papers the pointwise estimates cannot be improved by the Ritz method due to the definition of ε_3 (see [2] or [10] for a full explanation of the symbol) as the maximum of a set of surface integrals.

We shall determine an a priori inequality which can be used with the previously obtained pointwise inequalities of [2] to obtain new estimates on the solution functions and on the square of their gradients. This will be accomplished in a manner different from [2]. Moreover, we introduce only square integrable terms in our inequality so that under suitable conditions, such as the linearity of the h_i in (1.1) or the solution of the system of equations by the approximating functions, we are able to improve the new estimates obtained by the Ritz method.

* Received by the editors October 7, 1971, and in revised form May 12, 1972.

† Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916. This research was supported in part by a University of Tennessee Faculty Research Grant.

In addition, one can also deduce the uniqueness and stability of the solution set of (1.1) by means of this a priori inequality.

In [2] and [10], if the estimates were desired for points deep in the domain, one might be required to repeat the process given there to extend the results to a subdomain containing those points at which bounds were sought. This could compound the error considerably. We shall follow Payne [7] and introduce a family of level surfaces which may form subdomains that include those points at which bounds are sought in the initial application of the technique.

Finally, for simplicity, we shall consider only two equations in our system; the extension to m equations is immediate.

2. Notation and problem. Let D be a domain in Euclidean n-space with boundary B, a Lyapunov boundary, and let Σ be that portion of B on which Cauchy data is prescribed or, as presented below, measured within an allowable amount of error. We assume $\bar{\Sigma}$ is a C^1 surface.

Let

$$f(x) = \alpha, \qquad 0 < \alpha \leqq 1, \tag{2.1}$$

where $x = (x_1, \cdots, x_n)$, be a family of (not necessarily closed) surfaces which intersect D and form, for each α, a closed region D_α whose boundary consists only of points of Σ, denoted Σ_α, and points of the surface $f(x) = \alpha$, denoted S_α.

We assume that f is a C^2 function in $\bar{D}_1$ such that

(2.2)
(i) if $0 < \lambda < \mu \leqq 1$, then $D_\lambda \subset D_\mu$,
(ii) $|\text{grad}\, f| > \delta$ in D_1,
(iii) $\mathscr{L}_1 f \leqq 0$, $\mathscr{L}_2 f \leqq 0$, $|\mathscr{L}_1 f| \leqq c a_0 \delta^2$, $|\mathscr{L}_2 f| \leqq c b_0 \delta^2$, in D_1,

where c and δ are fixed positive constants. Here $\mathscr{L}_1$ and $\mathscr{L}_2$ are uniformly elliptic operators

$$\mathscr{L}_1 u = (a_{ij} u_{,i})_{,j}, \qquad \mathscr{L}_2 v = (b_{ij} v_{,i})_{,j}, \tag{2.3}$$

where the repeated indices denote summation, the comma notation indicates partial differentiation, and the coefficients are C^1 functions which satisfy

$$\begin{aligned} a_{ij} &= a_{ji}, \qquad a_0 \xi_i \xi_i \leqq a_{ij} \xi_i \xi_j \leqq a_1 \xi_i \xi_i, \\ b_{ij} &= b_{ji}, \qquad b_0 \xi_i \xi_i \leqq b_{ij} \xi_i \xi_j \leqq b_1 \xi_i \xi_i, \end{aligned} \tag{2.4}$$

in D for positive constants a_0, a_1, b_0, and b_1 and all real vectors $\xi = (\xi_1, \cdots, \xi_n)$. We shall assume that D_α, $0 < \alpha \leqq 1$, has nonzero volume and D_0 has zero volume.

The existence of such a family in the case $\mathscr{L}_1 = \mathscr{L}_2$ and the usefulness in forming regions D_α which may include points that are not close to Σ, but at which bounds are sought, was noted by Payne in [7]. We shall elaborate further on the situation $\mathscr{L}_1 \neq \mathscr{L}_2$ in § 4.

Consider the system

$$\begin{aligned} \mathscr{L}_1 u &= h_1(x, u, v, u_{,i}, v_{,i}), \\ \mathscr{L}_2 v &= h_2(x, u, v, u_{,i}, v_{,i}), \end{aligned} \tag{2.5}$$

where h_1 and h_2 satisfy uniform Lipschitz conditions in all but the x variables. Let u and v be C^2 functions which satisfy (2.5) in D and

$$\begin{aligned} &\int_\Sigma (u - u_0)^2 \, d\sigma \leqq \pi_1, \qquad \int_\Sigma (u_{,i} - u_i)(u_{,i} - u_i) \, d\sigma \leqq \pi_2, \\ &\int_\Sigma (v - v_0)^2 \, d\sigma \leqq \pi_3, \qquad \int_\Sigma (v_{,i} - v_i)(v_{,i} - v_i) \, d\sigma \leqq \pi_4, \end{aligned} \tag{2.6}$$

on Σ, where the quantities u_0, u_i, v_0, v_i are the respective measured values of u, $u_{,i}$, v, $v_{,i}$ on Σ and π_1, π_2, π_3, and π_4 are given bounds for the error in the measurement of the data. In addition we assume that u and v are uniformly bounded in D.

Let

$$U = u - \phi, \qquad V = v - \psi, \tag{2.7}$$

where ϕ and ψ are C^2 approximating functions. Then by (2.5) we have

$$\begin{aligned} \mathscr{L}_1 U &= h_1(x, u, v, u_{,i}, v_{,i}) - h_1(x, \phi, \psi, \phi_{,i}, \psi_{,i}) + A_1(\phi, \psi), \\ \mathscr{L}_2 V &= h_2(x, u, v, u_{,i}, v_{,i}) - h_2(x, \phi, \psi, \phi_{,i}, \psi_{,i}) + A_2(\phi, \psi), \end{aligned}$$

where

$$\begin{aligned} A_1 &= A_1(\phi, \psi) = h_1(x, \phi, \psi, \phi_{,i}, \psi_{,i}) - \mathscr{L}_1\phi, \\ A_2 &= A_2(\phi, \psi) = h_2(x, \phi, \psi, \phi_{,i}, \psi_{,i}) - \mathscr{L}_2\psi. \end{aligned}$$

In view of the Lipschitz assumption on h_1 and h_2 it follows that

$$\begin{aligned} |\mathscr{L}_1 U| &\leqq L_1|U| + L_2|V| + L_3|U_{,i}| + L_4|V_{,i}| + |A_1|, \\ |\mathscr{L}_2 V| &\leqq L_5|U| + L_6|V| + L_7|U_{,i}| + L_8|V_{,i}| + |A_2|, \end{aligned} \tag{2.8}$$

for constants $L_1, \cdots, L_8$, where $|U_{,i}|$ denotes the length of the gradient vector.

We now set

$$\begin{aligned} &\varepsilon_1 = \int_\Sigma U^2 \, d\sigma, \qquad \varepsilon_2 = \int_\Sigma V^2 \, d\sigma, \qquad \varepsilon_3 = \int_\Sigma U_{,i}U_{,i} \, d\sigma, \\ &\varepsilon_4 = \int_\Sigma V_{,i}V_{,i} \, d\sigma, \qquad \varepsilon_5 = \iint_{D_1} A_1^2 \, dx, \qquad \varepsilon_6 = \iint_{D_1} A_2^2 \, dx, \end{aligned} \tag{2.9}$$

where dx is the element of volume in D_1. Here ε_5 and ε_6 play the role of ε_3 in [2]. Finally, since u and v are assumed to be uniformly bounded, we have that

$$|U| \leqq M, \qquad |V| \leqq M, \tag{2.10}$$

for some prescribed constant M. We shall need these bounds in the final analysis.

We shall derive an a priori inequality of the form

$$\iint_{D_\alpha} (U^2 + V^2) \, dx \leqq KM^{2(1-d)}[k_i\varepsilon_i]^d, \tag{2.11}$$

where K and k_i are computable constants and d is a fixed constant between 0 and 1. Then using (2.11) with (4.3), (4.5), and (4.16) of [2], we arrive at the new pointwise estimates for U, V, $U_{,i}U_{,i}$, and $V_{,i}V_{,i}$ and ultimately for u, v, $u_{,i}u_{,i}$, and $v_{,i}v_{,i}$ by means of (2.7).

3. Derivation of inequalities. Let us define the functional F by

$$F(\alpha) = \int_0^\alpha (\alpha - \eta) \int\int_{D_\eta} [a_{ij}U_{,i}U_{,j} + U\mathscr{L}_1 U + b_{ij}V_{,i}V_{,j} + V\mathscr{L}_2 V]\, dx\, d\eta + k_i\varepsilon_i, \tag{3.1}$$

where $0 \leqq \alpha \leqq 1$ and the k_i are constants to be determined. First we show that F satisfies the differential inequality

$$FF'' - (F')^2 \geqq -C_1 FF' - C_2 F^2, \tag{3.2}$$

for computable constants C_1 and C_2, where the prime denotes differentiation with respect to α.

By differentiation,

$$F' = \int_0^\alpha \int\int_{D_\eta} [a_{ij}U_{,i}U_{,j} + U\mathscr{L}_1 U + b_{ij}V_{,i}V_{,j} + V\mathscr{L}_2 V]\, dx\, d\eta, \tag{3.3}$$

$$F'' = \int\int_{D_\alpha} [a_{ij}U_{,i}U_{,j} + U\mathscr{L}_1 U + b_{ij}V_{,i}V_{,j} + V\mathscr{L}_2 V]\, dx. \tag{3.4}$$

Using the generalized Green's identity in (3.3) and that $n_i = f_{,i}\,|\mathrm{grad}\, f|^{-1}$ on S_η, we have

$$F' = \int\int_{D_\alpha} [a_{ij}UU_{,i} + b_{ij}VV_{,i}]f_{,j}\, dx + \int_0^\alpha \int_{\Sigma_\eta} \left[U\left(\frac{\partial U}{\partial v}\right)_a + V\left(\frac{\partial V}{\partial v}\right)_b\right] d\sigma\, d\eta, \tag{3.5}$$

where

$$\left(\frac{\partial U}{\partial v}\right)_a = a_{ij}U_{,i}n_j, \qquad \left(\frac{\partial V}{\partial v}\right)_b = b_{ij}V_{,i}n_j.$$

Thus by the divergence theorem, we obtain

$$\begin{aligned} F' = {} & \frac{1}{2}\int_{S_\alpha + \Sigma_\alpha} \left[U^2\left(\frac{\partial f}{\partial v}\right)_a + V^2\left(\frac{\partial f}{\partial v}\right)_b\right] d\sigma + \int_0^\alpha \int_{\Sigma_\eta} \left[U\left(\frac{\partial U}{\partial v}\right)_a + V\left(\frac{\partial V}{\partial v}\right)_b\right] d\sigma\, d\eta \\ & - \frac{1}{2}\int\int_{D_\alpha} [U^2\mathscr{L}_1 f + V^2\mathscr{L}_2 f]\, dx. \end{aligned} \tag{3.6}$$

Now since

$$F(\alpha) = \int_0^\alpha F'(\eta)\, d\eta + k_i\varepsilon_i,$$

we get by (3.6) that

$$F(\alpha) = \frac{1}{2}\int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx + \frac{1}{2}\int_0^\alpha \int_{\Sigma_\eta} \left[U^2\left(\frac{\partial f}{\partial v}\right)_a + V^2\left(\frac{\partial f}{\partial v}\right)_b\right] d\sigma\, d\eta$$
$$+ \int_0^\alpha \int_0^\eta \int_{\Sigma_\zeta} \left[U\left(\frac{\partial U}{\partial v}\right)_a + V\left(\frac{\partial V}{\partial v}\right)_b\right] d\sigma\, d\zeta\, d\eta$$
$$- \frac{1}{2}\int_0^\alpha \int\int_{D_\eta} [U^2 \mathscr{L}_1 f + V^2 \mathscr{L}_2 f]\,dx\,d\eta + k_i\varepsilon_i,$$

where $\rho = a_{ij}f,_i f,_j$ and $\tau = b_{ij}f,_i f,_j$. Thus by (2.2) and the arithmetic mean–geometric mean inequality (abbreviated A–G inequality),

$$|xy| \leqq (1/2\gamma)x^2 + (\gamma/2)y^2, \qquad \gamma > 0,$$

there are computable constants θ_i such that

$$(3.7)\qquad \begin{aligned}(k_i - \theta_i)\varepsilon_i + \frac{1}{2}\int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx &\leqq F(\alpha)\\ &\leqq \frac{c+1}{2}\int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx + (k_i + \theta_i)\varepsilon_i.\end{aligned}$$

We choose $k_i > \theta_i$ to ensure the nonnegativity of the lower side.

Using (2.8), the A–G inequality, and (2.4) we obtain

$$(3.8)\qquad \begin{aligned}\left|\int\int_{D_\eta} (U\mathscr{L}_1 U + V\mathscr{L}_2 V)\,dx\right| &\leqq c_1 \int\int_{D_\eta} (\rho U^2 + \tau V^2)\,dx\\ &+ \frac{1}{2}\int\int_{D_\eta} (a_{ij}U,_i U,_j + b_{ij}V,_i V,_j)\,dx + \tfrac{1}{2}(\varepsilon_5 + \varepsilon_6),\end{aligned}$$

for a computable constant c_1. The A–G inequality and (3.7) allow us to conclude that there exists a determinable constant B_1 for which

$$(3.9)\qquad \left|\int_0^\alpha \int_{\Sigma_\eta} \left[U\left(\frac{\partial U}{\partial v}\right)_a + V\left(\frac{\partial V}{\partial v}\right)_b\right] d\sigma\, d\eta\right| \leqq B_1 F.$$

From (3.3), (3.8), and (3.7) it follows that

$$(3.10)\qquad \int_0^\alpha \int\int_{D_\eta} (a_{ij}U,_i U,_j + b_{ij}V,_i V,_j)\,dx\,d\eta \leqq 2F' + 2B_2 F,$$

for a computable constant B_2. Then by (3.8) and (3.10) we see that

$$(3.11)\qquad \left|\int_0^\alpha \int\int_{D_\eta} (U\mathscr{L}_1 U + V\mathscr{L}_2 V)\,dx\,d\eta\right| \leqq F' + 2B_2 F.$$

Thus we have, without recourse to an a priori inequality as was needed in [7],

$$(3.12)\qquad |F'| \leqq 3F' + 4B_2 F$$

by (3.10) and (3.11).

Furthermore, we have the inequality

$$\begin{aligned}(3.13)\qquad &\left|\int\int_{D_\alpha}[a_{ij}U_{,i}U_{,j} - 2\rho^{-1}(a_{ij}f_{,j}U_{,i})^2 + b_{ij}V_{,i}V_{,j} - 2\tau^{-1}(b_{ij}f_{,j}V_{,i})^2]\,dx\right|\\ &\leqq B_3F + B_4F',\end{aligned}$$

for computable constants B_3 and B_4. To establish (3.13), we utilize identities like (2.20) and (2.21) of [7] and then apply the A–G inequality, (2.8), and (3.7)–(3.10). We note that we shall need both an upper and lower estimate on the integral in question, whereas Payne needed only a lower estimate in his situation. Moreover, we again obtain this estimate without recourse to such an a priori inequality as was done in [7].

Finally we shall need the inequality

$$\begin{aligned}(3.14)\qquad (F')^2 \leqq &\left\{\int\int_{D_\alpha}(a_{ij}UU_{,i} + b_{ij}VV_{,i})f_{,j}\,dx\right\}^2\\ &+ 2F'\int_0^\alpha\int_{\Sigma_\eta}\left[U\left(\frac{\partial U}{\partial v}\right)_a + V\left(\frac{\partial V}{\partial v}\right)_b\right]d\sigma\,d\eta,\end{aligned}$$

which follows by means of (3.5).

We now form $FF'' - (F')^2$ and find

$$\begin{aligned}(3.15)\qquad FF'' - (F')^2 \geqq &\frac{1}{2}\int\int_{D_\alpha}(\rho U^2 + \tau V^2)\,dx\int\int_{D_\alpha}(a_{ij}U_{,i}U_{,j} + b_{ij}V_{,i}V_{,j})\,dx\\ &+ F\int\int_{D_\alpha}(U\mathscr{L}_1U + V\mathscr{L}_2V)\,dx - \int\int_{D_\alpha}(\rho U^2 + \tau V^2)\,dx\\ &\cdot\int\int_{D_\alpha}[\rho^{-1}(a_{ij}f_{,i}U_{,j})^2 + \tau^{-1}(b_{ij}f_{,i}V_{,j})^2]\,dx\\ &- \left\{\int\int_{D_\alpha}(a_{ij}UU_{,i} + b_{ij}VV_{,i})f_{,j}\,dx\right\}^2\\ &- 2F'\int_0^\alpha\int_{\Sigma_\eta}\left[U\left(\frac{\partial U}{\partial v}\right)_a + V\left(\frac{\partial V}{\partial v}\right)_b\right]d\sigma\,d\eta + \int\int_{D_\alpha}(\rho U^2 + \tau V^2)\,dx\\ &\cdot\int\int_{D_\alpha}[\rho^{-1}(a_{ij}f_{,i}U_{,j})^2 + \tau^{-1}(b_{ij}f_{,i}V_{,j})^2]\,dx.\end{aligned}$$

Here we find it expedient to add and subtract

$$(3.16)\qquad \int\int_{D_\alpha}(\rho U^2 + \tau V^2)\,dx\int\int_{D_\alpha}[\rho^{-1}(a_{ij}f_{,i}U_{,j})^2 + \tau^{-1}(b_{ij}f_{,i}V_{,j})^2]\,dx,$$

for reasons similar to those of Trytten in [10]. Basically, this is a consequence of the appearance of derivative terms in (2.8) and that the region of integration in

(3.4) is D_α. Consequently, we have

$$\begin{aligned} FF'' - (F')^2 \geqq \frac{1}{2}\int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx \int\int_{D_\alpha} [a_{ij}U_{,i}U_{,j} + b_{ij}V_{,i}V_{,j} \\ - 2\rho^{-1}(a_{ij}f_{,i}U_{,j})^2 - 2\tau^{-1}(b_{ij}f_{,i}V_{,j})^2]\,dx \\ - 2|F'|\left|\int_0^\alpha \int_{\Sigma_\eta} \left[U\left(\frac{\partial U}{\partial \nu}\right)_a + V\left(\frac{\partial V}{\partial \nu}\right)_b\right] d\sigma\,d\eta\right| + J, \end{aligned} \tag{3.17}$$

where

$$J = I - F\left|\int\int_{D_\alpha} (U\mathscr{L}_1 U + V\mathscr{L}_2 V)\,dx\right|, \tag{3.18}$$

and

$$\begin{aligned} I = \int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx \int\int_{D_\alpha} [\rho^{-1}(a_{ij}f_{,i}U_{,j})^2 + \tau^{-1}(b_{ij}f_{,i}V_{,j})^2]\,dx \\ - \left\{\int\int_{D_\alpha} (a_{ij}UU_{,i} + b_{ij}VV_{,i})f_{,j}\,dx\right\}^2, \end{aligned}$$

which is nonnegative by Schwarz's inequality.

We shall now show that $J \geqq -B_5F^2 - B_6FF'$. By Schwarz's inequality, (2.8), and (2.4), we have

$$\begin{aligned} \left\{\int\int_{D_\alpha} (U\mathscr{L}_1 U + V\mathscr{L}_2 V)\,dx\right\}^2 \leqq \int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx \left\{c_2 \int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx \right. \\ + c_3 \int\int_{D_\alpha} (a_{ij}U_{,i}U_{,j} + b_{ij}V_{,i}V_{,j})\,dx \\ \left. + c_4\varepsilon_5 + c_5\varepsilon_6 \right\}, \end{aligned}$$

for constants c_2, c_3, c_4, and c_5. Moreover, by adding and subtracting a similar expression in the braces and using (3.7), we can write

$$\begin{aligned} \left\{\int\int_{D_\alpha} (U\mathscr{L}_1 U + V\mathscr{L}_2 V)\,dx\right\}^2 \leqq 2F\left\{c_2 \int\int_{D_\alpha} (\rho U^2 + \tau V^2)\,dx + c_4\varepsilon_5 + c_5\varepsilon_6 \right. \\ + c_3 \left|\int\int_{D_\alpha} [a_{ij}U_{,i}U_{,j} + b_{ij}V_{,i}V_{,j} \right. \\ \left.\left. - 2\rho^{-1}(a_{ij}f_{,i}U_{,j})^2 - 2\tau^{-1}(b_{ij}f_{,i}V_{,j})^2]\,dx \right|\right\} \\ + 2c_3\left\{I + \left[\int\int_{D_\alpha} (a_{ij}UU_{,i} + b_{ij}VV_{,i})f_{,j}\,dx\right]^2\right\}. \end{aligned}$$

By means of (3.5) and (3.9),

$$\left| \int\int_{D_\alpha} (a_{ij}UU_{,i} + b_{ij}VV_{,i})f_{,j}\,dx \right| \leqq |F'| + B_1 F,$$

so that using (3.7), (3.13), and the A–G inequality we arrive at

$$\left\{ \int\int_{D_\alpha} (U\mathscr{L}_1 U + V\mathscr{L}_2 V)\,dx \right\}^2 \leqq 2c_3 I + F^2 + \{c_6 F + c_3(B_3 F + B_4 F')\}^2 + 2c_3(|F'| + B_1 F)^2 \tag{3.19}$$

for a positive constant c_6. Consequently, combining (3.19) and (3.18) and using the elementary inequality

$$(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2)^{1/2} \leqq |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|,$$

we arrive at

$$J \geqq I - F\{(2c_3 I)^{1/2} + F + c_6 F + c_3(B_3 F + B_4 F') + (2c_3)^{1/2}(|F'| + B_1 F)\}.$$

Now using

$$\gamma_1^2 - 2\gamma_1\gamma_2 \geqq -\gamma_2^2$$

on the first two terms, we conclude that by means of (3.12),

$$J \geqq -(1 + (1/2)c_3)F^2 - c_6 F^2 - c_3 F(B_3 F + B_4 F') - F(2c_3)^{1/2}(3F' + 4B_2 F + B_1 F),$$

that is, for computable constants B_5 and B_6,

$$J \geqq -B_5 F^2 - B_6 FF'. \tag{3.20}$$

Finally, using (3.13), (3.7), (3.12), (3.9), and (3.20) in (3.17) we see that (3.2) is satisfied.

Having verified (3.2), we define

$$\mathscr{F}(\beta) = \log F(\alpha)\beta^{-C_2/C_1^2}, \qquad \beta = \exp(-C_1\alpha), \tag{3.21}$$

and note that

$$\frac{d^2}{d\beta^2}\mathscr{F}(\beta) = \frac{C_2 F^2 + FF'' + C_1 FF' - (F')^2}{F^2 C_1^2 \beta^2} \geqq 0.$$

Consequently, $\mathscr{F}(\beta)$ is a convex function so that by Jensen's inequality [1],

$$F(\alpha) \leqq K_0[F(0)]^d[F(1)]^{1-d}, \tag{3.22}$$

where K_0 is a computable constant and d is a fixed number between 0 and 1. Thus by means of (3.7), (2.10), (2.2), and (3.1) we obtain the desired inequality (2.11).

4. Existence of level surfaces. We now consider the question of the existence of a function f which satisfies the conditions (2.2) in the case where $\mathscr{L}_1$ and $\mathscr{L}_2$ are different elliptic operators. We shall answer this question by exhibiting a function which meets the required conditions under certain assumptions on the coefficients of the differential operators.

We choose the origin O of a coordinate system in the exterior of D such that there exists a ball $K(\sigma)$ of radius σ and center O whose intersection with B is nonempty and contains no points of $B - \Sigma$. Let $r_0 > 0$ denote the radius of the ball $K(r_0)$ which is tangent to B at some point of Σ and R_0 denote the radius of the largest ball satisfying the conditions set forth for $K(\sigma)$. Let r represent the distance from O to a point P in $D \cap K(R_0)$ so that $r_0 \leqq r \leqq R_0$.

Now consider the function

$$f(x) = [r_0^{-\gamma} - r^{-\gamma}]/[r_0^{-\gamma} - R_0^{-\gamma}], \tag{4.1}$$

where γ is an undetermined positive constant. It follows readily that f satisfies the requirements of (2.1) and (2.2), provided we choose γ sufficiently large. Explicitly, if we choose γ such that

$$\begin{aligned} |a_{ij,i}x_j + a_{ii}| &\leqq (\gamma + 2)a_0, \\ |b_{ij,i}x_j + b_{ii}| &\leqq (\gamma + 2)b_0, \end{aligned} \tag{4.2}$$

for x in $D \cap K(R_0)$, then we can determine δ and c so that (2.2) is satisfied. Obviously, the family of surfaces determined by (4.1) may not be the best (nor the conditions (4.2) the weakest possible), but we have established that a family of level surfaces, such as is required, does exist.

Acknowledgment. The author would like to acknowledge conversations he had with L. E. Payne and H. A. Levine with regard to a function f satisfying (2.2). The above family is a result of these discussions which took place during the Symposium on Non-Well-Posed Problems and Logarithmic Convexity at Heriot–Watt University in Edinburgh, Scotland, in March 1972. The author also acknowledges the travel support of the National Science Foundation under Grant GP-33110 to attend this symposium.

REFERENCES

[1] E. F. BECKENBACH AND R. E. BELLMAN, *Inequalities*, Springer, Berlin, 1961.

[2] J. CONLAN AND G. N. TRYTTEN, *Pointwise bounds in the Cauchy problem for elliptic systems of partial differential equations*, Arch. Rational Mech. Anal., 22 (1966), pp. 143–152.

[3] J. HADAMARD, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale University Press, New Haven, 1923.

[4] F. JOHN, *A note on "improper" problems in partial differential equations*, Comm. Pure Appl. Math., 8 (1955), pp. 494–495.

[5] M. M. LAVRENTIEV, *Some Improperly Posed Problems of Mathematical Physics*, Springer-Verlag, New York, 1967.

[6] L. E. PAYNE, *Bounds in the Cauchy problem for the Laplace equation*, Arch. Rational Mech. Anal., 5 (1960), pp. 35–45.

[7] ———, *On a priori bounds in the Cauchy problem for elliptic equations*, this Journal, 1 (1970), pp. 82–89.

[8] C. PUCCI, *Some Topics in Parabolic and Elliptic Equations*, Lecture Ser. No. 36, Institute for Fluid Dynamics and Applied Math., College Park, Md., 1958.

[9] ———, *Sui problemi di Cauchy non "ben posti,"* Rend. Acad. Naz. Lincei, 18 (1955), pp. 473–477.

[10] G. N. TRYTTEN, *Pointwise bounds for solutions in the Cauchy problem for elliptic equations*, Arch. Rational Mech. Anal., 13 (1963), pp. 222–224.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

A CORRECTION TO LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES*

THOMAS J. OSLER†

Abstract. This paper calls attention to an error in the proofs of various extensions of the "Leibniz rule" for the fractional derivative of the product of two functions published previously by the author. The error occurs at a step where integration and summation must be interchanged, and justified. The justification requires that a new restriction be added to the functions involved. The new restriction, however, is a natural one, and in no way affects applications of the Leibniz rule previously published.

1. Introduction. Previously the author published papers which contained proofs and applications of generalized Leibniz rules for the fractional derivative of the product of two functions, $D^\alpha u(z)v(z)$ (and more generally $D^\alpha f(z, z)$), [1], [2], [5]. While exploring further generalizations, the author discovered an error in the proofs of Leibniz rule given in [1], [2], [5]. The error occurs at a stage where summation and integration are interchanged ($\int \sum = \sum \int$). This interchange does not seem to be valid unless an additional restriction is added to the hypotheses under consideration. It is the purpose of this note to state this added restriction, and to demonstrate that the interchange ($\int \sum = \sum \int$) is then valid. Fortunately the new restriction does not affect any of the applications of Leibniz rule to infinite series expansions given in [1], [2], [3], [5].

2. The new restriction. The new restriction is as follows.

Restriction. Let the singularity (if any) of $f(z, w)$ at $z = w = 0$ be such that $|f(z, w)| \leqq M|z|^p|w|^q$ for all z and w considered, where M is constant and p, q, and $p + q$ are in the interval $(-1, \infty)$.
This Restriction should be added to the hypotheses of Theorems 4.1 and 5.1 of [1]; to the hypothesis of Theorem 1, p. 664 of [2] (in which $f(z, w)$ is $u(z)v(w)$); to the hypothesis of Theorem 4.1 of [5]; and to Theorem 1, p. 290 of [3] (in which $f(z, w)$ is $u(h^{-1}(z))v(h^{-1}(w))$.

In [1], [2], [5] we required the behavior of $f(z, w)$ near the origin to be such that $\oint_C f(z, z)\,dz$, $\oint_C f(z, w)\,dz$, and $\oint_C f(z, w)\,dw$ vanish over any closed path C through the origin. Our new Restriction is stronger. Nevertheless, a function $f(z, w)$ having a singularity of the type $z^p w^q$ would still have the same restrictions placed on p and q by the integrals just mentioned as it would by our new Restriction. This is the reason why the series expansions considered in our applications of Leibniz rule are not affected by the new Restriction.

3. The corrected proof. The extended Leibniz rule given in [5] is more general than that given in [1], [2]. Thus we show where the error in reasoning occurs in [5] and how our new Restriction corrects it. The correction needed in [1], [2] is simply a special case of the one we are about to consider.

* Received by the editors March 6, 1972.

† Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York. Now at Department of Mathematics, Glassboro State College, Glassboro, New Jersey 08028.

Our restriction is needed to show that we can integrate the infinite series (4.2) of [5] term by term over the contour $C(z)$. The reason stated in [5] for this term by term integration is incorrect, as this series is not a Fourier series as the author previously assumed. According to [4, p. 44], the infinite series (4.2) of [5] can be written as a finite sum with two remainder terms in the form

$$\frac{\Gamma(\alpha+1)f(\xi,\xi)}{2\pi i(\xi-z)^{\alpha+1}} = \sum_{n=-N}^{N} \{\text{same terms as in (4.2) of [5]}\} + R_\varepsilon(N) + R_{-\delta}(N),$$

where

$$R_\varepsilon(N) = \frac{a\Gamma(\alpha+1)\theta(\xi;z)^{\gamma+a}}{-4\pi^2(\xi-z)^{\alpha+1}} \int_{C_\varepsilon} \frac{\theta(t;z)^{-\gamma-1}\{\theta(\xi;z)/\theta(t;z)\}^{aN} f(\xi,t)\theta_t(t;z)\,dt}{\theta(t;z)^a - \theta(\xi;z)^a},$$

and

$$R_{-\delta}(N) = \frac{a\Gamma(\alpha+1)\theta(\xi;z)^{\gamma+a}}{-4\pi^2(\xi-z)^{\alpha+1}} \int_{C_{-\delta}} \frac{\theta(t:z)^{a-\gamma-1}\{\theta(t;z)/\theta(\xi;z)\}^{aN} f(\xi,t)\theta_t(t;z)\,dt}{\theta(\xi;z)^a - \theta(t;z)^a}.$$

The contours of integration C_x ($x = \varepsilon$ and $-\delta$) are shown in Fig. 1, and we note that each consists of three parts: $C_x = C_1(x) + C_2(x) + C_3(x)$, where

$C_1(x)$ starts at $t = 0$ and continues to the point where $|\theta(t;z)| = |\theta(0;z)| + x$ along the curve defined by $\arg\theta(t;z) = \arg\theta(0;z)$,

$C_2(x) = \{t \mid |\theta(t;z)| = |\theta(0;z)| + x\}$,

$C_3(x) = C_1(x)$ traversed in the opposite direction.

Notice that in the notation of [5], $C(z) = C_2(0)$.

To show that we can integrate (4.2) of [5] term by term over $C(z)$ we must show that both

$$\int_{C(z)} R_\varepsilon(N)\,d\xi \quad \text{and} \quad \int_{C(z)} R_{-\delta}(N)\,d\xi$$

approach zero as N approaches infinity. Both remainders are examined in the same way; thus we shall only examine

$$\left|\int_{C(z)} R_{-\delta}(N)\,d\xi\right| \leqq \left|\int_{C(z)}\int_{C_1(-\delta)+C_3(-\delta)}\right| + \left|\int_{C(z)}\int_{C_2(-\delta)}\right|.$$

The last term above can be made arbitrarily small by taking N sufficiently large for fixed δ since

$$\left|\int_{C(z)}\int_{C_2(-\delta)}\right| \leqq \left|\frac{a\Gamma(\alpha+1)M\theta(0;z)^\gamma}{4\pi^2}\right| \max_{\xi\in C(z)} |(\xi-z)^{-\alpha-1}| \left|\frac{|\theta(0;z)|-\delta}{\theta(0;z)}\right|^{aN}$$
$$\cdot \int_{C(z)}\int_{C_2(-\delta)} \frac{|\theta(t;z)^{a-\gamma-1}\theta_t(t;z)|\,|\xi|^p|t|^q|dt|\,|d\xi|}{1-|(\theta(0;z)-\delta)/\theta(0;z)|^a}.$$

Thus we must show that

$$I = \left|\int_{C(z)}\int_{C_1(-\delta)+C_3(-\delta)}\right|$$

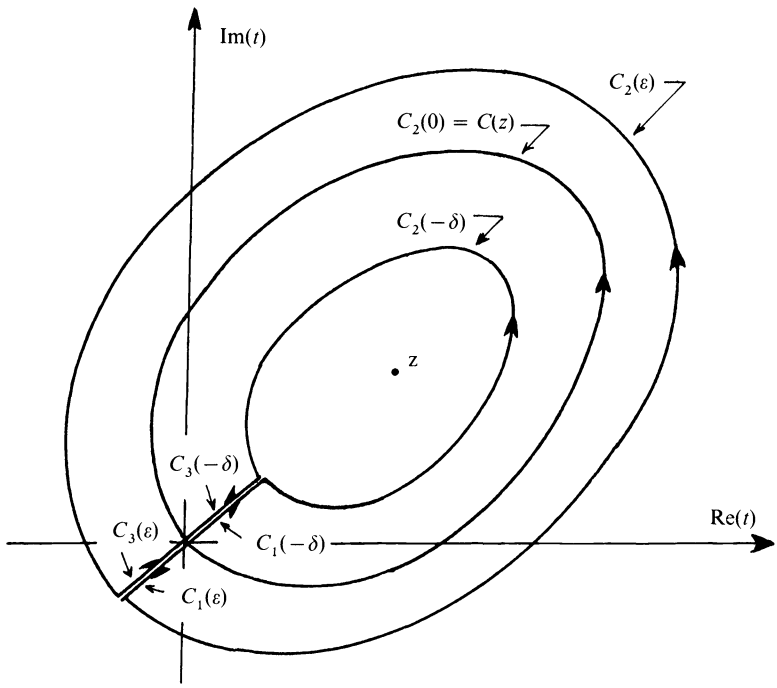

FIG. 1. *Contours of integration*

can be made arbitrarily small:

$$I \leqq \left| \frac{2aM\Gamma(\alpha + 1)\theta(0; z)^{a+\gamma}}{4\pi^2} \right| \max_{\substack{t \in C_1(-\delta) \\ \xi \in C(z)}} \left| \frac{\theta(t; z)^{a-\gamma-1}\theta_t(t; z)}{(\xi - z)^{\alpha+1}} \right|$$

$$\cdot \int_{\xi \in C(z)} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt|\,|d\xi|}{|\theta(\xi; z)^a - \theta(t; z)^a|}.$$

If this last integral exists, we can make I arbitrarily small by taking δ sufficiently small, that is, making the length of $C_1(-\delta)$ small. Note that N does not appear in this integral.

$\theta(\xi; z)^a - \theta(t; z)^a$ has a simple zero at $\xi = t \neq z$ by the argument presented in [4, p. 43]. Thus, $\theta(\xi; z)^a - \theta(t; z)^a = G(\xi, t)(t - \xi)$, where $G(\xi, t)$ is not zero on the contours of integration. Thus we must show that

$$I' = \int_{\xi \in C(z)} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt|\,|d\xi|}{|t - \xi|}$$

exists. But

$$I' = \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{t \in C_1(-\delta)} + \int_{\substack{\xi \in C(z) \\ |\xi| \geqq \delta}} \int_{t \in C_1(-\delta)}$$

The second term on the right-hand side exists because $|t - \xi|$ does not approach zero. Thus our attention focuses finally on the critical integral,

$$I'' = \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt|\,|d\xi|}{|t - \xi|},$$

which must be shown to exist. The contours $C(z)$ and $C_1(-\delta)$ are smooth and intersect at right angles at the origin. (This is because $C_1(-\delta)$ defines constant argument and $C(z)$ constant modulus of θ at the origin.) Since the contours of integration for I'' are short and nearly straight line segments, we know that β exists such that $0 < \beta \leqq |\arg(t) - \arg(\xi)| \leqq \beta + \pi/2$. Then it is clear that

$$|t - \xi| \geqq ||t| - |\xi| e^{i\beta}|,$$

and we have

$$I'' \leqq \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{t \in C_1(-\delta)} \frac{|\xi|^p |t|^q |dt|\,|d\xi|}{||t| - |\xi| e^{i\beta}|}.$$

Let $|t| = u|\xi|$. We obtain

$$I'' \leqq \int_{\substack{\xi \in C(z) \\ |\xi| \leqq \delta}} \int_{u=0}^{\infty} \frac{u^q |\xi|^{p+q}\, du |d\xi|}{|u - e^{i\beta}|}$$

which exists. Thus we have shown that we can integrate (4.2) of [5] term by term provided we add the new Restriction.

REFERENCES

[1] T. J. Osler, *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, Doctoral thesis, New York University, New York, 1970.

[2] ———, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM J. Appl. Math., 18 (1970), pp. 658–674.

[3] ———, *The fractional derivative of a composite function*, this Journal, 1 (1970), pp. 288–293.

[4] ———, *Taylor's series generalized for fractional derivatives and applications*, this Journal, 2 (1971), pp. 37–48.

[5] ———, *A further extension of the Leibniz rule to fractional derivatives and its relation to Parseval's formula*, this Journal, 3 (1972), pp. 1–15.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

SOME NONOSCILLATION THEOREMS FOR A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION*

STIG-OLOF LONDEN†

Abstract. We investigate the equation

$$[p(t)x'(t)]' + q(t)g(x(t)) = f(t)$$

and give sufficient hypotheses for the approach to zero of all nonoscillatory solutions. The conditions are related to earlier theorems of Bhatia and Hammett.

1. Introduction. In a recent paper, Hammett [2] considered the second order nonlinear, nonhomogeneous differential equation

$$(1) \qquad [p(t)x'(t)]' + q(t)g(x(t)) = f(t), \qquad 0 \leqq t < \infty,$$

where $p(t), q(t) \in C[0, \infty)$, $p(t) > 0$, $0 \leqq t < \infty$, $g(x) \in C(-\infty, \infty)$, and proved the following.

THEOREM 1. *Let*

$$(2) \qquad \int_0^\infty \frac{1}{p(\tau)} d\tau = \infty,$$

$$(3) \qquad p(t) > k > 0, \qquad q(t) > k > 0, \qquad 0 \leqq t < \infty,$$

for some constant k,

$$(4) \qquad xg(x) > 0, \qquad x \neq 0,$$

$$(5) \qquad g'(x) \geqq 0, \qquad |x| < \infty,$$

$$(6) \qquad f(t) \in C[0, \infty) \cap L_1[0, \infty).$$

If $x(t)$ *is a nonoscillatory solution of* (1), *then* $\lim_{t \to \infty} x(t) = 0$.

The proof of Theorem 1 makes use of a previous result of Bhatia [1] on the homogeneous version of (1).

In the present note we give some extensions of Hammett's result. In Theorem 2 we replace condition (3) by the weaker assumption (10). Also, $g(x)$ is not required to be monotonically nondecreasing.

Theorems 3 and 4 were motivated by the case when $p(t), q(t) \to 0, t \to \infty$. However, when allowing for this possibility we do instead have to require that $f(t)$ be small also compared to $p(t)$ and $q(t)$, in a sense made precise in the theorems. That $f(t)$ be integrable is not enough. Observe that Theorem 4 covers an equation mentioned by Hammett but not covered by his result, namely

$$[t^{-1}x']' + t^{-1}x(t) = t^{-2} + 3t^{-4},$$

which has the nonoscillatory solution $x(t) = t^{-1}$.

* Received by the editors February 15, 1972, and in revised form July 20, 1972.

† Department of Mathematics, Helsinki University of Technology, Otaniemi, Finland.

Note that (24) is trivially satisfied if $q(t)$ is sufficiently smooth and monotonically nonincreasing. Condition (25) is needed only to insure one-sided boundedness of $x'(t)$ and may be replaced by any other assumption accomplishing the same purpose.

2. Theorems and proofs.

THEOREM 2. *Let* (2), (4) *and* (6) *hold. In addition let*

$$q(t) \geqq 0, \qquad 0 \leqq t < \infty, \tag{7}$$

$$\int_0^\infty q(\tau)\,d\tau = \infty, \tag{8}$$

$$\liminf_{x\to\infty} g(x) > 0, \qquad \limsup_{x\to-\infty} g(x) < 0, \tag{9}$$

and suppose that there exists $\varepsilon > 0$ *such that if* $P \overset{\text{def}}{=} \{t \mid 0 \leqq t < \infty, p(t)q(t) < \varepsilon\}$, *then*

$$\int_P \left[\frac{\varepsilon}{p(\tau)} - q(\tau)\right] d\tau < \infty. \tag{10}$$

Finally let $x(t)$ *be a nonoscillatory solution of* (1) *on* $[0, \infty)$. *Then* $\lim_{t\to\infty} x(t) = 0$.

Proof of Theorem 2. Without loss of generality let $x(t) > 0$, $0 \leqq t < \infty$. Suppose $\liminf_{t\to\infty} x(t) > 0$. Then by (4) and the first part of (9), $g(x(t)) \geqq \delta$, $0 \leqq t < \infty$, for some positive δ. From (1) and (7) one therefore has

$$p(t)x'(t) - p(0)x'(0) \leqq -\delta \int_0^t q(\tau)\,d\tau + \int_0^t f(\tau)\,d\tau, \qquad t > 0, \tag{11}$$

which by (6) and (8) gives

$$\lim_{t\to\infty} p(t)x'(t) = -\infty. \tag{12}$$

Combining (2) and (12) yields $\lim_{t\to\infty} x(t) = -\infty$. Hence,

$$\liminf_{t\to\infty} x(t) = 0. \tag{13}$$

We assert that $\limsup_{t\to\infty} x(t) = 0$. Suppose not. Then

$$\limsup_{t\to\infty} x(t) \geqq \delta_1 > 0. \tag{14}$$

By (13) and (14) there exists $\{\hat{t}_n\}$, $\lim_{n\to\infty} \hat{t}_n = \infty$, such that $x'(\hat{t}_n) = 0$. Integrating (1), using (6) and (7) together with some straightforward arguments therefore gives

$$\lim_{t\to\infty} \int_0^t q(\tau)g(x(\tau))\,d\tau < \infty, \tag{15}$$

and

$$\lim_{t\to\infty} p(t)x'(t) = 0. \tag{16}$$

By (13) and (14) there exist $\{t_n\}$, $\{t'_n\}$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} t'_n = \infty$, such that

$$(17)\qquad x(t_n) = \frac{\delta_1}{2}, \qquad x(t'_n) = \frac{\delta_1}{4}, \qquad \frac{\delta_1}{4} \leqq x(t) \leqq \frac{\delta_1}{2}, \quad t'_n \leqq t \leqq t_n.$$

From (16) it follows that

$$(18)\qquad x'(t) \leqq \varepsilon_n/p(t), \qquad t'_n \leqq t < \infty,$$

where $\varepsilon_n \to 0$, $n \to \infty$. Let $G(x) = \int_0^x g(u)\,du$. After multiplying (18) by $g(x(t))$ and integrating over $[t'_n, t_n]$ one has, by (4) and (17),

$$(19)\qquad 0 < \delta_2 = G(x(t_n)) - G(x(t'_n)) \leqq \varepsilon_n \int_{t'_n}^{t_n} \frac{g(x(\tau))}{p(\tau)}\,d\tau,$$

for some δ_2. Thus,

$$(20)\qquad \lim_{n\to\infty} \int_{t'_n}^{t_n} \frac{g(x(\tau))}{p(\tau)}\,d\tau = \infty.$$

Invoking (15) and (20) gives

$$(21)\qquad \lim_{n\to\infty} \int_{t'_n}^{t_n} g(x(\tau))\left[\frac{\hat{\varepsilon}}{p(\tau)} - q(\tau)\right] d\tau = \infty \quad \text{for any } \hat{\varepsilon} > 0.$$

But by (4), (10) and (17),

$$\int_{t'_n}^{t_n} g(x(\tau))\left[\frac{\varepsilon}{p(\tau)} - q(\tau)\right] d\tau \leqq \int_{t'_n}^{t_n} g(x(\tau))\left[\frac{\varepsilon}{p(\tau)} - q(\tau)\right]^+ d\tau$$
$$\leqq \delta_3 \int_P \left[\frac{\varepsilon}{p(\tau)} - q(\tau)\right] d\tau < \infty,$$

which violates (21). Thus, $\limsup_{t\to\infty} x(t) = 0$.

This completes the proof.

THEOREM 3. *Let* (2), (4), (6), (8) *and* (9) *hold. In addition let*

$$(22)\qquad q(t) \in C^1[0, \infty), \qquad q(t) > 0, \quad 0 \leqq t < \infty,$$

$$(23)\qquad f(t)[q(t)]^{-1} \in L_1[0, \infty).$$

Let $Q \overset{\text{def}}{=} \{t \mid 0 \leqq t < \infty, q'(t) > 0\}$ *and let*

$$(24)\qquad \int_Q \left|\left(\frac{1}{q(\tau)}\right)'\right| d\tau < \infty.$$

Also let

$$(25)\qquad \limsup_{t\to\infty} \frac{\int_t^\infty |f(\tau)|\,d\tau}{p(t)} < \infty.$$

Finally let $x(t)$ *be a nonoscillatory solution of* (1) *on* $[0, \infty)$. *Then* $\lim_{t\to\infty} x(t) = 0$.

Proof of Theorem 3. Let $x(t) > 0, 0 \leqq t \leqq \infty$. Using the present hypothesis and arguing as in the proof of Theorem 2, one again obtains (13) and (16).

Suppose $\limsup_{t\to\infty} x(t) \geqq \delta_1 > 0$ for some δ_1. Then there exists $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, such that $x(t_n) = \delta_1/2$. Choose any such sequence and define

$$t'_n = \max\{t \mid 0 \leqq t < \infty, t < t_n, x(t) \geqq 3\delta_1/4, x'(t) = 0\},$$
$$t''_n = \min\{t \mid 0 \leqq t < \infty, t_n < t, x(t) \leqq \delta_1/4, x'(t) = 0\},$$
$$S_n = \{t \mid t'_n < t < t''_n, x'(t) < 0\}.$$

The set S_n may then be written as the union of disjoint open intervals, $S_n = \bigcup_i (t'_{in}, t''_{in})$, where

(26) $$x'(t'_{in}) = x'(t''_{in}) = 0, \qquad x'(t) < 0, \quad t'_{in} < t < t''_{in}.$$

Dividing (1) by $q(t)$, integrating over $[t'_{in}, t''_{in}]$, performing an integration by parts, using (22) and the first part of (26) gives

(27) $$\int_{t'_{in}}^{t''_{in}} g(x(\tau))\,d\tau = \int_{t'_{in}}^{t''_{in}} \frac{f(\tau)}{q(\tau)}\,d\tau - \int_{t'_{in}}^{t''_{in}} \frac{p(\tau)x'(\tau)q'(\tau)}{q^2(\tau)}\,d\tau.$$

Consider the integrand in the last term in (27). On $\{\tau \mid t'_{in} \leqq \tau \leqq t''_{in}, q'(\tau) \leqq 0\}$ the integrand is nonnegative. On $\{\tau \mid t'_{in} \leqq \tau \leqq t''_{in}, q'(\tau) > 0\}$ we use (16) and (24). Hence, by (23) and (27),

(28) $$\limsup_{n\to\infty} \int_{S_n} g(x(\tau))\,d\tau = 0,$$

which of course implies

(29) $$\limsup_{n\to\infty} \int_{T_n} g(x(\tau))\,d\tau = 0,$$

where

$$T_n = S_n \cap \left\{t \,\middle|\, \frac{\delta_1}{4} \leqq x(t) \leqq \frac{3\delta_1}{4}\right\}.$$

Note that

(30) $$\int_{T_n} x'(\tau)\,d\tau \leqq -\frac{\delta_1}{2}.$$

Integrating (1) over $[t, \hat{t}_n]$, where $\hat{t}_n$ again is such that $x'(\hat{t}_n) = 0$, gives

(31) $$x'(t) \geqq -\frac{1}{p(t)} \int_t^\infty |f(\tau)|\,d\tau,$$

for all sufficiently large t. From (25), (30) and (31) one concludes that $m(T_n) \geqq \delta_2$ (m is the Lebesgue measure) for some positive δ_2. This fact, combined with (4) and the definition of T_n, yields a contradiction to (29).

This completes the proof.

THEOREM 4. *Let* (2), (4), (6), (8), (9), (22), (24) *and* (25) *hold. In addition let*

$$\limsup_{t\to\infty} p(t)/q(t) < \infty, \tag{32}$$

$$\lim_{t\to\infty} |f(t)|/q(t) = 0. \tag{33}$$

Finally let $x(t)$ *be a nonoscillatory solution of* (1) *on* $[0, \infty)$. *Then* $\lim_{t\to\infty} x(t) = 0$.

Proof of Theorem 4. Let $x(t) > 0, 0 \leqq t < \infty$. As in the proof of Theorem 2 one obtains (13) and (16). Suppose $\limsup_{t\to\infty} x(t) \geqq \delta_1 > 0$ for some δ_1. Then it follows from (13) that there exists $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, such that

$$x(t_n) \geqq \delta_1/2, \qquad x'(t_n) < 0. \tag{34}$$

Choose any such sequence and let

$$t'_n = \max\{t | t < t_n, x'(t) = 0\},$$

$$t''_n = \min\{t | t > t_n, x'(t) = 0\}.$$

Suppose that there exist a subsequence $\{n_k\}$ of $\{n\}$ and $\delta_2 > 0$ such that $x(t''_{n_k}) \geqq \delta_2$. Then by (9) there exists $\delta_3 > 0$ such that $g(x(t)) \geqq \delta_3$, $t'_{n_k} \leqq t \leqq t''_{n_k}$. After integrating (1) over $[t'_{n_k}, t''_{n_k}]$ one therefore has

$$\delta_3 \int_{t'_{n_k}}^{t''_{n_k}} q(\tau)\, d\tau \leqq \int_{t'_{n_k}}^{t''_{n_k}} f(\tau)\, d\tau,$$

which violates (33) if n_k is sufficiently large. Thus

$$\limsup_{n\to\infty} x(t''_n) = 0. \tag{35}$$

Dividing (1) by $q(t)$, integrating over $[t'_n, t], t'_n < t \leqq t''_n$, and performing an integration by parts gives

$$\int_{t'_n}^{t} g(x(\tau))\, d\tau = \int_{t'_n}^{t} \frac{f(\tau)}{q(\tau)}\, d\tau - \frac{p(t)x'(t)}{q(t)} - \int_{t'_n}^{t} \frac{p(\tau)x'(\tau)q'(\tau)}{q^2(\tau)}\, d\tau. \tag{36}$$

If $t'''_n = \max\{t | t < t''_n, x(t) = \delta_1/2\}$, then (33), (36) and the fact that $g(x(t))$ is bounded away from zero on $[t'_n, t'''_n]$, together imply

$$\int_{t'''_n}^{t} g(x(\tau))\, d\tau \leqq \int_{t'''_n}^{t} \frac{f(\tau)}{q(\tau)}\, d\tau - \frac{p(t)x'(t)}{q(t)} - \int_{t'_n}^{t} \frac{p(\tau)x'(\tau)q'(\tau)}{q^2(\tau)}\, d\tau, \tag{37}$$

for all sufficiently large n. Suppose there exist $T < \infty$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $\sup[t''_{n_k} - t'''_{n_k}] \leqq T$. As (31) is still valid one has by (25),

$$x'(t) \geqq -k, \qquad 0 \leqq t < \infty, \tag{38}$$

and therefore the left side of (37), with $t = t''_{n_k}$, is bounded away from zero. But using (16), (24), (33) and the existence of T one has that the right side tends to zero with increasing n. This provides a contradiction. Thus $\lim_{n\to\infty}[t''_n - t'''_n] = \infty$.

By (16), (24), (32) and (37),

$$\int_{t'''_n}^{t} g(x(\tau))\, d\tau \leqq \int_{t'''_n}^{t} \frac{f(\tau)}{q(\tau)}\, d\tau - Kx'(t) + \varepsilon_n, \qquad t'''_n \leqq t \leqq t''_n, \tag{39}$$

for some constant K, and some sequence $\{\varepsilon_n\}, \varepsilon_n \to 0, n \to \infty$. Integrating (39) gives, for $t_n''' + 1 \leqq t \leqq t_n''$,

$$\int_{t_n'''+1}^{t} \left[\int_{t_n'''}^{\tau} g(x(s))\,ds\right] d\tau \leqq \int_{t_n'''+1}^{t} \varepsilon_n[\tau - t_n''']\,d\tau + \frac{K\delta_1}{2} + \varepsilon_n[t - [t_n''' + 1]], \tag{40}$$

where we have assumed, without loss of generality, $f(s) \leqq \varepsilon_n q(s)$, $t_n''' \leqq s \leqq t_n''$. Choose $\bar{t}_n$ such that $t_n''' + 1 < \bar{t}_n \leqq t_n''$, and such that if $T_n = \bar{t}_n - t_n''' - 1$, $\eta_n = \varepsilon_n T_n$, then

$$\lim_{n\to\infty} T_n = \infty, \qquad \lim_{n\to\infty} \eta_n = 0. \tag{41}$$

By (38),

$$\int_{t_n'''}^{t_n'''+1} g(x(s))\,ds \geqq \delta_4, \tag{42}$$

for some δ_4, and thus by (40), (42),

$$\delta_4 T_n \leqq \eta_n T_n + K_1,$$

for some constant K_1, which by (41) is impossible. $\operatorname{Lim\,sup}_{t\to\infty} x(t) = 0$ follows. This completes the proof.

REFERENCES

[1] N. P. BHATIA, *Some oscillation theorems for second order differential equations*, J. Math. Anal. Appl., 15 (1966), pp. 442–446.

[2] M. E. HAMMETT, *Nonoscillation properties of a nonlinear differential equation*, Proc. Amer. Math. Soc., 30 (1971), pp. 92–96.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

ASYMPTOTIC BEHAVIOR OF A CLASS OF INTEGRAL TRANSFORMS*

K. SONI AND R. P. SONI†

Abstract. Let $F(x) = \int_0^\infty k(xt)f(t)\,dt$ and let $M(k, s) = \int_0^\infty t^{s-1}k(t)\,dt$. The result, $f(t) \sim t^{-\alpha}$ as $t \to 0\,(\infty)$ implies $F(x) \sim M(k, 1-\alpha)x^{\alpha-1}$ as $x \to \infty(0)$, is proved under rather general conditions on the transform kernel $k(t)$ and under different sets of conditions on $f(t)$. Some of the results obtained extend those given by Titchmarsh for the Fourier sine and cosine transform. The case when $f(t) \sim t^{-\alpha}(\log t)^\beta$ is also discussed. It is further proved that if $M(k, 1-\alpha)$ does not exist but $M(k^*, 1-\alpha)$ does, where $k^*(t)$ is some regular mean of $k(t)$, then $f(t) \sim t^{-\alpha}$ implies $F^*(x) \sim M(k^*, 1-\alpha)x^{\alpha-1}$, $F^*(x)$ being the corresponding mean of $F(x)$. This extends some known results for the Hankel transform. Finally, the importance of the regular summability technique for obtaining the asymptotic behavior is demonstrated by some examples.

1. Introduction. Let F be the integral transform of a function f with respect to a kernel k and let $F(x)$ be defined by

$$F(x) = \int_0^\infty k(xy)f(y)\,dy, \qquad 0 < x < \infty; \tag{1}$$

or equivalently, by

$$F(x) = x^{-1}\int_0^\infty k(y)f(y/x)\,dy. \tag{2}$$

If $\phi(x) = x^\alpha f(x)$ is bounded and the integral

$$A(\alpha) = \int_0^\infty x^{-\alpha}k(x)\,dx \tag{3}$$

converges absolutely then, by the Lebesgue dominated convergence theorem, $\phi(x) \to c$ as $x \to 0$ $(x \to \infty)$ implies that

$$F(x) \sim cA(\alpha)x^{\alpha-1}, \qquad x \to \infty\ (x \to 0). \tag{4}$$

We note that the condition that $\phi(x)$ tend to a limit can be relaxed. In case $x \to \infty$, it can indeed be replaced by the condition that some Cesaro mean of $\phi(x)$ [13, p. 111] tends to a limit. The corresponding conclusion then follows from a Tauberian theorem of Wiener [21, p. 213]. On the other hand, if the assumption that the integral in (3) converges absolutely is replaced by the weaker condition that $A(\alpha)$ exists, the asymptotic behavior of $F(x)$ may no longer be given by (4). For example, consider the Hankel transform of the function

$$f(x) = x^{\nu+1}\phi(x), \qquad -1 < \nu < -1/2, \qquad \phi(x) = \exp(ix^2).$$

* Received by the editors May 21, 1971, and in revised form August 6, 1972. The work of the first author was supported in part by the University of Tennessee Faculty Research Award, Summer 1970.

† Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916.

By [5, p. 30, (12)],

$$F(x) = \int_0^\infty J_\nu(xy) y^{\nu+1} e^{iy^2}\, dy$$
$$= 2^{-\nu-1} x^\nu \exp\,[i(\nu + 1)\pi/2 - ix^2/4].$$

Obviously $\phi(x)$ is bounded, $\phi(0+) = 1$ but $F(x)$ does not satisfy (4) as $x \to \infty$.

An elementary result in the theory of the Fourier transform states that if $\phi(x)$ is of bounded variation in $0 \leqq x \leqq \delta$ for some $\delta > 0$, then

$$\int_0^\delta y^{-1}\phi(y) \sin xy\, dy \sim \frac{\pi}{2}\phi(0+), \qquad x \to \infty.$$

An extension of this result was given by Titchmarsh [20, p. 172]. For the Fourier kernels $\cos x$ and $\sin x$, he proved that if $\phi(x)$ is of bounded variation in $0 \leqq x < \infty$ and $0 < \alpha < 1$, then $F(x)$ does satisfy (4). In these cases $A(\alpha)$ exists but $x^{-\alpha}k(x)$ is not absolutely integrable in $(0, \infty)$. We obtain results which are similar to those of Titchmarsh under different sets of conditions on $\phi(x)$. Our results are applicable to the integral transforms whose kernel may satisfy rather general conditions. In particular, they apply to the Hankel transform.

The techniques used to obtain the dominant behavior or an asymptotic expansion of an integral transform differ widely depending upon the form of the kernel $k(x, y)$. Even when the transform is of the convolution type as in (1), the usefulness of a technique depends to a large extent upon the specific kernel and the singularities of f. For a comparison of the techniques, see [3, p. 50], [4], [7]–[9], [12], [17] and [19]. Doetsch [2, vol. 2, pp. 131–134] has shown that when f and k have the Mellin transforms $\mathcal{M}f(s)$ and $\mathcal{M}k(s)$ which behave in a suitable manner in the complex s-plane, Parseval's relation for the Mellin transform provides a fairly straightforward device for obtaining the asymptotic expansion of (2). Recently, Handelsman and Lew [10], [11] and [12] have studied this technique extensively.

However, we point out that in the case of oscillatory kernels like the Fourier and the Hankel, the advantage is rather limited. The Mellin transform of these kernels does not permit the shifting of the line of integration effectively unless heavy restrictions are imposed on f [12, Ex. 3].

In the second half of this paper, we consider "regular summability" in connection with the asymptotic behavior of a function. Let

$$G(x) = \int_0^\infty p(x, y) f(y)\, dy. \tag{5}$$

The kernel $p(x, y)$ is a "regular" summability kernel if (5) defines a regular transformation, that is, $f(x)$ essentially bounded and $f(x) \to M$ $(x \to \infty)$ implies $G(x) \to M$ $(x \to \infty)$. If $f(x)$ does not have a limit but $G(x)$ does, that limit is regarded as the limit of f in the extended sense. For a set of sufficient conditions that p be regular, see [13, p. 50]. It is clear that if $\phi(x) = x^\alpha f(x)$ is bounded and $p(x, y)(x/y)^\alpha$ is regular, then $G(x) \sim \phi(\infty)x^{-\alpha}$, $(x \to \infty)$, provided that $\phi(\infty)$ exists.

In (1) if f is absolutely integrable and k is essentially bounded, $F(x)$ certainly exists for all $x > 0$; but if $A(\alpha)$ defined by (3) does not exist, (4) is meaningless.

In fact, $x^{1-\alpha}F(x)$ need not have a limit as $x \to 0$ or ∞. In such a case, we may be interested in finding the limit in some extended sense. This is highly desirable when we consider the inversion of a transform. For instance, in the inversion of the Fourier transform of an absolutely integrable function f, we need the asymptotic behavior of

$$F(u,x) = \pi^{-1}\int_0^\infty [f(u+y) + f(u-y)]y^{-1}\sin xy\,dy.$$

$F(u,x)$ has the limit $(1/2)[f(u+) + f(u-)]$, $x \to \infty$, provided that f is of bounded variation at u; otherwise the limit may not exist. It is well known, however, that the above integral is summable $(C, 1)$ to $f(u)$ as $x \to \infty$, at all points u in the Lebesgue set of f [6, p. 14].

C. Nasim [16] has given the behavior of $F(x)$ near infinity in the extended sense, in terms of the behavior of $f(x)$ near zero ($f(x) \sim cx^{-\alpha}$), when F is the Hankel transform of f. His proofs are based on Parseval's relation for the Hankel transform and the corresponding regular summability kernels are related to certain Riesz means. From his proofs, however, it is not clear whether there is any definite relationship between the transform kernel and a "suitable" summability kernel. We prove a result which gives the behavior of the transform in the extended sense under certain conditions on the summability kernel. These conditions are directly related to the transform kernel. Nasim's results follow as special cases.

Finally, we work out a few examples where we obtain the dominant behavior of a transform with the help of regular summability methods. The technique is not new. Recently Shimshon Zimering [23] made a very effective use of it to obtain the behavior of the Stieltjes transform of a function f when $f \sim cx^{-\alpha}$, $\alpha \leqq 1, x \to \infty$. Our object is to show that in many cases this technique provides a simple alternative when the behavior of f is such that some other techniques, namely those involving the Mellin or the Fourier transform, may not give the best results.

2. Main results. Let f and k be measurable functions of the real variable t, $0 \leqq t < \infty$; and let F be defined by (1). We shall assume that f and k are real, since otherwise we can replace (1) by four similar integrals and consider each one of them separately.

THEOREM 2.1. *If*

(i) $f(t) = t^{-\alpha}\phi(t)$, *where* $\phi(t)$ *is of bounded variation in* $0 \leqq t < \infty$,

(ii) $t^{-\alpha}k(t)$ *is absolutely integrable in every finite interval*,

(iii) $$\int_0^{\to\infty} t^{-\alpha}k(t)\,dt = A(\alpha) = A,$$

then $F(x)$ *is defined for every* $x > 0$ *and*

$$F(x) \sim \begin{cases} A\phi(0+)x^{\alpha-1}, & x \to \infty, \\ A\phi(\infty)x^{\alpha-1}, & x \to 0+. \end{cases} \tag{6}$$

Proof. Let $h(t)$ be defined by

$$h(t) = \int_0^t u^{-\alpha}k(u)\,du. \tag{7}$$

By the assumptions (ii) and (iii), $h(t)$ is continuous and uniformly bounded in $0 \leqq t < \infty$; $h(0) = 0$ and $h(\infty) = A$. Integrating (1) by parts,

$$\begin{aligned} x^{1-\alpha}F(x) &= \int_0^\infty x(xt)^{-\alpha}k(xt)\phi(t)\,dt \\ &= A\phi(\infty) - \int_0^\infty h(xt)\,d\phi(t). \end{aligned}$$

By the dominated convergence theorem [22, p. 48] it follows that

$$\lim_{x\to 0+} x^{1-\alpha}F(x) = A\phi(\infty),$$

whereas

$$\begin{aligned} \lim_{x\to\infty} x^{1-\alpha}F(x) &= A\phi(\infty) - A\int_0^\infty d\phi(t) \\ &= A\phi(0+). \end{aligned}$$

THEOREM 2.2. *Let* $\alpha > 0$ *and let* $k(t)$ *satisfy the assumptions* (ii) *and* (iii) *of Theorem* 2.1. *If*

(i) $f(t) \to 0$ *as* $t \to \infty$, $t^{\alpha+1}f'(t)$ *is bounded for* $t > 0$ *and* $f'(t)$ *is integrable in every finite interval not ending at* $t = 0$,

(ii) $\int_1^\infty u^{-1}|h_1(u)|\,du < \infty$, *where*

$$h_1(t) = \int_t^\infty u^{-\alpha}k(u)\,du;$$

then

$$f(t) \sim ct^{-\alpha}, \quad t \to 0+ \Rightarrow F(x) \sim cAx^{\alpha-1}, \quad x \to \infty, \tag{8}$$

and similarly,

$$f(t) \sim ct^{-\alpha}, \quad t \to \infty \Rightarrow F(x) \sim cAx^{\alpha-1}, \quad x \to 0+. \tag{9}$$

Proof. Let $\phi(t) = t^\alpha f(t)$. By the assumption (i), $\phi(t)$ and $t\phi'(t)$ are bounded in $0 < t < \infty$. Define $h(t)$ as in (7) so that $h_1(t) = A - h(t)$.

$$\begin{aligned} x^{1-\alpha}F(x) &= \int_0^\infty x(xt)^{-\alpha}k(xt)\phi(t)\,dt \\ &= \int_0^\Delta \phi\left(\frac{t}{x}\right)dh(t) + \int_\Delta^\infty \phi\left(\frac{t}{x}\right)t^{-\alpha}k(t)\,dt. \end{aligned} \tag{10}$$

Integrating by parts,

$$\int_{\Delta}^{\infty} \phi\left(\frac{t}{x}\right) t^{-\alpha} k(t)\, dt = h_1(\Delta)\phi\left(\frac{\Delta}{x}\right) + \int_{\Delta}^{\infty} x^{-1} h_1(t)\phi'\left(\frac{t}{x}\right) dt$$

$$= O\{h_1(\Delta)\} + O\left\{\int_{\Delta}^{\infty} t^{-1}|h_1(t)|\, dt\right\}.$$

Hence, the second integral on the right in (10) can be made arbitrarily small uniformly in x by taking Δ large enough. As for the first integral, for a fixed Δ, $h(t)$ is absolutely continuous in $0 \leqq t \leqq \Delta$ and $\phi(t)$ is bounded and continuous in $0 < t < \infty$. By the dominated convergence theorem, if $\phi(0+)$ exists,

$$\lim_{x \to \infty} \int_0^{\Delta} \phi\left(\frac{t}{x}\right) dh(t) = \phi(0+)h(\Delta),$$

while if $\phi(\infty)$ exists,

$$\lim_{x \to 0+} \int_0^{\Delta} \phi\left(\frac{t}{x}\right) dh(t) = \phi(\infty)h(\Delta).$$

This proves the theorem.

In the next theorem we consider functions which are absolutely integrable except perhaps in some neighborhood of the origin. If $F(x)$ is the Fourier sine or cosine transform of an absolutely integrable function f, then by the Riemann–Lebesgue theorem, $F(x) \to 0$ as $x \to \infty$. For our purpose, we shall need the following result which gives the behavior of the transform for large x when the kernel is essentially bounded. This result can be obtained directly from Wiener's theorem [21, p. 213]. We give here an elementary proof of this result.

LEMMA. *Let $f(t) \in L(0, \infty)$. If $k(t)$ is essentially bounded and $k_1(t) = o(t)$, $t \to \infty$, where*

(11) $$k_1(t) = \int_0^t k(u)\, du,$$

then $F(x) = o(1), x \to \infty$. If $k_1(t) \sim ct, t \to \infty$, then

$$F(x) \sim c\int_0^{\infty} f(t)\, dt, \ x \to \infty.$$

Proof. Given $\varepsilon > 0$, first choose X so large that

(12) $$\int_X^{\infty} |f(t)|\, dt < \frac{\varepsilon}{M},$$

M being the essential upper bound of $|k|$. Next, let $\phi(t)$ be an absolutely continuous function defined in $0 \leqq t \leqq X$ such that

(13) $$\int_0^X |f(t) - \phi(t)|\, dt < \frac{\varepsilon}{M}.$$

Since

$$
(14)\qquad
\begin{aligned}
|F(x)| &= \left| \int_0^\infty f(t)k(xt)\,dt \right| \\
&\leqq \int_0^X |f(t) - \phi(t)||k(xt)|\,dt + \int_X^\infty |f(t)k(xt)|\,dt \\
&\quad + \left| \int_0^X \phi(t)k(xt)\,dt \right| \\
&\leqq 2\varepsilon + \left| \int_0^X \phi(t)k(xt)\,dt \right|,
\end{aligned}
$$

we have only to prove that if $k_1(t) = o(t)$, then the last integral tends to zero.

$$\int_0^X \phi(t)k(xt)\,dt = x^{-1}k_1(xX)\phi(X) - \int_0^X (xt)^{-1}k_1(xt)t\,d\phi(t).$$

Obviously, the integrated part tends to zero as $x \to \infty$. Next, let

$$\xi(t) = \int_0^t u\,d\phi(u).$$

Since $t^{-1}k_1(t)$ is continuous and uniformly bounded in $0 < t < \infty$ and $\phi(t)$ is absolutely continuous in $0 \leqq t \leqq X$, by the dominated convergence theorem,

$$\lim_{x\to\infty} \int_0^X (xt)^{-1}k_1(xt)\,d\xi(t) = 0.$$

This proves the first part of the Lemma. If $k_1(t) \sim ct, t \to \infty$, consider the kernel $k^*(t) = k(t) - c$. It is essentially bounded and

$$
\begin{aligned}
k_1^*(t) &= \int_0^t [k(t) - c]\,dt \\
&= o(t), \qquad t \to \infty.
\end{aligned}
$$

Hence,

$$\int_0^\infty [k(xt) - c]f(t)\,dt = o(1).$$

THEOREM 2.3. *Let $f(t) \in L(a, \infty)$ for every $a > 0$ and let $k(t)$ be essentially bounded and $k_1(t) = o(t), t \to \infty$. Further let $k(t)$ satisfy the assumptions* (ii) *and* (iii) *of Theorem* 2.1. *If $f(t) = t^{-\alpha}\phi(t)$, where $\phi(t)$ is of bounded variation in some neighborhood of the origin, then*

$$(15)\qquad F(x) = [A\phi(0+) + o(1)]x^{\alpha-1} + o(1), \qquad x \to \infty.$$

Proof. Choose $\delta > 0$ so that $\phi(t)$ is of bounded variation in $(0, \delta)$. Now define $f_1(t)$ and $f_2(t)$ as follows:

$$f_1(t) = \begin{cases} f(t), & 0 \leqq t \leqq \delta, \\ 0, & \delta < t < \infty, \end{cases}$$
$$f_2(t) = f(t) - f_1(t), \qquad 0 \leqq t < \infty.$$

Since $t^\alpha f_1(t)$ has bounded variation in $0 \leqq t < \infty$ and $f_2(t) \in L(0, \infty)$, by Theorem 2.1 and the Lemma, it follows that

$$\begin{aligned} F(x) &= \int_0^\infty k(xt) f_1(t)\,dt + \int_0^\infty k(xt) f_2(t)\,dt \\ &= [A\phi(0+) + o(1)]x^{\alpha-1} + o(1), \qquad x \to \infty. \end{aligned}$$

This proves (15).

Remarks. We observe that each one of the three theorems has independent domain of validity although these domains overlap. For example, for $t \geqq 0$, let

$$\begin{aligned} \phi_1(t) &= (1 + t^2)^{-1} \operatorname{sgn}(\sin t), \\ \phi_2(t) &= t^{-1} \sin t, \\ \phi_3(t) &= t^{-2} \sin t^2, \qquad \alpha = 0. \end{aligned}$$

The function $\phi(t) = \phi_1(t)$ satisfies the assumptions of Theorem 2.1 but not those of Theorem 2.2, whereas $\phi_2(t)$ and $\phi_3(t)$ do not satisfy the assumptions of Theorem 2.1 but $\phi_2(t)$ satisfies those of Theorem 2.2 and $\phi_3(t)$, those of Theorem 2.3.

The asymptotic behavior of $F(x)$ in (15) has two terms. These reflect the behavior of $f(t)$ near the origin and away from the origin respectively. The first term dominates if $\alpha \geqq 1$. If $\alpha < 1$, the assumption on ϕ can obviously be weakened; we need only assume that it is bounded near the origin. In that case, $f(t) \in L(0, \infty)$ and $F(x) = o(1)$, $x \to \infty$ by the Lemma. If the kernel is such that $\int_0^\infty t^{-\alpha} k(t)\,dt$ converges absolutely, then the assumption on ϕ can again be replaced by the weaker assumption that $\phi(t) \to \phi(0+)$, $t \to 0$.

Finally, in each of the theorems if $A = 0$, the results are valid when A is replaced by $o(1)$. The same is true when either $\phi(0+)$ or $\phi(\infty)$ happens to be zero.

Applications (a). The asymptotic behavior, as well as the asymptotic series expansion for the Hankel transform, has been discussed in a large number of papers. The leading term, however, can be given under much weaker hypotheses [15]. For example, let $k(t) = \sqrt{t}\, J_\nu(t)$ and let $f(t) = t^{-\alpha}\phi(t)$, where $\phi(t)$ is of bounded variation in $(0, \infty)$. By Theorem 2.1, if $0 < \alpha < \nu + 3/2$,

$$F(x) \sim 2^{-\alpha+1/2} \frac{\Gamma(\nu/2 - \alpha/2 + 3/4)}{\Gamma(\nu/2 + \alpha/2 + 1/4)} x^{\alpha-1} \begin{cases} \phi(0+), & x \to \infty, \\ \phi(\infty), & x \to 0+. \end{cases} \tag{16}$$

For $\nu = 1/2$, (16) gives the behavior of the sine transform when $0 < \alpha < 2$ and for $\nu = -1/2$ that of the cosine transform when $0 < \alpha < 1$ [20, p. 172].

(b) Erdélyi [3, p. 50] proved that if $\phi(t)$ is N times continuously differentiable for $\alpha \leqq t \leqq \beta$ and $0 < \lambda \leqq 1$, $0 < \mu \leqq 1$, then

$$\begin{aligned} F(x) &= \int_\alpha^\beta e^{ixt}(t-\alpha)^{\lambda-1}(\beta-t)^{\mu-1}\phi(t)\,dt \\ &= B_N(x) - A_N(x) + O(x^{-N}), \qquad x \to \infty. \end{aligned} \tag{17}$$

$A_N(x)$ and $B_N(x)$ are given explicitly in terms of the decreasing exponents of x. Our interest is primarily in the case $N = 0$. If $\phi(t)$ is bounded, we can apply the Riemann–Lebesgue theorem to conclude that $F(x) = o(1)$. If, however, $\phi(t)$ is of bounded variation in $\alpha \leqq t \leqq \beta$, we can do better. By a change of variable, (17) can be written as follows:

$$\begin{aligned} F(x) &= \int_0^{1/2(\beta-\alpha)} e^{ix(\alpha+u)}u^{\lambda-1}(\beta-\alpha-u)^{\mu-1}\phi(\alpha+u)\,du \\ &\quad + \int_0^{1/2(\beta-\alpha)} e^{ix(\beta-u)}(\beta-\alpha-u)^{\lambda-1}u^{\mu-1}\phi(\beta-u)\,du \\ &= e^{i\alpha x}\int_0^\infty e^{ixu}u^{\lambda-1}\phi_1(u)\,du \\ &\quad + e^{i\beta x}\int_0^\infty e^{-ixu}u^{\mu-1}\phi_2(u)\,du, \end{aligned}$$

where $\phi_1(u)$ and $\phi_2(u)$ are of bounded variation in $0 \leqq u < \infty$, $\phi_1(0+) = (\beta-\alpha)^{\mu-1}\phi(\alpha+)$ and $\phi_2(0+) = (\beta-\alpha)^{\lambda-1}\phi(\beta-)$. By Theorem 2.1, for $0 < \lambda < 1$, $0 < \mu < 1$,

$$F(x) \sim \begin{cases} x^{-\lambda}\, e^{ix\alpha + i\lambda\pi/2}\Gamma(\lambda)(\beta-\alpha)^{\mu-1}\phi(\alpha+), & \lambda < \mu, \\ x^{-\mu}\, e^{ix\beta - i\mu\pi/2}\Gamma(\mu)(\beta-\alpha)^{\lambda-1}\phi(\beta-), & \mu < \lambda, \\ x^{-\lambda}\Gamma(\lambda)(\beta-\alpha)^{\lambda-1}\left[\begin{array}{l} e^{ix\alpha+i\lambda\pi/2}\phi(\alpha+) \\ + e^{ix\beta - i\lambda\pi/2}\phi(\beta-)\end{array}\right], & \lambda = \mu. \end{cases} \tag{18}$$

In a similar manner, we can obtain the behavior of $F(x)$ if $\phi(t)$ satisfies the conditions of Theorem 2.2.

(c) The asymptotic series expansion for the Bessel functions is usually given by means of contour integration. The dominant behavior of $J_\nu(x)$ can, however, be obtained in an elementary manner by using (18). We consider the function

$$F(x) = \int_{-1}^1 e^{ixt}(1-t^2)^{\lambda+n-1}\,dt, \qquad n \geqq 0, \quad 0 < \lambda \leqq 1.$$

Applying integration by parts n times, by [1, p. 21],

$$F(x) = i^n x^{-n}\int_{-1}^1 e^{ixt}(1-t)^{\lambda-1}(1+t)^{\lambda-1}\phi(t)\,dt, \tag{19}$$

where

$$\phi(t) = \sum_{0 \leqq p \leqq n}\binom{n}{p}\frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda-p)}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda+p)}(-1)^p(1-t)^{n-p}(1+t)^p.$$

Obviously, the total variation of $\phi(t)$ is bounded in $-1 \leqq t \leqq 1$;

$$\phi(-1+) = (-1)^n\phi(1-) = 2^n\Gamma(n + \lambda)[\Gamma(\lambda)]^{-1}.$$

If $0 < \lambda < 1$, by (18),

$$F(x) \sim x^{-n-\lambda}\Gamma(n + \lambda)2^{n+\lambda}\cos(x - (\lambda + n)\pi/2).$$

If $\lambda = 1$, (19) integrated by parts gives the same results as above. The behavior of $J_\nu(x)$ is obtained by changing $(n + \lambda)$ to $\nu + 1/2$ in the integral representation

$$J_\nu(x) = \pi^{-1/2}[\Gamma(\nu + 1/2)]^{-1}(x/2)^\nu \int_{-1}^{1} e^{ixt}(1 - t^2)^{\nu - 1/2}\,dt, \quad \nu > -1/2.$$

3. Logarithmic behavior. The theorems of § 2 can be extended in an obvious manner to the functions which behave as $x^{-\alpha}(\log x)^\beta$ near zero or infinity so long as β is a positive integer. Let

$$f^*(t) = t^{-\alpha}(\log t)^m\phi(t), \tag{20}$$

$$\begin{aligned} F^*(x) &= \int_0^\infty k(xt)f^*(t)\,dt \\ &= \sum_{p=0}^{m}\binom{m}{p}(-\log x)^{m-p}\int_0^\infty k_p(xt)t^{-\alpha}\phi(t)\,dt, \end{aligned} \tag{21}$$

where

$$k_p(t) = k(t)(\log t)^p. \tag{22}$$

If $f(t) = t^{-\alpha}\phi(t)$ satisfies the assumptions on f given in any one of the theorems in § 2 and each $k_p(t)$ satisfies the assumptions on $k(t)$ in the same theorem, then

$$F^*(x) \sim x^{\alpha-1}C\sum_{p=0}^{m}\binom{m}{p}(-\log x)^{m-p}A_p,$$

C corresponds to $\phi(0+)$ or $\phi(\infty)$ as the case may be and

$$A_p = \int_0^{\to\infty} t^{-\alpha}k_p(t)\,dt = \int_0^{\to\infty} t^{-\alpha}(\log t)^p k(t)\,dt. \tag{23}$$

In particular,

$$F^*(x) \sim Cx^{\alpha-1}(-\log x)^m A_0. \tag{24}$$

The absolute convergence of the integrals in (23) is not required. The situation is quite different if $f^*(t) \sim ct^{-\alpha}|\log t|^\beta$ and β is an arbitrary real number. The Mellin transform technique is very useful when β has positive integral values, but the nonintegral values of β ($\beta > -1$) correspond to branch points in the complex plane of the Mellin transform, and so shifting of the line of integration in general presents a problem. Using a very different device, Erdélyi [4] obtained asymptotic series expansions for the integrals

$$\int_0^c (-\log t)^\lambda t^{-\alpha}e^{-\rho t}\,dt, \qquad 0 < c < 1, \quad \rho \to \infty,$$

and

$$\int_c^\infty (\log t)^\lambda t^{-\alpha} e^{-\rho t}\, dt, \qquad 1 < c < \infty, \quad \rho \to 0.$$

If the integral $\int_0^\infty k(t)t^{-s}\, dt$ converges absolutely in the interval $\alpha - \delta < s < \alpha + \delta$ for some $\delta > 0$ and $\phi(x) \to \phi(\infty)$ as $x \to \infty$ then, for $c > 1$,

$$\int_c^\infty k(xt)t^{-\alpha}(\log t)^\beta \phi(t)\, dt \sim x^{\alpha-1}(-\log x)^\beta \phi(\infty) \int_0^\infty k(t)t^{-\alpha}\, dt, \qquad x \to 0+.$$

If no such interval of convergence exists, under certain conditions, we may still be able to give some estimate of the behavior.

THEOREM 3.1. *Let* $f(t) = t^{-\alpha}|\log t|^\beta \phi(t)$, $\beta = n + \gamma$, *n being a positive integer or zero*, $0 < \gamma < 1$. *Define* $k_p(t)$ *by* (22) *for* $0 \leqq p \leqq n + 1$, *and let each* $k_p(t)$ *satisfy the assumptions on the kernel in Theorem* 2.1. *If* $\phi(t)$ *is of bounded variation* (*b.v.*) *in* (c, ∞), $c > 1$, *then*

$$F(x) = \int_c^\infty k(xt) f(t)\, dt = o(x^{\alpha-1}(\log 1/x)^{n+1}), \qquad x \to 0+;$$

and if $\phi(t)$ *is of b.v. in* $(0, c)$, $0 < c < 1$, *then*

$$F(x) = \int_0^c k(xt) f(t)\, dt = o(x^{\alpha-1}(\log x)^{n+1}), \qquad x \to \infty.$$

Proof. We note that $\psi(t) = \phi(t)|\log t|^{\gamma-1}$ is of bounded variation in $(0, c)$, $c < 1$, as well as in (c, ∞), $c > 1$. Also $\psi(0+) = \psi(\infty) = 0$. The conclusion now follows from Theorem 2.1 and (24).

4. Summability and asymptotic behavior. If $\omega(x) \geqq 0$ and $\int_0^\infty \omega(x)\, dx = 1$, then $T^{-1}\omega(x/T)$ is a totally regular summability kernel [13, p. 53]. We define the mean or average of $F(x)$ with respect to $\omega(x)$ as

(25) $$\mathscr{m}(F(x), T) = T^{-1} \int_0^\infty F(x)\omega\left(\frac{x}{T}\right) dx$$

(whenever the integral exists) and consider the behavior of this mean when $F(x)$ itself is a transform defined by (1). If $k(x)$ is essentially bounded and $f(x) \in L(0, \infty)$,

(26) $$\begin{aligned} \mathscr{m}(F(x), T) &= \int_0^\infty \omega(x)F(xT)\, dx \\ &= \int_0^\infty f(y)\, dy \int_0^\infty \omega(x)k(xTy)\, dx \\ &= \int_0^\infty f(y)K(Ty)\, dy, \end{aligned}$$

where $K(Ty) = \mathscr{m}(k(x), Ty)$. Thus the mean of $F(x)$ is the K-transform of $f(x)$, $K(x)$ being the mean of $k(x)$. In the following theorem, however, we do not assume any conditions on $\omega(x)$ except those explicitly stated.

THEOREM 4.1. *Let $f(x) \in L(a, \infty)$ for every $a > 0$. If*

(i) *for every $T > 0$, $\omega(x)f(y)k(xyT)$ is absolutely integrable in the domain $x > 0, y > 0$;*

(ii) *$f(x) = x^\beta \phi(x)$, where $\phi(x)$ is of bounded variation in $0 \leqq x \leqq \delta$ for some $\delta > 0$, $\phi(0+) \neq 0$, and $\beta + 1 \geqq 0$;*

(iii) *$\int_0^{\to\infty} x^\beta K(x)\,dx$ converges, $K(x) = x^{-1}\int_0^\infty k(u)\omega(u/x)\,du$;*

(iv) *$x^{\beta+1}K(x)$ is bounded; $\int_0^x u^{\beta+1}K(u)\,du = o(x), x \to \infty$;*

then

$$T^{-1}\int_0^\infty \omega\left(\frac{x}{T}\right)F(x)\,dx \sim \phi(0+)T^{-\beta-1}\int_0^{\to\infty} x^\beta K(x)\,dx, \qquad T \to \infty. \tag{27}$$

If the integral in (iii) converges absolutely, it is sufficient to assume that $\phi(x) \to \phi(0+)$ in (ii).

Proof. First we note that by (i) and (ii), $x^\beta K(x)$ is absolutely integrable in every finite interval. By (i) again, the change in the order of integration is justified so that

$$T^\beta\int_0^\infty \omega\left(\frac{x}{T}\right)F(x)\,dx = T^{\beta+1}\left(\int_0^\delta + \int_\delta^\infty f(y)K(Ty)\,dy\right) \qquad (0 < \delta < \infty)$$

$$= I_1 + I_2.$$

By Theorem 2.3,

$$I_1 \sim \phi(0+)\int_0^\infty y^\beta K(y)\,dy, \qquad T \to \infty,$$

while by assumptions (iv) and the Lemma of § 2,

$$I_2 = \int_\delta^\infty y^{-\beta-1}f(y)(Ty)^{\beta+1}K(Ty)\,dy$$

$$= o(1) \quad \text{as } T \to \infty.$$

This proves (27). The last assertion follows by applying the dominated convergence theorem to I_1.

We note that in the particular case when $T^{-1}\omega(x/T)$ is a regular summability kernel, (27) gives the behavior of the mean of $F(x)$ in terms of an integrability condition on the mean of $k(x)$. In the above theorem, we assume that $\beta + 1 \geqq 0$. If $k(x)$ is bounded, the case $\beta < -1$ is uninteresting because in that case, the behavior of $F(x)$ can be given directly.

5. Examples on summability and asymptotic behavior. First we give two applications of Theorem 4.1.

Example 1. Let $k(x) = J_\nu(2\sqrt{x})$, $\nu > -1$, and

$$\omega(x) = \begin{cases} cx^{\nu/2}(1-x)^{\mu-1}, & 0 < x < 1, \quad \mu > 0, \\ 0, & x > 1, \end{cases}$$

$$c^{-1} = \frac{\Gamma(\nu/2+1)\Gamma(\mu)}{\Gamma(\nu/2+\mu+1)}.$$

By [5, p. 194, (63)],

$$K(x) = \frac{\Gamma(\nu/2 + \mu + 1)}{\Gamma(\nu/2 + 1)} x^{-\mu/2} J_{\mu+\nu}(2\sqrt{x})$$

and

$$\int_0^\infty x^\beta K(x)\,dx = \frac{\Gamma(\nu/2 + \mu + 1)\Gamma(\beta + \nu/2 + 1)}{\Gamma(\nu/2 + 1)\Gamma(\mu + \nu/2 - \beta)}.$$

The above integral converges when $-1 - \nu/2 < \beta < \mu/2 - 1/4$. However, this is no restriction because if $f(x) \sim x^\beta \phi(0+)$ as $x \to 0$, $F(x)$ is defined only when $-1 - \nu/2 < \beta$. On the other hand, we can always take μ so large that $\beta < \mu/2 - 1/4$. Finally $x^{\beta+1}K(x)$ satisfies conditions (iv) of Theorem 4.1 if $-1 - \nu/2 \leqq \beta \leqq \mu/2 - 3/4$. Hence,

$$\mathscr{m}(F(x), T) \sim \phi(0+)T^{-\beta-1}\frac{\Gamma(\nu/2 + \mu + 1)\Gamma(\beta + \nu/2 + 1)}{\Gamma(\nu/2 + 1)\Gamma(\mu + \nu/2 - \beta)}, \quad -1 - \nu/2 < \beta \leqq \mu/2 - 3/4. \tag{28}$$

Since

$$\mathscr{m}(F(x), T) = \frac{\Gamma(\nu/2 + \mu + 1)}{\Gamma(\nu/2 + 1)\Gamma(\mu)} T^{-\nu/2-1} \int_0^T x^{\nu/2}\left(1 - \frac{x}{T}\right)^{\mu-1} F(x)\,dx,$$

(28) can be restated as

$$\lim_{T\to\infty} T^{-\nu/2+\beta} \int_0^T x^{\nu/2}\left(1 - \frac{x}{T}\right)^{\mu-1} F(x)\,dx = \phi(0+)\frac{\Gamma(\mu)\Gamma(\beta + \nu/2 + 1)}{\Gamma(\mu + \nu/2 - \beta)},$$
$$-1 - \nu/2 < \beta \leqq \mu/2 - 3/4, \quad \nu > -1, \quad \mu > 0.$$

If we take $\beta = \nu/2$, we extend the domain of ν for the validity of Theorem 1 in [16]. If we take $\beta = -1/4$ and $\mu = 1$, we obtain the second theorem of [16].

Example 2. Consider the well-known integral [6, pp. 12–14],

$$F(u, x) = \int_0^\delta [f(u + y) + f(u - y)]y^{-1} \sin xy\,dy, \tag{29}$$

u fixed, $f(u \pm y) \in L(0, \delta)$, $\delta > 0$. The (C, λ) limit of $F(u, x)$ involves some tedious calculations. Instead of that we consider

$$\omega(x) = \begin{cases} c(1 - x^2)^{\lambda-1}, & 0 \leqq x < 1, \lambda > 0, \\ 0, & x \geqq 1, \end{cases}$$
$$c^{-1} = 2^{-1}\sqrt{\pi}\Gamma(\lambda)[\Gamma(\lambda + 1/2)]^{-1}.$$

$T^{-1}\omega(x/T)$ is a regular summability kernel. By [14, p. 113],

$$K(y) = c\int_0^1 (1 - x^2)^{\lambda-1} \sin xy\,dx$$
$$= \Gamma(\lambda + 1/2)2^{\lambda-1/2}y^{-\lambda+1/2}\mathbf{H}_{\lambda-1/2}(y).$$

$\mathbf{H}_\nu(x)$ is the Struve function of order ν. In (29), $\beta = -1$. Considering the behavior of $\mathbf{H}_{\lambda-1/2}(x)$ near zero and infinity [14, pp. 113–115], we conclude that $y^{-1}K(y)$

is absolutely integrable in $(0, \infty)$ and $K(y)$ satisfies the conditions (iv) of Theorem 4.1. Also, it is easy to show that

$$\int_0^\infty y^{-1}K(y)\,dy = \frac{\pi}{2}.$$

Hence, by Theorem 4.1, if $f(u+0)$ and $f(u-0)$ exist,

$$\mathscr{m}(F(u,x),T) \sim \frac{\pi}{2}[f(u+)+f(u-)], \qquad T\to\infty. \tag{30}$$

If $\lambda = 1$, (30) is a consequence of the $(C, 1)$ limit of (29).

In the following examples, we make use of the regular summability method. Even though in each case the integral converges absolutely, the Lebesgue dominated convergence theorem or the Mellin convolution technique cannot be used with advantage. We need the following result due to Erdélyi [4]. Let

$$H(\rho,\alpha,\beta,\lambda) = \int_0^\infty t^{\lambda-1}\exp[\alpha t^\beta - \rho t]\,dt. \tag{31}$$

If either (i) $\alpha > 0, 0 < \beta < 1, \lambda > 0$ and $\rho \to 0+$ or (ii) $\alpha < 0, \beta < 0, \lambda$ arbitrary and $\rho\to\infty$, then

$$H(\rho,\alpha,\beta,\lambda) \sim \sqrt{\frac{2\pi}{1-\beta}}(\alpha\beta)^{(\lambda-1/2)/(1-\beta)}\rho^{(\beta/2-\lambda)/(1-\beta)}\,e^{-(1-\beta^{-1})z}, \tag{32}$$

where

$$z = (\alpha\beta)^{1/(1-\beta)}\rho^{-\beta/(1-\beta)}.$$

Example 3. Let $f(t) = e^{-a/t^b}t^{\lambda-1}\phi(t)$. If $\phi(t)$ is essentially bounded and $\phi(t)\to\phi(0+)$ as $t\to 0+, a>0, b>0, \lambda$ real, then

$$\int_0^\infty e^{-st}f(t)\,dt \sim H(s,-a,-b,\lambda)\phi(0+), \qquad s\to\infty. \tag{33}$$

In particular,

$$\int_0^\infty e^{-st}\,e^{-a/t}t^{\lambda-1}\phi(t)\,dt \sim \sqrt{\pi}a^{1/2(\lambda-1/2)}s^{-1/2(\lambda+1/2)}\,e^{-2\sqrt{as}}\phi(0+), \qquad s\to\infty.$$

Example 4. Let $f(t) = t^{\lambda-1}\,e^{\alpha t^\beta}\phi(t)$. If $\phi(t)$ is essentially bounded and $\phi(t)\to c$ as $t\to\infty, \alpha>0, 0<\beta<1, \lambda>0$, then

$$\int_0^\infty e^{-t/s}f(t)\,dt \sim cH(s^{-1},\alpha,\beta,\lambda), \qquad s\to\infty. \tag{34}$$

In particular, for $\beta = 1/2, \alpha = 2\sqrt{\gamma}$,

$$\int_0^\infty e^{-t/s}\,e^{2\sqrt{\gamma t}}t^{\lambda-1}\phi(t)\,dt \sim 2\pi^{1/2}\gamma^{\lambda-1/2}s^{2\lambda-1/2}\,e^{\gamma s}, \qquad s\to\infty. \tag{35}$$

Proof of (33) *and* (34). We note that by a change of variable,

$$\int_0^\infty e^{-st}\,e^{-a/t^b}t^{\lambda-1}\phi(t)\,dt = \int_0^\infty e^{-s/u}\,e^{-au^b}u^{-\lambda-1}\phi\left(\frac{1}{u}\right)du.$$

Let

(36) $$k_1(s,t) = e^{-s/t}\, e^{-at^b} t^{-\lambda-1}[H(s, -a, -b, \lambda)]^{-1},$$

and

(37) $$k_2(s,t) = e^{-t/s}\, e^{\alpha t^\beta} t^{\lambda-1}[H(s^{-1}, \alpha, \beta, \lambda)]^{-1}.$$

For $i = 1, 2$, $k_i(s,t) \geqq 0$ and $\int_0^\infty k_i(s,t)\,dt \to 1$ as $s \to \infty$. Furthermore, it is not difficult to verify that

$$\int_0^T k_i(s,t)\,dt \to 0 \quad \text{as } s \to \infty$$

for every fixed $T > 0$. Therefore, $k_i(s,t)$, $i = 1, 2$, are regular summability kernels [13, p. 50]. Hence, (33) and (34) follow.

In (33), $H(s, -a, -b, \lambda)$ has exponential decay. If we tried to obtain the behavior in terms of negative powers of s, we shall only get $\int_0^\infty e^{-st} f(t)\,dt = o(s^{-n})$ as $s \to \infty$. In (34), the function $f(t)$ has exponential increase and so the Mellin or the Fourier transform techniques cannot be used. In the next example, we estimate the behavior of a function with the help of the summability kernel $k_2(s,t)$.

Example 5. It was proved in [18] that

(38) $$g(x) = \frac{d}{dx}\int_0^x I_0[2\sqrt{k(x-t)}]f(t)\,dt$$

is an $L_2(0,\infty)$ solution of the integral equation

(39) $$f(x) = \frac{d}{dx}\int_0^x J_0[2\sqrt{k(x-t)}]g(t)\,dt$$

if and only if $f(x)$ is in $L_2(0,\infty)$ and the Hankel transform of $f(x)$ vanishes in $(0,k)$. Here we investigate the behavior of $g(x)$ under the assumption that the Laplace transform of $|f|$ exists for $s > 0$. Let

(40) $$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \qquad s > 0.$$

It is easy to show that the Laplace transform of $g(x)$ exists and that

(41) $$G(s) = \int_0^\infty e^{-st} g(t)\,dt = e^{k/s} F(s), \qquad s > 0.$$

Let

$$h(x) = e^{-2\sqrt{\gamma x}} g(x).$$

Since $k_2(s,t)$ (defined by (37)) is a totally regular summability kernel, using the continuous analogue of [13, Thm. 9], we obtain

$$\varliminf_{x\to\infty} h(x) \leqq \varlimsup_{s\to\infty} \int_0^\infty k_2(s,t) h(t)\,dt \leqq \varlimsup_{x\to\infty} h(x).$$

In the particular case when $\alpha = 2\sqrt{\gamma}, \beta = 1/2, \lambda = 1$, by (41),

$$\int_0^\infty k_2(s,t)h(t)\,dt = [H(s^{-1}, 2\sqrt{\gamma}, 1/2, 1)]^{-1} \int_0^\infty e^{-t/s} g(t)\,dt$$
$$= [2\sqrt{\pi\gamma} s^{3/2}\, e^{\gamma s}]^{-1}\, e^{ks} F(1/s).$$

Hence,

$$\varliminf_{x\to\infty} e^{-2\sqrt{\gamma x}} g(x) \leqq \overline{\lim_{s\to\infty}}\, 2^{-1}(\pi\gamma)^{-1/2} s^{-3/2}\, e^{(k-\gamma)s} F(1/s) \tag{42}$$
$$\leqq \overline{\lim_{x\to\infty}}\, e^{-2\sqrt{\gamma x}} g(x).$$

In [18], it was pointed out that if the Hankel transform of $f(x)$ does not vanish in $(0, k)$, it is possible for the solution $g(x)$ to be $O(x^{-\alpha} e^{2\sqrt{kx}})$ for some $\alpha > 0$ as $x \to \infty$, whereas (42) shows that so long as $F(s)$ has a zero of only finite order at the origin $s = 0$, $e^{-2\sqrt{\gamma x}} g(x)$ is unbounded as $x \to \infty$ for every γ, $0 < \gamma < k$. The connection between these two results is apparent if we note that under certain conditions,

$$\int_0^\infty e^{-sx}\,dx \int_0^\infty J_0(2\sqrt{xy}) f(y)\,dy = s^{-1} F\left(\frac{1}{s}\right).$$

Acknowledgment. We wish to thank Dr. John S. Lew for his many valuable comments that considerably improved the presentation. Among his many suggestions were the use of Stieltjes integrals and the present form of the proofs of Theorems 2.1–2.3.

REFERENCES

[1] E. T. Copson, *Asymptotic Expansions*, Cambridge University Press, Cambridge, 1965.
[2] G. Doetsch, *Handbuch der Laplace-transformation*, vols. I, II, III, Birkhauser, Basel, 1950–1956.
[3] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956.
[4] ———, *General asymptotic expansions of Laplace-integrals*, Arch. Rational Mech. Anal., 7 (1961), pp. 1–20.
[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, vol. 2, McGraw-Hill, New York, 1954.
[6] R. R. Goldberg, *Fourier Transforms*, Cambridge University Press, Cambridge, 1961.
[7] J. L. Griffith, *A theorem concerning the asymptotic behavior of Hankel transforms*, J. Proc. Roy. Soc. New South Wales, 88 (1955), pp. 61–65.
[8] ———, *On the asymptotic behavior of Hankel transforms*, Ibid., 88 (1955), pp. 71–76.
[9] C. C. Grosjean, *On the series expansion of certain types of Fourier integrals in the neighborhood of the origin*, Bull. Soc. Math. Belg., 17 (1965), pp. 251–418.
[10] R. A. Handelsman and J. S. Lew, *Asymptotic expansion of Laplace transforms near the origin*, this Journal, 1 (1970), pp. 118–130.
[11] ———, *Asymptotic expansion of a class of integral transforms via Mellin transforms*, Arch. Rational Mech. Anal., 35 (1969/70), pp. 382–396.
[12] ———, *Asymptotic expansions of a class of integral transforms with algebraically dominated kernels*, J. Math. Anal. Appl., 35 (1971), pp. 405–433.
[13] G. H. Hardy, *Divergent Series*, Oxford University Press, London, 1956.
[14] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.
[15] G. E. McKelvey, *Hankel transform: Asymptotic behavior and Lipschitz conditions*, Master's thesis, The University of Tennessee, Knoxville, 1971.

[16] C. NASIM, *Some results relating the behavior of Fourier transforms near the origin and at infinity*, Canad. J. Math., 21 (1969), pp. 942–950.

[17] PAUL W. SCHMIDT, *Asymptotic expansion of certain integrals containing the Bessel function* $J_0(x)$, J. Mathematical Phys., 5 (1964), pp. 1183–1184.

[18] K. SONI, *An integral equation with Bessel function kernel*, Duke Math. J., 38 (1971), pp. 175–180.

[19] F. STENGER, *Transform methods for obtaining asymptotic expansions of definite integrals*, this Journal, 3 (1972), pp. 20–30.

[20] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Oxford University Press, London, 1948.

[21] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

[22] A. C. ZAANEN, *Linear Analysis*, North-Holland, Amsterdam, 1964.

[23] S. ZIMERING, *Some asymptotic behavior of Stieltjes transforms*, J. Mathematical Phys., 10 (1969), pp. 181–183.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

UNIFORM ASYMPTOTIC EXPANSIONS OF CERTAIN CLASSES OF MEIJER G-FUNCTIONS FOR A LARGE PARAMETER*

JERRY L. FIELDS†

Abstract. Asymptotic estimates of certain Meijer G-functions are derived using contour integration techniques. Making use of these results, a basis of solutions is found for a particular difference equation satisfied by certain hypergeometric functions, known as extended Jacobi functions. Asymptotic expansions for these functions are then derived.

Introduction. In Theorems 1 and 2 of this paper, uniform asymptotic expansions for two classes of Meijer G-functions [3],

$$g_n(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G_{p+2,q}^{q,1}\left(w \middle| \begin{matrix} 1-n-\lambda, a_1, \cdots, a_p, n+1 \\ b_1, \cdots, b_q \end{matrix}\right),$$

(A)

$$l_{n,j}(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G_{p+2,q}^{q,2}\left(w \middle| \begin{matrix} 1-n-\lambda, a_j, a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_p, n+1 \\ b_1, \cdots, b_q \end{matrix}\right),$$

$$\beta = q - p \geqq 3, \quad j = 1, \cdots, p,$$

are derived for n large, essentially positive, and w suitably restricted. In particular, if w is nonzero and independent of n, the expansion for $g_n(w)$ is derived when $|\arg[wn(n+\lambda)]| < \pi[\beta + 1]$.

While studying the asymptotic behavior of the coefficients of the expansion of a Meijer G-function in a series of Jacobi polynomials of inverted argument, Wimp [16] determined the asymptotic representation for the special case

$$\frac{\Gamma(n)}{\Gamma(n+1-\lambda)} g_{n-\lambda}(w) = (-1)^n G_{p+3,q+1}^{q+1,1}\left(w \middle| \begin{matrix} 1, a_1, \cdots, a_p, n+1-\lambda, 1-n \\ 1, b_1, \cdots, b_q \end{matrix}\right),$$

$$\beta = q - p \geqq 3, \quad n = 0, 1, 2, \cdots,$$

when $n \to \infty$, w is independent of n, and $|\arg w| < \pi\beta/2$. In the same context, Miller [14] and Németh [15] investigated special $\beta = 3$ cases. Miller's results were derived for $n^2 w \to \infty$, $w = O(n)$, $n \to \infty$ and $|\arg w| < \pi\beta/2$.

Alternately, in a series of papers [17], [7], [8], Wimp, Luke and Fields showed that the functions (A) satisfied a certain explicit linear difference equation,

$$\text{(B)} \qquad \mathscr{M}_n(w)\{y_n(w)\} = 0,$$

---

* Received by the editors February 10, 1972.

† Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada. This work was initiated for the United States Air Force, AF 33(615)-2764, while the author was at the Midwest Research Institute in Kansas City, Missouri, and completed at the University of Alberta under National Research Council of Canada Grant NRC A 7549. The major portion was completed while the author was a visiting professor at the National Tsing Hua University, Taiwan, Republic of China.

which is also satisfied by the extended Jacobi functions,

$$ {}_{p+3}F_q\left(\begin{matrix} -n, n+\lambda, 1-a_1, \cdots, 1-a_p, 1 \\ 1-b_1, \cdots, 1-b_q \end{matrix}\middle| w\right), \qquad n \text{ arbitrary}, $$

whose asymptotic properties with respect to n were discussed in [4], [5]. As a basis of solutions for (B) can be constructed from the $g_n(w)$ and $l_{n,j}(w)$, these asymptotic properties can be deduced from Theorems 1 and 2 (see § 3).

The results of Theorems 1 and 2 have already been used in [9] to derive rational approximations to certain Meijer G-functions.

1. Asymptotic expansions for $g_n(w)$ and $l_{n,j}(w)$. In this section, we will principally be interested in using the method of steepest descent [2] to find the asymptotic expansion of contour integrals of the form

$$ J(\Omega, w) = \int_C t^\gamma \exp\{-\Omega(t + \rho^{-1}t^{-\rho})\} G(t, \Omega, w)\, dt, $$

where γ, $\rho(>0)$ are constants, the complex parameter Ω is large, and C is a suitable contour in the t-plane containing $t = 1$. To this end, we describe the steepest descent curves through $t = 1$,

$$ \Gamma_\phi : \operatorname{Im}\{e^{i\phi}\sqrt{t - 1 + \rho^{-1}(t^{-\rho} - 1)}\} \equiv 0, \qquad -\pi \leqq 2\phi \leqq \pi, $$

where Γ_0 is the nonnegative real axis with positive orientation, and Γ_ϕ is required to vary continuously with $2\phi(\neq \pm\pi)$ for ρ fixed, $t \neq 0, \infty$. Then for $2|\phi| < \pi$, Γ_ϕ starts at $t = 0 \cdot \exp\{i2\phi\rho^{-1}\}$, passes through $t = 1$ at angle $(-\phi)$ with Γ_0 and ends at $t = \infty \cdot \exp\{-i2\phi\}$. Alternatively, in polar coordinates, $t = r\,e^{i\theta}$, these contours are among the solutions of

$$ r \sin(2\phi + \theta) + \rho^{-1}r^{-\rho} \sin(2\phi - \rho\theta) = (1 + \rho^{-1}) \sin(2\phi). $$

The contours Γ_ϕ, $2\phi = \pm\pi$, are singular (see Figs. 1–4).

Much of what follows is just a uniform treatment of Wright [18].

PROPOSITION 1 (Wright). *Let*

$$ f(t) = t - 1 + \rho^{-1}(t^{-\rho} - 1), \qquad \rho > 0, $$

$$ g(t) = \sqrt{\frac{2f(t)}{(\rho + 1)(t - 1)^2}}, \qquad g(1) = 1, $$

$$ x(t) = \sqrt{((\rho + 1)/2)}(t - 1)g(t) = \sqrt{f(t)}, $$

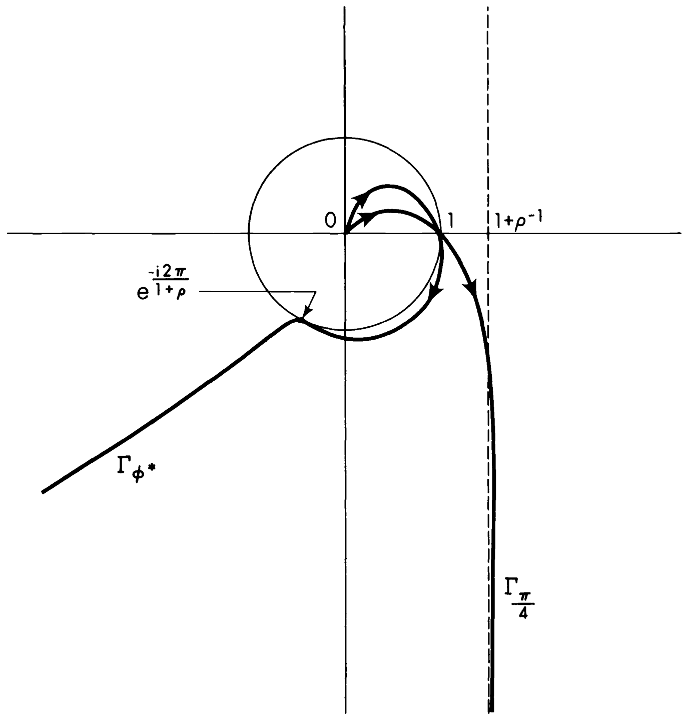

FIG. 1. *t-plane*, $1 < \rho$

and

$$\phi^* = \frac{\pi}{4} + \frac{\pi}{2+2\rho}\min(1,\rho) \leqq \frac{\pi}{2}.$$

Then for $|\phi| \leqq \phi^* - \varepsilon, \varepsilon > 0$,

(i) $x(t)$ *is an analytic function of* t *for* $t \in \Gamma_\phi, t \neq 0$,

(ii) $x(t)$ *maps the* t *contour* Γ_ϕ *onto the* x *contour*

$$\Gamma'_\phi : x = Re^{-i\phi}, \qquad -\infty < R < \infty,$$

(iii)

$$x(e^{i2\pi/(\rho+1)}) = -\sqrt{\frac{2(\rho+1)}{\rho}\sin\left(\frac{\rho\pi}{1+\rho}\right)}\exp\left\{-i\left(\frac{\pi}{4} + \frac{\rho\pi}{2+2\rho}\right)\right\},$$

$$x(e^{-i2\pi/(\rho+1)}) = \sqrt{\frac{2(\rho+1)}{\rho}\sin\left(\frac{\pi}{1+\rho}\right)}\exp\left\{-i\left(\frac{\pi}{4} + \frac{\pi}{2+2\rho}\right)\right\},$$

(iv) $t = t(x)$ *is an analytic function of* x *for* $x \in \Gamma'_\phi$,

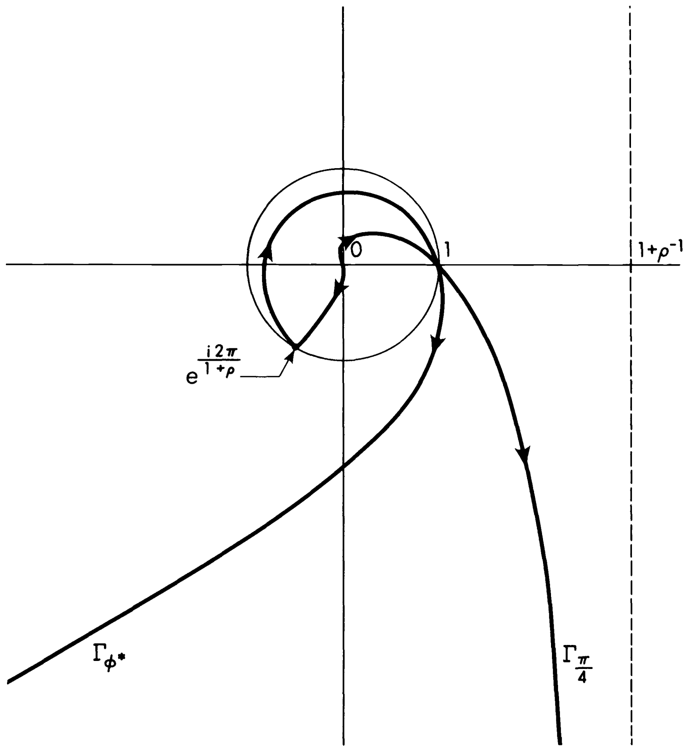

FIG. 2. *t-plane*, $0 < \rho < 1$

(v) $$t = 1 + \sqrt{\frac{2}{\rho + 1}}x + \frac{\rho + 2}{3(\rho + 1)}x^2 + \frac{(\rho + 2)(2\rho + 1)}{18(\rho + 1)\sqrt{2(\rho + 1)}}x^3 + O(x^4),$$

$$x \to 0,$$

(vi) $dt/dx = 2x/(1 - t^{-\rho-1}) = O((1 + |x|))$, *uniformly for* $x \in \Gamma'_\phi$,

(vii) $t^\gamma = O((1 + |x|)^{\gamma_0}), \gamma_0 = \max(2 \operatorname{Re} \gamma, -2\rho^{-1} \operatorname{Re} \gamma) \geqq 0$, *uniformly for* $x \in \Gamma'_\phi$.

Proof. Clearly $x(t)$ is analytic at $t = 1$. As the zeros of $f'(t) = 1 - t^{-\rho-1}$ all lie on the unit t-circle and simple computations show that $f'(t) \neq 0, t \neq 1$, $t \in \Gamma_\phi, |\phi| \leqq \phi^* - \varepsilon$, an application of Rolle's theorem to the real-valued function $e^{i2\phi}f(t)$ on Γ_ϕ implies that $f(t) \neq 0, t \neq 1, t \in \Gamma_\phi, |\phi| \leqq \phi^* - \varepsilon$. Properties (i)–(vi) then follow by explicit computation. For (vii), we note that

$$|t|^{-\rho} \leqq \rho|x|^2 + 2\rho + 1 \leqq \rho(|x| + \sqrt{(2\rho + 1)/\rho})^2, \qquad |t| \leqq 1,$$

$$|t| \leqq |x|^2 + 1 + 2\rho^{-1} \leqq (|x| + \sqrt{(\rho + 2)/\rho})^2, \qquad |t| \geqq 1.$$

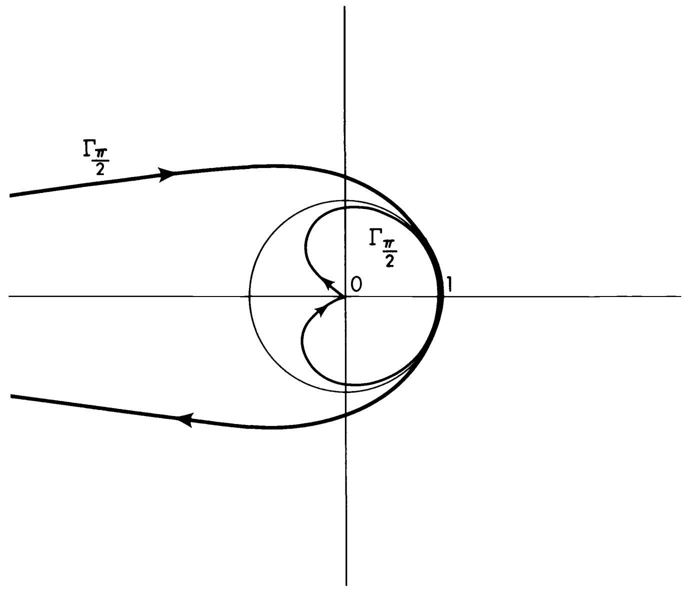

FIG. 3. *t-plane*, ▬, $0 < \rho < 1$, —, $1 < \rho$

PROPOSITION 2. *Let* $t, x, \Gamma_\phi, \phi^*$ *have the same significance as in Proposition* 1, *and let*

$$J(\Omega, w) = \int_{\Gamma_\phi} t^\gamma \exp\{-\Omega(t + \rho^{-1}t^{-\rho})\} H(t, \Omega, w)\, dt, \tag{1.1}$$

$$|\arg \Omega - 2\phi| \leqq \pi/2 - \varepsilon, \quad |\phi| \leqq \phi^* - \varepsilon, \quad \varepsilon > 0,$$

where Ω *is a large parameter,* γ *is a complex constant and* $H(t, \Omega, w)$ *is an analytic function of t except, perhaps, at* $t = 0$ *and* $t = \infty$. *Assume that for all m sufficiently large,* ϕ *fixed,* $|\phi| \leqq \phi^* - \varepsilon$ *and* $y = x\sqrt{\Omega}$,

$$H(t, \Omega, w) = 1 + \sum_{j=1}^{m-1} Q_j(y, w)\Omega^{-j/2} + O(\Omega^{-m/2}(1 + |w|)^{m\Delta}(1 + |y|)^m\, e^{|cy^2|} \tag{1.2}$$

$$\Omega \to \infty, \quad \textit{uniformly for } x \in \Gamma'_\phi, \quad |c| < \sin \varepsilon,$$

where c, Δ *are constants, and* $Q_j(y, w)$ *is a polynomial in* y, w *whose degrees in* y *and* w, $\partial_y Q_j(y, w)$ *and* $\partial_w Q_j(y, w)$, *are* $\leqq j$ *and* $j\Delta$, *respectively. Then there exists a sequence of polynomials* $\tilde{Q}_j(y, w)$ *satisfying the same degree estimates as the* $Q_j(y, w)$ *such that*

$$J(\Omega, w) = \sqrt{\frac{2\pi}{(\rho + 1)\Omega}}\, e^{-(1+\rho^{-1})\Omega} \tag{1.3}$$

$$\cdot \left\{1 + \sum_{j=1}^{m-1} \Omega^{-j/2} \int_{-\infty}^{\infty} e^{-y^2} \tilde{Q}_j(y, w)\, dy + O\left(\left[\frac{(1 + |w|)^{2\Delta}}{\Omega}\right]^{m/2}\right)\right\},$$

$$\Omega \to \infty, -\pi/2 + 2\phi + \varepsilon \leqq \arg \Omega \leqq \pi/2 + 2\phi - \varepsilon.$$

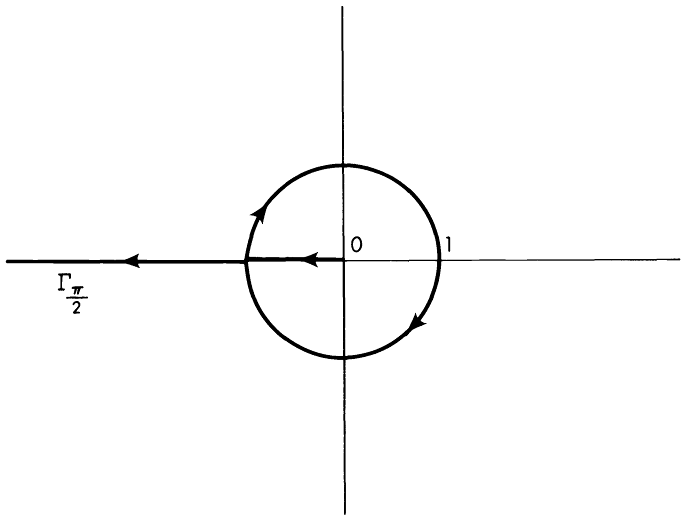

FIG. 4. *t-plane*, $\rho = 1$

In particular, if (1.2) *is valid for* $\phi = 0, \pm(\phi^* - \varepsilon)$, (1.3) *is valid uniformly for* $|\arg \Omega| \leqq \pi[1 + (\min(1, \rho))/(1 + \rho)] - 3\varepsilon$.

Proof. The saddle points of $\exp\{-\Omega(t + \rho^{-1}t^{-\rho})\}$ occur at $t = \exp\{i2\pi q(1 + \rho)^{-1}\}$, q an integer. In the range of Ω under consideration, only the saddle point at $t = 1$ is pertinent. From Proposition 1 with $t = t(x)$, it follows that there exist numbers d_j such that for $m \geqq 1 + \gamma_0$,

$$\sqrt{2^{-1}(\rho + 1)}t^{\gamma}\frac{dt}{dx} = 1 + \sum_{j=1}^{m-1} d_j x^j + O(x^m),$$

$$\text{uniformly for } x \in \Gamma'_{\phi}, \quad |\phi| \leqq \phi^* - \varepsilon, \tag{1.4}$$

$$d_1 = \frac{2(\rho + 3\gamma + 2)}{3(\rho + 1)}\sqrt{\frac{\rho + 1}{2}}, \qquad d_2 = \frac{(\rho + 2)(2\rho + 1) + 12\gamma(\gamma + \rho + 1)}{12(\rho + 1)}.$$

Then for m sufficiently large, (1.2) and (1.4) can be multiplied together to yield, $y = x\sqrt{\Omega}$,

$$\sqrt{\frac{\rho + 1}{2}}t^{\gamma}\frac{dt}{dx}H(t, \Omega, w)$$

$$= 1 + \sum_{j=1}^{m-1} \Gamma(1/2)\tilde{Q}_j(y, w)\Omega^{-j/2} + O(\Omega^{-m/2}(1 + |w|)^{m\Delta}(1 + |y|)^m e^{|cy^2|}),$$

$$\Omega \to \infty, \quad \text{uniformly for } x \in \Gamma'_{\phi}, \tag{1.5}$$

$$\Gamma(1/2)\tilde{Q}_j(y, w) = \sum_{k=0}^{j} d_k y^k Q_{j-k}(y, w), \qquad d_0 = Q_0(y, w) = 1,$$

where $\partial_y \tilde{Q}_j(y, w) \leqq j$ and $\partial_w \tilde{Q}_j(y, w) \leqq j\Delta$. Substituting (1.5) into (1.1) with $2\omega = \arg \Omega - 2\phi$, one obtains

$$J(\Omega, w) = \sqrt{\frac{2\pi}{(\rho + 1)\Omega}}\, e^{-(1+\rho^{-1})\Omega} \left\{1 + \sum_{j=1}^{m-1} \Omega^{-j/2} \int_{-\infty e^{i\omega}}^{\infty e^{i\omega}} e^{-y^2} \tilde{Q}_j(y, w)\, dy + R_m\right\},$$

(1.6)

$$R_m = \Omega^{-m/2}(1 + |w|)^{m\Delta} \int_{-\infty e^{i\omega}}^{\infty e^{i\omega}} \exp\left[-y^2(1 - |c|\, e^{-i2\omega})\right](1 + |y|)^m O(1)\, dy,$$

which clearly reduces to (1.3) for m sufficiently large. But, as the jth term in (1.6) can be estimated explicitly by $O(\Omega^{-j/2}(1 + |w|)^{j\Delta})$, $\Omega \to \infty$, (1.3) is valid for arbitrary m.

Remark 1. If $\tilde{Q}_j(-y, w) = (-1)^j \tilde{Q}_j(y, w)$, then all the odd terms in (1.3) are zero, and it is more natural to replace m by $2m$.

Remark 2. If $H(t, \Omega, w)$ is identically 1, Proposition 2 with $\arg \Omega = 0$ is related, under a change of variable, to a problem proposed in the SIAM Review by McNeil [13] and solved by Lansing [11]. Also, in this special case,

$$J(\Omega, w) = \frac{\Omega^{-1-\gamma}}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(-u)\Gamma(1 + \gamma - \rho u)\left(\frac{\Omega^{1+\rho}}{\rho}\right)^u du,$$

where the contour runs from $-i\infty$ to $i\infty$ and lies to the left of the poles of $\Gamma(-u)\Gamma(1 + \gamma - \rho u)$. Thus $J(\Omega, w)$ is then a Wright generalized hypergeometric function and the results of Proposition 2 in this special case can be deduced from Braaksma [1].

In what follows, n will always be taken as a large parameter such that $\arg n \to 0$ as $n \to \infty$. All unqualified order estimates are tacitly assumed to satisfy the condition "as $n \to \infty$."

THEOREM 1. *Let the parameters a_j, b_j, λ be independent of the large parameter n and satisfy the conditions*

$$a_j - a_k \neq \text{an integer}, \qquad j, k = 1, \cdots, p, \quad j \neq k.$$

Then for $\beta = q - p \geqq 3$, and

$$w = o(N^{2\mu}), \quad (wN^2)^{-1} = o(1),$$

$$N^2 = n(n + \lambda), \qquad 2\mu = \max\left(\tfrac{2}{3}, \tfrac{4}{5}\beta - 2\right) < \beta - 2,$$

there exist polynomials $S_j(w)$ such that for m arbitrary,

$$g_n(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G^{q,1}_{p+2,q}\left(w \middle| \begin{matrix} 1-n-\lambda, a_1, \cdots, a_p, n+1 \\ b_1, \cdots, b_q \end{matrix}\right)$$

$$= \sqrt{\frac{(2\pi)^{\beta-1}}{\beta}} \Omega^{\beta\gamma} \exp\{-\beta\Omega + 2\Omega P(v_0)\}$$

$$\cdot \left\{1 + \sum_{j=1}^{m-1} S_j(w)\Omega^{-j} + O(\Omega^{-m}(1+|w|)^{2\Delta m})\right\},$$

$$\Omega \to \infty, \quad |\arg \Omega^\beta| \leqq \pi[\beta+1] - \varepsilon, \quad \varepsilon > 0,$$

$$\Omega^\beta = wN^2, \quad 2\beta\gamma = 1 - \beta + 2B_1 - 2A_1,$$

$$P(v_0) = \frac{\beta-2}{2}\left[1 - \left(\frac{\sinh v_0}{v_0}\right)^{-2/(\beta-2)}\right]$$

$$= \frac{(v_0)^2}{6} - \frac{(\beta+3)(v_0)^4}{180(\beta-2)} + O((v_0)^6), \qquad v_0 \to 0,$$

$$v_0 = \sqrt{w\Omega^{2-\beta}}, \quad \Delta = \min\left(\frac{2}{3}, \frac{1}{2\beta-5}\right),$$

$$S_1(w) = \frac{[3(1-\lambda) + 2(A_1 - B_1)]w}{6\Gamma(4-\beta)} + A_2 - B_2$$

$$+ \frac{(B_1 - A_1)}{2\beta}\{\beta(B_1 + A_1) + (A_1 - B_1)\} + \frac{1-\beta^2}{24\beta},$$

$$\prod_{j=1}^{p}(x + a_j) = \sum_{j=0}^{p} A_j x^{p-j}, \qquad \prod_{j=1}^{q}(x + b_j) = \sum_{j=0}^{q} B_j x^{q-j},$$

where the degree of $S_j(w) \leqq 2\Delta j$, and Ω, N are positive when w, n, and λ are positive.

Proof. From the fundamental Mellin–Barnes integral representation

$$g_n(w) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} w^s \frac{\Gamma(n+\lambda+s)\Gamma(n+1)\prod_{j=1}^{q}\Gamma(b_j - s)}{\Gamma(n+\lambda)\Gamma(n+1-s)\prod_{j=1}^{p}\Gamma(a_j - s)}\,ds, \quad w > 0, \tag{1.7}$$

the contour running from $-i\infty$ to $i\infty$, separating the poles of $\Gamma(n+\lambda+s)$ from the poles of the $\Gamma(b_j - s)$, and the beta integral

$$\int_0^\infty e^{-u(2x+y-1)}(\sinh u)^{y-1}\,du = 2^{-y}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \operatorname{Re} x, \quad \operatorname{Re} y > 0, \tag{1.8}$$

it follows that

$$g_n(w) = \frac{2\Gamma(1/2)\Gamma(n+1)}{\Gamma(n+\lambda)} \int_0^\infty e^{-v(2n+\lambda)} [\sinh v]^{-\lambda} G_0(X(v,w))\, dv,$$

(1.9)
$$w > 0, \quad X(v,w) = w(\sinh v)^{-2},$$

$$G_0(X) = G^{q,0}_{p+2,q}\left(X \middle| \begin{matrix} a_1, \cdots, a_p, & \frac{1-\lambda}{2}, \frac{2-\lambda}{2} \\ b_1, \cdots, b_q & \end{matrix}\right).$$

As

(1.10)
$$G_0(X) = \sum_{j=1}^{q} X^{b_j} F_j(X),$$

where the $F_j(X)$ are entire functions of X, and

$$G_0(X) = \sqrt{\frac{(2\pi)^{\nu-1}}{\nu}} X^{\tau} \exp\{-\nu X^{1/\nu}\}\left\{1 + \sum_{j=1}^{m-1} K_j X^{-j/\nu} + X^{-m/\nu} E_m(X)\right\},$$

$$|X| \geqq 1, \qquad |\arg X| \leqq \pi[\nu + \min(1, \nu/2) - \eta], \quad \eta > 0,$$

(1.11)
$$\nu = \beta - 2 \geqq 1, \qquad 2\nu\tau = 2\beta\gamma + 2\lambda - 1 = 1 - \nu + 2B_1 - 2\tilde{A}_1,$$
$$K_1 = \tilde{A}_2 - B_2 + \frac{(B_1 - \tilde{A}_1)}{2\nu}[\nu(\tilde{A}_1 + B_1) + \tilde{A}_1 - B_1] + \frac{1-\nu^2}{24\nu},$$

$$\left(x + \frac{1-\lambda}{2}\right)\left(x + \frac{2-\lambda}{2}\right) \sum_{j=1}^{p} (x + a_j) = \sum_{j=0}^{p+2} \tilde{A}_j x^{p+2-j},$$

where the remaining K_j are polynomials in $\tilde{A}_j$, B_j independent of X, and $E_m(X)$ is analytic and bounded in the above sector [12], the contour in (1.9) can be deformed to represent $g_n(w)$ for different ranges of $\arg w$. In particular, if we set $v = tv_0$,

(1.12)
$$v_0 = \sqrt{w\Omega^{-\nu}} = (wN^{-\nu})^{1/\beta} = \Omega N^{-1} = o(1),$$
$$N^{-2} = w\Omega^{-\beta}, \qquad 2n + \lambda = \sqrt{4N^2 + \lambda^2}, \qquad \rho = 2\nu^{-1},$$

and

$$H(t, \Omega, w) = \sqrt{\frac{\nu}{(2\pi)^{\nu-1}}} \frac{N^{\lambda-1}\Gamma(n+1)}{\Gamma(n+\lambda)} [X(v,w)]^{-\tau}[Y(v)]^{-\lambda-2\tau}$$

(1.13)
$$\cdot \exp\{v(2N - 2n - \lambda) + 2\rho^{-1}\Omega t^{-\rho} - 2\Omega P(v_0)\} G_0(X(v,w)),$$

$$X(v,w) = wv^{-2}[Y(v)]^{-2}, \quad Y(v) = v^{-1}\sinh v,$$

equation (1.9) can be rewritten in the form

$$g_n(w) = \sqrt{\rho}(2\pi)^{1/\rho}\Omega^{\beta\gamma+1/2}\, e^{2\Omega P(v_0)} \int_{\Gamma_\phi} t^{-\lambda-2\tau} \exp\{-2\Omega(t + \rho^{-1}t^{-\rho})\} H(t, \Omega, w)\, dt, \tag{1.14}$$

$$|\arg \Omega - 2\phi| \leqq \pi/2 - \varepsilon, \quad |\phi| < \pi/2,$$

which can be identified with $J(2\Omega, w)$ in Proposition 2. To complete the identification with Proposition 2, it would only be necessary to show that $H(t, \Omega, w)$ can be expanded in a series of the form (1.2) for $t \in \Gamma_\phi$, $\phi = 0, \pm(\phi^* - \varepsilon)$. The analysis associated with Γ_0 is relatively straightforward, but technical problems arise in connection with the other contours. To simplify these technical problems, we introduce the modified contour $\tilde{\Gamma}_\phi$ as is indicated in Fig. 5.

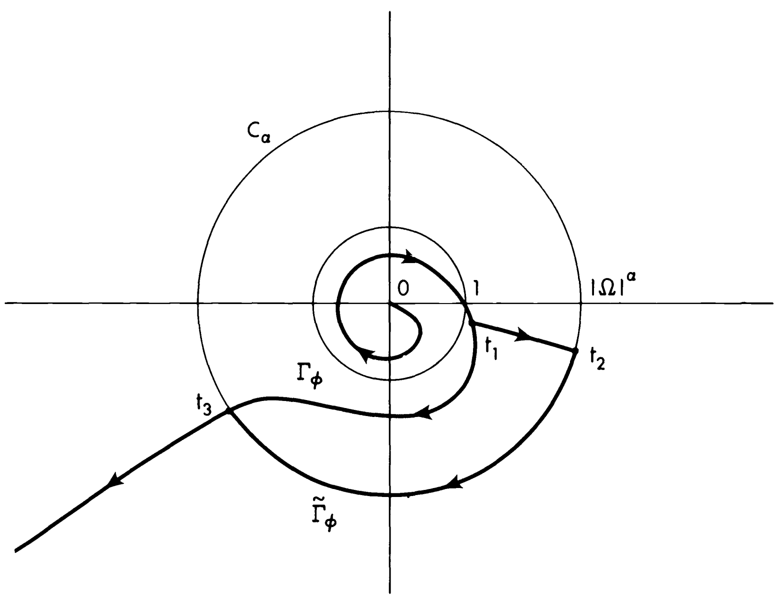

FIG. 5. *t-plane*

Then $\tilde{\Gamma}_\phi$ varies from Γ_ϕ only in the path taken from the points t_1 to t_3. From t_1 to t_2, $\tilde{\Gamma}_\phi$ is a straight line with $\arg t_1 = \arg t_2$, and from t_2 to t_3, $\tilde{\Gamma}_\phi$ is an arc of the circle $C_\alpha : |t| = |\Omega|^\alpha$, $8\alpha = \min(2, v)$. The point t_1 is independent of the large parameter, and taken sufficiently close to $t = 1$ to satisfy certain conditions that occur in the body of the proof. Clearly, the proof of Proposition 2 remains valid for these modified contours, provided the expansions (1.2) are valid uniformly for $t \in \tilde{\Gamma}_\phi$. A simple computation shows that C_α is always interior to $C_0 : |tv_0| = |v| \leqq 1$. The following lemma collects various estimates pertinent to C_α and C_0.

LEMMA 1. *With the above notation*, $8\alpha = \min(2, v)$, $|\arg \Omega - 2\phi| \leqq \pi/2 - \varepsilon$, *and* $|\arg w| < \pi[\beta + 1]$, *the following estimates are easily proved.*

$$v = tv_0 = o(1), \qquad v^{-1}\sinh v = 1 + o(1), \qquad |t| \leqq |\Omega|^\alpha < |v_0|^{-1}.$$
$$X = \Omega^\nu t^{-2}\{1 + o(1)\}, \qquad |t| \leqq |\Omega|^\alpha < |v_0|^{-1}.$$
$$\arg t = -2\phi + o(1), \qquad X = o(y^{2\nu}), \qquad |t| \geqq |\Omega|^\alpha, \quad t \in \Gamma_\phi.$$
$$4 \leqq 5|v^{-1}\sinh v| \leqq 6, \qquad 16\,|\arg(v^{-1}\sinh v)| < \pi, \qquad |tv_0| = |v| \leqq 1.$$
$$2\,|\arg v| \leqq \pi - \varepsilon, \qquad |e^{-2v}| \leqq e^{-\sin\varepsilon} < 1, \qquad |tv_0| = |v| \geqq 1, \quad t \in \Gamma_\phi.$$
$$2^{-1}(1 - e^{-\varepsilon/2}\, e^{\varepsilon|v|/4} \leqq |\sinh v| \leqq e^{|v|}, \qquad |tv_0| = |v| \geqq 1, \quad t \in \Gamma_\phi.$$
$$|\arg(v^{-1}\sinh v)| < \pi + |v|, \qquad |tv_0| = |v| \geqq 1, \quad t \in \Gamma_\phi.$$
$$t = O(1) + o(y^2), \qquad |\arg X| < \pi[\beta + 3] + 2|v|, \qquad \textit{uniformly for } t \in \tilde{\Gamma}_\phi.$$
$$X^b = O(e^{c|y^2|}), \quad b \textit{ a constant}, \quad 6c = \sin\varepsilon, \quad |t| \geqq |\Omega|^\alpha, \quad \textit{uniformly for } t \in \tilde{\Gamma}_\phi.$$

Thus X is large for t interior to C_α, and small for t exterior to C_0 near ∞. For $X = e^{-i2\gamma}$,

$$v = tv_0 = \log\{W + \sqrt{1 + W^2}\}, \qquad W = e^{i\gamma}\sqrt{w}, \tag{1.15}$$

so that the points at which $|X| = 1$ are exterior to C_α. Since

$$\beta \arg \Omega = \arg w + 2\arg N = \arg w + o(1),$$

it is sufficient to establish the theorem for $|\arg\Omega| \leqq \pi[1 + \beta^{-1}] - \varepsilon$. As the product of two series of the form (1.2) is again a series of the same form, provided the sum of the corresponding $|c|$'s is less than $\sin\varepsilon$, it is sufficient to show that each factor of $H(t, \Omega, w) = \prod_{j=1}^5 H_j$,

$$H_1 = \frac{N^{\lambda-1}\Gamma(n+1)}{\Gamma(n+\lambda)},$$
$$H_2 = \exp\{\nu(wv^{-2})^{1/\nu} - \nu X^{1/\nu} - 2\Omega P(v_0)\},$$
$$H_3 = \exp\{v(2N - 2n - \lambda)\},$$
$$H_4 = (v^{-1}\sinh v)^{-\lambda - 2\tau},$$
$$H_5 = \sqrt{\nu(2\pi)^{1-\nu}}X^{-\tau}\exp\{\nu X^{1/\nu}\}G_0(X),$$

can be written in the form

$$1 + \sum_{j=1}^{2m-1} Q_j(y, w)\zeta^j + O(\zeta^{2m}(1 + |w|)^{2\Delta m}(1 + |y|)^{2m}\, e^{c|y^2|}),$$

$$t = t(x), \qquad x = y\zeta, \qquad \zeta\sqrt{2\Omega} = 1, \qquad v = tv_0, \qquad 6c = \sin\varepsilon, \tag{1.16}$$
$$\partial_y Q_j(y, w) \leqq j, \qquad \partial_w Q_j(y, w) \leqq j\Delta,$$
uniformly for $t \in \tilde{\Gamma}_\phi$, $\phi = 0, \pm(\phi^* - \varepsilon)$.

Moreover, the H_j expansions are invariant under $(y, \zeta) \to (-y, -\zeta)$, so that

$$Q_j(y, w) = (-1)^j Q_j(-y, w), \tag{1.17}$$

and this property is preserved under multiplication of series (1.16). In what follows, $Q_j(y, w)$ will be used generically to denote a polynomial in y, w satisfying the restrictions in (1.16) and (1.17). Also, $q_{k,j}(v_0)$ and $q_j(v_0)$ will be used to denote analytic functions of v_0^2 at $v_0 = 0$. The following relations are basic:

$$\begin{aligned} &N^{-2} = 2^{\beta}\zeta^{2\beta}w, \qquad (v_0)^2 = 2^{\nu}\zeta^{2\nu}w, \\ (1.18) \qquad &\zeta w^{\Delta} = o(1), \qquad w^{2\Delta} = o(\Omega), \qquad w\zeta^{2\beta} = o(1), \\ &\Omega = o(N^{(2\mu+2)/\beta}) = o(N), \qquad 2\beta\Delta\mu = 1 + \mu. \end{aligned}$$

As $t = t(x)$ is analytic at $x = 0$, $t(0) = 1$, there exists a positive number x_0 independent of w and N such that in $|x| \leqq 2x_0$, $|t(x) - 1| \leqq 2^{-1}$ and $X^{-1/\nu}$, H_j, $j = 2, 3, 4$, are analytic functions of x. Since at $t = 1$, $|\arg X| \leqq \pi[\nu + \nu\beta^{-1} + o(1)] < \pi[\nu + \min(1, \nu/2) - 2\eta]$, for η sufficiently small, x_0 can also be chosen sufficiently small such that $|\arg X| \leqq \pi[\nu + \min(1, \nu/2) - \eta]$ for all $t(x)$, $|x| \leqq x_0$. Finally, $t_1 \in \Gamma_\phi$ is chosen interior to the image of $0 < |x| \leqq x_0$.

In [6], it is shown that there exist constants $e_1, \cdots, e_{m-1}$ such that

$$H_1 = N^{\lambda-1}\frac{\Gamma(n+1)}{\Gamma(n+\lambda)} = 1 + \sum_{j=1}^{m-1} e_j N^{-2j} + O(N^{-2m}),$$

which is clearly a series of the form (1.16), with $Q_j(y, w) = 0$, $1 \leqq j \leqq 2\beta - 1$. The analysis of H_2 is much more typical, and we proceed in some detail. Rewrite H_2 as

$$H_2 = \exp\{2\Omega[P(t, v_0) - P(1, v_0)]\}, \qquad P(1, v_0) = P(v_0),$$

$$\begin{aligned} P(t, v_0) &= \rho^{-1}t^{-\rho}[1 - (v^{-1}\sinh v)^{-\rho}] \\ &= \frac{(v_0)^2}{6}t^{2-\rho} - \frac{(5\rho + 2)(v_0)^4}{360}t^{4-\rho} + \frac{(35\rho^2 + 42\rho + 16)(v_0)^6}{45360}t^{6-\rho} + \cdots, \end{aligned}$$

where the series converges for $|tv_0| < \pi$, which includes $|t - 1| \leqq 2^{-1}$. Then for $|x| \leqq x_0$,

$$\begin{aligned} H_2 &= 1 + \sum_{j=1}^{2m-1} x^j \sum_{k=1}^{j} (\Omega v_0^{2\mu_1})^k q_{k,j}(v_0) + O\left(x^{2m} \max_{|x| = 2x_0} |H_2|\right), \\ \mu_1 &= \max(1, 3 - \nu). \end{aligned}$$

Expanding the $q_{k,j}(v_0)$ in powers of $(v_0)^2$, one obtains

$$\begin{aligned} H_2 = 1 &+ \sum_{j=3}^{2m-1} Q_j(y, w)\zeta^j + O(\zeta^{2m}(1 + |w|)^{2m\Delta}(1 + |y|)^{2m}) \\ (1.19) \qquad &+ O(1) \max_{2|t-1| \leqq 1} |H_2|. \end{aligned}$$

Since

$$2^{\nu\mu_1}(\zeta w^{\Delta})^{-1+2\nu\mu_1} = \sqrt{2\Omega}(v_0)^{2\mu_1} = o(1),$$

and

$$\max_{2|t-1|\leqq 1} |\log H_2| = O\left(\Omega(t-1) \max_{2|u-1|\leqq 1} \left|\frac{\partial P(u, v_0)}{\partial u}\right|\right)$$
$$= o(\sqrt{2\Omega} x) = o(y) \leqq 1 + c|y|^2,$$

equation (1.19) can be written in the form

$$(1.20) \qquad \begin{aligned} H_2 &= 1 + \sum_{j=3}^{2m-1} Q_j(y, w)\zeta^j + R_{m,2}, \\ R_{m,2} &= O(\zeta^{2m}(1 + |w|)^{2m\Delta}(1 + |y|)^{2m} e^{c|y^2|}), \qquad |x| \leqq x_0, \end{aligned}$$

where the $Q_j(y, w)$ satisfy the required degree estimates. To estimate $R_{m,2}$ for $|x| \geqq x_0 > 0$, $t \in \tilde{\Gamma}_\phi$, we note that

$$\begin{aligned} Q_j(y, w)\zeta^j &= O((1 + |w|)^{j\Delta}(|\zeta| + |x|)^j) \\ &= O(x^{2m}(1 + |w|)^{2m\Delta}), \qquad 0 \leqq j \leqq 2m, \end{aligned}$$

and that

$$\begin{aligned} |R_{m,2}| &\leqq |H_2| + \sum_{j=0}^{2m-1} |Q_j(y, w)\zeta^j| \\ &\leqq |H_2| + O(\zeta^{2m}(1 + |w|)^{2m\Delta}(1 + |y|)^{2m}), \end{aligned}$$

so that it is sufficient to estimate H_2 for $|x| \geqq x_0$, $t \in \tilde{\Gamma}_\phi$. First, choose M such that $|u^\rho P(u, v_0)| \leqq M$ for $|u| \leqq |v_0|^{-1}$, and then $\tilde{t}$ such that $M < 2^{-1}c(x_0)^2(\tilde{t})^\rho$. Thus,

$$\begin{aligned} |\log H_2| &= O(\Omega(v_0)^2 x^2 x_0^{-2}) \\ &= o(y^2) \leqq c|y|^2, \qquad |x| \geqq x_0, \quad |t| \leqq \tilde{t}, \end{aligned}$$

and

$$\begin{aligned} |\log H_2| &\leqq |2\Omega t^{-\rho} M| + O(\Omega(v_0)^2) \leqq 2^{-1}c|2\Omega x^2| + o(\Omega x^2) \\ &\leqq c|y|^2, \qquad \tilde{t} \leqq |t| \leqq |v_0|^{-1}, \quad |x| \geqq x_0, \end{aligned}$$

whereas

$$\begin{aligned} |\log H_2| &= O(\Omega t^{-\rho}) + O(\Omega(v_0)^2) \\ &= o(y^2) \leqq c|y|^2, \qquad |v_0|^{-1} \leqq |t|, \quad |x| \geqq x_0. \end{aligned}$$

Combining these estimates, it is clear that (1.20) is actually valid uniformly for $t \in \tilde{\Gamma}_\phi$, $\phi = 0, \pm(\phi^* - \varepsilon)$, $|\arg \Omega - 2\phi| \leqq \pi/2 - \varepsilon$.

Next, consider

$$\begin{aligned} H_3 &= e^{t\Lambda}, \\ \Lambda &= 2\Omega\{1 - \sqrt{1 + \lambda^2(2N)^{-2}}\} \\ &= -\lambda^2 w 2^{\nu-1}\zeta^{2\nu+2}\{1 + O(w\zeta^{2\beta})\} = O(N^{-1}). \end{aligned}$$

For $|x| \leqq x_0$,

$$H_3 = e^{\Lambda} + \sum_{j=1}^{2m-1} x^j \Lambda p_j(\Lambda)\, e^{\Lambda} + O\left(x^{2m} \max_{|x|=2x_0} |H_3|\right),$$

where the $p_j(\Lambda)$ are polynomials in Λ of degree $j-1$. Expanding these quantities in terms of ζ, we can write H_3 in the form

$$H_3 = 1 + \sum_{j=4}^{2m-1} Q_j(y, w)\zeta^j + R_{m,3},$$

$$R_{m,3} = O(\zeta^{2m}(1+|w|)^{2m\Delta}(1+|y|)^{2m} e^{c|y^2|}), \qquad |x| \leqq x_0.$$

Just as for H_2, the expansion for $|x| \geqq x_0$ depends basically on establishing proper estimates for H_3, that is,

$$\begin{aligned} |\log H_3| &= O(\Omega t N^{-2}) \\ &= O(N^{-1}) \leqq 1 + c|y|^2 && (|t| \leqq 1) \\ &= O(N^{-1}) + O(\Omega x^2 N^{-2}) = 1 + c|y|^2 && (|t| \geqq 1), \end{aligned}$$

which implies the expansion for H_3.

Penultimately, for $|x| \leqq x_0$,

$$H_4 = \left(\frac{\sinh v_0}{v_0}\right)^{-\lambda-2\tau} + \sum_{j=1}^{2m-1} q_j(v_0)x^j + O\left(x^{2m} \max_{|x|=2x_0} |H_4|\right),$$

where $q_j(v_0) = (v_0)^{2[(j+1)/2]}O(1)$. Expanding in powers of ζ, one obtains

$$H_4 = 1 + \frac{(\lambda+2\tau)2^{\nu-1}}{-3} w\zeta^{2\nu} + \sum_{j=2\nu+1}^{2m-1} Q_j(y, w)\zeta^j + R_{m,4},$$

$$R_{m,4} = O(\zeta^{2m}(1+|w|)^{2m\Delta}(1+|y|)^{2m} e^{c|y^2|}), \qquad |x| \leqq x_0.$$

For $|x| \geqq x_0$, $|tv_0| \leqq 1$, we note that $H_4 = O(1)$, and that for $|t| \geqq |v_0|^{-1}$,

$$\begin{aligned} \log|H_4| &= O(1) + O(v) \\ &= O(1) + o(x^2) \leqq O(1) + c|y|^2, \end{aligned}$$

which implies an expansion (1.16).

Finally, consider H_5. For $t \in \tilde{\Gamma}_\phi$, between $t = 0 \cdot \exp\{i2\phi\rho^{-1}\}$ and t_2,

$$X \in S_0 = \{X \,|\, |X| \geqq 1, |\arg|X| \leqq \pi[\nu + \min(1, \nu/2) - \eta]\},$$

so that H_5 is described adequately by (1.11), that is,

$$H_5 = 1 + \sum_{j=1}^{m-1} K_j X^{-j/\nu} + X^{-m/\nu}E_m(X), \tag{1.21}$$

$$X^{-1/\nu} = 2\zeta^2 t^\rho (v^{-1}\sinh v)^\rho,$$

and $E_m(X)$ is bounded in S_0. Under the restriction $|x| \leqq x_0$,

$$X^{-j/\nu} = (2\zeta^2)^j \sum_{k=0}^{2m-1} x^k q_{k,j}(v_0) + O\left(x^{2m} \max_{|x|=2x_0} |X^{-j/\nu}|\right),$$

and (1.21) becomes

$$(1.22)\qquad \begin{aligned} H_5 &= 1 + 2K_1\zeta^2 + \sum_{j=3}^{2m-1} Q_j(y, w)\zeta^j + R_{m,5}, \\ R_{m,5} &= O(\zeta^{2m}(1 + |w|)^{2m\Delta}(1 + |y|)^{2m} e^{c|y^2|}), \qquad |x| \leqq x_0. \end{aligned}$$

Clearly, $H_5 = O(1)$ for $X \in S_0$ and if we can obtain proper estimates for H_5 when t is between t_2 and $t = \infty \cdot \exp\{-i^2\phi\}$ on $\tilde{\Gamma}_\phi$, it follows as before that (1.22) is actually valid uniformly for $t \in \tilde{\Gamma}_\phi$. As $\arg X$ changes rapidly on $\tilde{\Gamma}_\phi$, $|t| \geqq |\Omega|^\alpha$, it is necessary to describe the asymptotic behavior of $G_0(X)$ in an arbitrarily wide sector (see [10]).

LEMMA 2. *Assume the a_j, b_j parameter conditions of Theorem 1. Let k_0 be the largest integer $\leqq (v + 2)/4$, $\eta > 0$, and*

$$S_{k,v} = \{X \mid |X| \geqq 1, \pi[v - 3 + 2k + \eta] \leqq \arg X \leqq \pi[v/2 + 2k + 2k_0 - \eta]\}, \qquad v \geqq 2, \quad k_0 \geqq 1,$$

$$S_{k,1} = \{X \mid |X| \geqq 1, \pi[2k - 3/2 + \eta] \leqq \arg X \leqq \pi[2k + 1/2 - \eta]\}, \qquad v = 1, \quad k_0 = 0,$$

$$S_{k,1}^{+} = \{X \mid |X| \geqq 1, \pi[2k - 1/2 + \eta] \leqq \arg X \leqq \pi[2k + 3/2 - \eta]\}, \qquad v = 1, \quad k_0 = 0.$$

Then for $X \in S_{k,v}$, or $S_{k,1}^{+}$,

$$G_0(X) = \sum_{j=1}^{p+2} C_j(k, s)L_j(X\, e^{i\pi(1-2k)}) + \sum_{h=1}^{v} D_h(k, s)G(X\, e^{i\pi(2-2s-2h)}),$$

$$s = k + k_0 \quad \text{if} \quad X \in S_{k,v}; \qquad s = k + 1 \quad \text{if} \quad X \in S_{k,1}^{+},$$

$$G(W) = W^\tau \exp\{-vW^{1/v}\}\{1 + O(W^{-1})\}, \qquad W \to \infty,$$

$$|\arg W| \leqq \pi[v + \min(1, v/2) - \eta],$$

$$L_j(W) = W^{-1+a_j}\{1 + O(W^{-1})\}, \qquad W \to \infty,$$

$$|\arg W| \leqq \pi[v/2 + 1 - \eta],$$

$$\tau \text{ as in } (1.11), \quad a_{p+1} = \frac{1 - \lambda}{2}, \quad a_{p+2} = \frac{2 - \lambda}{2},$$

where the connecting constants $C_j(k, s), D_h(k, s)$ satisfy the conditions

$$|C_j(k, s)| \leqq M\, e^{|k|\Delta_0}, \qquad |D_h(k, s)| \leqq M\, e^{|k|\Delta_0},$$

$$s = k + k_0, \quad \text{or} \quad s = k + 1,$$

the positive numbers M, Δ_0 depending only on a_j, b_j, λ and v.

From Lemma 1 and (1.15) it follows that $|\arg X| \leqq \theta = O(1) + O(\log(1 + |w|))$, for $t \in \tilde{\Gamma}_\phi$. Hence, $G_0(X)$ must be described in a sector S of central angle θ which requires at most $[\theta/\pi]$ expansions of the form given in Lemma 2. Moreover, the pertinent connecting constants $C_j(k, s), D_h(k, s)$ are uniformly bounded by numbers of the form

$$O(\exp[O(\Delta_0 \log(1 + |w|)]) = O((1 + |w|)^{2\Delta m})$$

for m sufficiently large, that is, $m \geqq m_0$. Combining these estimates, for $t \in \tilde{\Gamma}_\phi$, $|X| \geqq 1$, $|t| \geqq |\Omega|^\alpha$,

$$(1.23) \quad H_5 = O((1 + |w|)^{2m\Delta} X^{c_0} e^{\nu X^{1/\nu}}) + \sum_{h=1}^{\nu} O((1 + |w|)^{2m\Delta} \exp\{c_h X^{1/\nu}\}),$$

where the c_h are bounded real numbers dependent only on the a_j, b_j and λ. In view of Lemma 1, (1.23) implies the desired estimate for H_5 in $|X| \geqq 1$, $t \in \tilde{\Gamma}_\phi$. In the final range, $|X| \leqq 1$, $t \in \tilde{\Gamma}_\phi$, (1.10) implies

$$H_5 = O(X^b),$$

b being a constant, which again implies a proper estimate for H_5. Thus, the expansion for H_5 is established, and Proposition 2 is applicable to establish the theorem.

Remark 3. Theorem 1 remains true even if $a_j - a_k$ is an integer, as this situation merely introduces terms of the form $(\log X)^r$, r a positive integer, into (1.23).

THEOREM 2. *Let the parameters a_k, b_k, λ be independent of the large parameter n and satisfy the conditions*

$$a_j - a_k \neq \textit{an integer}, \qquad k = 1, \cdots, p, \quad k \neq j,$$

for j an integer, $1 \leqq j \leqq p$. Then for $\beta = q - p > 2$, $(wN^2)^{-1} = o(1)$, $N^2 = n(n + \lambda)$ and m arbitrary,

$$\begin{aligned} &l_{n,j}(w) \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G^{q,2}_{p+2,q}\left(w \left| \begin{matrix} 1-n-\lambda, a_j, a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_p, n+1 \\ b_1, \cdots, b_q \end{matrix} \right.\right) \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} \sum_{k=0}^{m-1} (-1)^k C_{k,j} \frac{\Gamma(-k+n+\lambda-1+a_j)}{\Gamma(k+n+2-a_j)} w^{-k-1+a_j} + O([wN^2]^{-m-1-a_j}) \\ &\sim C_{0,j} \frac{\Gamma(n+\lambda-1+a_j)\Gamma(n+1)}{\Gamma(n+\lambda)\Gamma(n+2-a_j)} w^{-1+a_j} \\ &\quad {}_{q+1}F_{p+2}\left(\begin{matrix} 1+b_1-a_j, \cdots, 1+b_q-a_j, 1 \\ 1+a_1-a_j, \cdots, 1+a_p-a_j, n+2-a_j, -n-\lambda+2-a_j \end{matrix} \left| \frac{1}{w} \right.\right), \end{aligned}$$

$$wN^2 \to \infty, \qquad |\arg(wN^2)| \leqq \frac{\pi}{2}[\beta + 2] - \varepsilon, \quad C_{k,j} = \frac{\prod_{r=1}^{q} \Gamma(k+1+b_r-a_j)}{\prod_{r=1}^{p} \Gamma(k+1+a_r-a_j)}.$$

Proof. It follows from (1.8), and the Mellin–Barnes integral representation for $l_{n,j}(w)$, that for n sufficiently large,

$$l_{n,j}(w) = 2\frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(n+\lambda)} \int_0^\infty e^{-v(2n+\lambda)} (\sinh v)^{-\lambda} G_1(X)\, dv,$$

$$(1.24) \qquad \operatorname{Re}(\lambda + 2a_j) < 3, \qquad 0 < w, \qquad X(v, w) = w(\sinh v)^{-2},$$

$$G_1(X) = G^{q,1}_{p+2,q}\left(X \left| \begin{matrix} a_j, a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_p, \dfrac{1-\lambda}{2}, \dfrac{2-\lambda}{2} \\ b_1, \cdots, b_q \end{matrix} \right.\right).$$

Since

$$G_1(X) = \sum_{k=0}^{m-1} (-1)^k \frac{C_{k,j}X^{-k-1+a_j}}{\Gamma(k - a_j + (3-\lambda)/2)\Gamma(k - a_j + (4-\lambda)/2)} + G_{1,m}(X),$$

$$(1.25)\quad G_{1,m}(X) = (-1)^m G^{q+1,1}_{p+3,q+1}\left(X \middle| \begin{matrix} -m + a_j, a_1, \cdots, a_p, \frac{1-\lambda}{2}, \frac{2-\lambda}{2} \\ -m + a_j, b_1, \cdots, b_q \end{matrix}\right)$$

$$= F_{0,m}(X)X^{-m+a_j} + \sum_{k=1}^{q} X^{b_k}F_{k,m}(X),$$

where the $F_{k,m}(X)$ are entire functions of X, and

$$(1.26)\quad \begin{gathered} G_{1,m}(X) = O(X^{-m-1+a_j}), \\ X \in T_0 = \{X \mid |X| \geqq 1, |\arg X| \leqq (\pi/2)\beta - \varepsilon, \varepsilon > 0\}, \end{gathered}$$

equation (1.24) can be rewritten in the form

$$l_{n,j}(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} \sum_{k=0}^{m-1} (-1)^k \frac{C_{k,j}\Gamma(-k+n+\lambda-1+a_j)}{\Gamma(k+n+2-a_j)} w^{-k-1+a_j} + R_m(N, w),$$

$$(1.27)\quad \begin{gathered} R_m(N, w) = 2\frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(n+\lambda)} \int_{C_\phi} e^{-v(2n+\lambda)} (\sinh v)^{-\lambda} G_{1,m}(X)\, dv, \\ \operatorname{Re}(\lambda + 2a_j) < 2m + 3, \qquad |\arg w - 2\phi| \leqq \frac{\pi\beta}{2} - 2\varepsilon, \qquad |\phi| \leqq \frac{\pi}{2} - \varepsilon, \\ C_\phi : v = \log(t + \sqrt{1 + t^2}), \quad \text{or} \quad t = \sinh v = ue^{i\phi}, \\ 0 \leqq u < \infty, \qquad 2|\phi| < \pi, \end{gathered}$$

and $C_0 = [0, \infty)$.

As $v \in C_\phi$, $|X| \geqq 1$, implies $X \in T_0$, the estimates for $|X| \leqq 1$, and $|X| \geqq 1$ in (1.25) and (1.26) can be combined and written as

$$G_{1,m}(X) = O(X^{-m-1+a_j}), \qquad m \geqq m_0 \geqq \max_k \{|b_k - a_j|\},$$

uniformly for $v \in C_\phi$, $|\phi| \leqq \pi/2 - \varepsilon$.

Breaking C_ϕ into

$$C'_\phi = \{v \in C_\phi \mid |v| \leqq 1\}, \qquad C''_\phi = \{v \in C_\phi \mid |v| \geqq 1\},$$

it is easily seen that

$$\begin{aligned} R_m(n, w) &= O(N^{1-\lambda} \int_{C'_\phi} |e^{-2Nv} v^{2m+2-\lambda-2a_j} w^{-m-1+a_j}| \cdot |dv|) \\ &\quad + O\left(N^{1-\lambda} \int_{C''_\phi} |e^{-2nv+bv} w^{-m-1+a_j}| \cdot |dv|\right) \\ &= O([N^2 w]^{-m-1+a_j}) + O(w^{-m-1+a_j} e^{-cN}), \end{aligned}$$

where b, c are positive constants. As each of the m terms in (1.27) can be estimated explicitly, the tentative assumption $m \geqq m_0$ can be dropped.

Remark 4. If $a_j - b_k$ is a positive integer, some of the initial $C_{k,j}$ vanish.

Remark 5. With minor modifications in the proof, Theorem 2 is valid with $\beta = 2$.

The analytic continuations of $g_n(w)$ and $l_{n,j}(w)$ are related as follows.

THEOREM 3. *Under the restrictions of Theorem* 1 *on the parameters* a_j,

$$l_{n,j}(w) = e^{i\pi 2a_j} l_{n,j}(w\,e^{-i\pi 2}) + (-2\pi i)\,e^{i\pi a_j} g_n(w\,e^{-i\pi}),$$

and there exist constants $C_j(k)$ *and* $D_h(k)$ *such that*

$$g_n(w) = \sum_{j=1}^{p} C_j(k) l_{n,j}(w\,e^{i\pi(1-2k)}) + \sum_{h=1}^{\beta} D_h(k) g_n(w\,e^{-i\pi 2h}),$$

$$C_j(k) = (-1)^{\beta+1}(2\pi i)^{\beta}\,e^{i\pi(B_1-A_1)}\,e^{i\pi(2k-\beta-1)a_j} \cdot \prod_{\substack{r=1\\ r\neq j}}^{p} \{\Gamma(a_r - a_j)\Gamma(1 - a_r + a_j)\} \Big/ \prod_{r=1}^{q} \{\Gamma(b_r - a_j)\Gamma(1 - b_r + a_j)\},$$

$$D_h(k) = (-1)^{\beta+1}\,e^{i\pi 2(B_1-A_1)} = e^{i\pi 2\beta\gamma}, \qquad 1 \leqq k < \beta,$$

where A_1, B_1 *are defined in Theorem* 1.

Proof. From the partial fraction decomposition

$$\frac{\prod_{j=1}^{q}(y-\beta_j)}{\prod_{j=1}^{p}(y-\alpha_j)} = \sum_{j=1}^{p} \frac{c_{j,k}y^k}{y-\alpha_j} + \sum_{h=0}^{\beta} d_{h,k}y^h, \qquad d_{\beta,k} = 1, \qquad d_{0,k} = (-1)^{\beta}\frac{\prod_{j=1}^{q}\beta_j}{\prod_{j=1}^{p}\alpha_j},$$

$$\alpha_j \neq \alpha_r, \quad j \neq r, \quad 0 < k < \beta = q - p,$$

with $y = e^{-i\pi 2s}$, $\beta_j = e^{-i\pi 2b_j}$ and $\alpha_j = e^{-i\pi 2a_j}$, it follows that there exist $C_j(k)$, $D_h(k)$ such that

$$(-1)^{\beta+1}(2\pi i)^{\beta} e^{i\pi(B_1-A_1)}\,e^{-i\beta s}\frac{\prod_{j=1}^{p}\{\Gamma(a_j - s)\Gamma(1 - a_j + s)\}}{\prod_{j=1}^{q}\{\Gamma(b_j - s)\Gamma(1 - b_j + s)\}}$$

$$= \sum_{j=1}^{p} C_j(k)\,e^{i\pi(1-2k)}\Gamma(a_j - s)\Gamma(1 - a_j + s) - 1 + \sum_{h=1}^{\beta} D_h(k)\,e^{-i\pi 2hs}.$$

Multiplying this identity by

$$w^s\frac{\prod_{r=1}^{q}\Gamma(b_r - s)}{\prod_{r=1}^{p}\Gamma(a_r - s)}\,\frac{\Gamma(n + \lambda + s)}{\Gamma(n + 1 - s)}$$

and integrating along the contour C_+ which separates the poles of $\Gamma(b_r - s)$ from those of $\Gamma(1 - a_j + s)$, one obtains an expansion of $G_{p+2,q}^{0,p+1}(w\,e^{-i\pi\beta}) \equiv 0$, which reduces to the expansion of $g_n(w)$ in Theorem 3. The $l_{n,j}(w)$ expansion similarly follows from the reflection formula for the gamma function.

To show how this extends Theorem 1, we proceed as follows. Denote by $G(\Omega)$, $L_j(\Omega)$, the formal series in Theorems 1, 2, representing $g_n(w)$, $l_{n,j}(w)$, respectively. We have shown that

$$(1.28)\qquad \begin{aligned} g_n(w\,e^{-i\pi 2h}) &\sim G(\Omega\,e^{-i2\pi h/\beta}), && |\beta \arg\Omega - 2\pi h| \leqq \pi[\beta + 1] - \varepsilon,\\ l_{n,j}(w\,e^{i\pi(1-2k)}) &\sim L_j(\Omega\,e^{i\pi(1-2k)/\beta}), && |\beta \arg\Omega + \pi(1 - 2k)| \leqq (\pi/2)[\beta + 2] - \varepsilon. \end{aligned}$$

Although the $g_n(w)$ expansion in Theorem 3 is valid for all w, we can use it to estimate $g_n(w)$ asymptotically only when the inequalities in (1.28) are satisfied for $h = 1, \cdots, \beta$, for example, as they are when

$$k = [(\beta + 1)/2], \quad \pi[\beta - 1] + \varepsilon \leqq \beta \arg \Omega \leqq \pi[\beta + 3] - \varepsilon, \qquad \varepsilon > 0.$$

Then

$$g_n(w) \sim \sum_{j=1}^{p} C_j(k) L_j(\Omega\, e^{i\pi(1-2k)/\beta}) + \sum_{h=1}^{\beta} D_h(k) G(\Omega\, e^{-i\pi 2h/\beta}),$$

for this range. From Theorem 2, the $\sum_{j=1}^{p}$ series contributes only terms of algebraic (or logarithmic) growth as $n \to \infty$, and as such, are subdominant to the exponential growth terms of $\sum_{h=1}^{\beta}$. Moreover, as

$$\begin{aligned} |\Omega|^{-1} \operatorname{Re}\{\Omega\, e^{-i\pi 2h/\beta} - \Omega\, e^{-i\pi 2}\} &= \cos\{\arg \Omega - 2\pi h \beta^{-1}\} - \cos\{\arg \Omega\} \\ &= 2 \sin\{\arg \Omega - \pi h \beta^{-1}\} \sin\{\pi h \beta^{-1}\} > 0, \\ h = 1, \cdots, \beta - 1, &\quad \pi[\beta - 1] < \beta \arg \Omega < \pi[\beta + 1], \end{aligned}$$

it is clear that $D_\beta(k)G(\Omega\, e^{-i\pi 2})$ is dominant in $|\beta \arg \Omega - \pi\beta| < \pi$, but for $\arg \Omega = \pi[\beta + 1]$, the $h = \beta$ and $h = 1$ terms are comparable, while the $h = 1$ term is dominant for $\beta \arg \Omega = \pi[\beta + 1] + \varepsilon$, $\varepsilon > 0$. Systematic use of the results in Theorem 3 permit an asymptotic description of $g_n(w)$ and $l_{n,j}(w)$ in an arbitrarily wide sector [10].

Finally, we state the following, simple comparison result.

THEOREM 4. *Let*

$$g(w) = G^{q,0}_{p,q}\left(w \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right),$$

$$l_j(w) = G^{q,1}_{p,q}\left(w \,\middle|\, \begin{matrix} a_j, a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right).$$

Then under the conditions of Theorem 1,

$$\begin{gathered} g_n(w) = g(\Omega^\beta) \exp\{2\Omega P(v_0)\}\{1 + \Omega^{-1}[S_1(w) - S_1(0)] + O(\Omega^{-2}(1 + |w|)^{4\Delta})\}, \\ \Omega \to \infty, \quad |\arg \Omega^\beta| \leqq \pi[\beta + 1] - \varepsilon, \quad \varepsilon > 0, \\ l_{n,j}(w) = l_j(\Omega^\beta)\{1 + O(\Omega^{-\beta}(1 + |w|))\}, \\ \Omega \to \infty, \quad |\arg \Omega^\beta| \leqq \pi[\beta + 2]/2 - \varepsilon, \quad \varepsilon > 0, \quad \Omega^\beta = wn(n + \lambda). \end{gathered}$$

2. A difference equation for $g_n(w)$ and $l_{n,j}(w)$. We begin with the elementary operators

$$\mathscr{V}_n(\lambda, \mu) = \frac{(n + \lambda - 1)(n - \mu)}{2n + \lambda - 1}\xi^0 - \frac{n(n + \lambda - 1 + \mu)}{2n + \lambda - 1}\xi^{-1},$$

$$\mathscr{V}_n(\lambda) = \frac{(n + \lambda - 1)}{2n + \lambda - 1}\xi^0 + \frac{n}{2n + \lambda - 1}\xi^{-1},$$

where ξ^{-j} is the shift operator on n, that is, $\xi^{-j}y_n = y_{n-j}$. These operators satisfy the functional equations

$$\mathscr{V}_n(\lambda, \mu)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\} = \frac{(n+\lambda-1)_s}{(n+1)_{-s}}(s-\mu),$$

$$\mathscr{V}_n(\lambda)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\} = \frac{(n+\lambda-1)_s}{(n+1)_{-s}},$$

$$(\sigma)_\omega = \frac{\Gamma(\sigma+\omega)}{\Gamma(\sigma)}, \tag{2.1}$$

while

$$\mathscr{M}_n(w) = \prod_{j=1}^{q} \mathscr{V}_n(\lambda + 1 - j, b_j)$$
$$- wn(n+\lambda-q)\xi^{-1} \prod_{j=1}^{\beta-2} \mathscr{V}_n(\lambda+1-p-j) \prod_{j=1}^{p} \mathscr{V}_n(\lambda+1-j, -1+a_j),$$

$$\prod_{j=1}^{r} P_j = P_r P_{r-1} \cdots P_1, \quad \beta = q - p \geqq 3,$$

satisfies the equation

$$\mathscr{M}_n(w)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\} = \frac{(n+\lambda-q)_s}{(n+1)_{-s}} \prod_{j=1}^{q}(s-b_j)$$
$$- w\frac{(n+\lambda-q)_{s+1}}{(n+1)_{-s-1}} \prod_{j=1}^{p}(s+1-a_j).$$

Note that if s is equal to an integer r, $(-1)^r(-n)_r(n+1)_{-r} = 1$. The notation in (2.1) will be used extensively.

Clearly, $\mathscr{M}_n(w)$ is a difference operator of order q, and can be written in the form

$$\mathscr{M}_n(w) = \sum_{j=0}^{q} \{C_j(n,\lambda) + wD_j(n,\lambda)\}\xi^{-j}.$$

Using the above functional equation for $\mathscr{M}_n(w)$, it is easy to see that the constants $C_j(n,\lambda)$, $D_j(n,\lambda)$ satisfy the relations,

$$\prod_{j=1}^{q}(s-b_j) = \sum_{j=0}^{q} C_j(n,\lambda)\frac{(s-n)_j(s+n+\lambda-q)_{q-j}}{(-n)_j(n+\lambda-q)_{q-j}},$$

$$(s-n)(s+n+\lambda-q)\prod_{j=1}^{p}(s+1-a_j) = \sum_{j=0}^{q} D_j(n,\lambda)\frac{(s-n)_j(s+n+\lambda-q)_{q-j}}{(-n)_j(n+\lambda-q)_{q-j}}.$$

Explicit expressions for the $C_j(n,\lambda)$, $D_j(n,\lambda)$ can then be deduced from [7, Lemma 2.1].

THEOREM 5. *Under the conditions of Theorem* 1 *the functions* $g_n(w\, e^{i\pi(\beta+2m)})$, $l_{n,j}(w\, e^{i\pi(\beta-1+2m)})$, *m an integer*, $j = 1, \cdots, p$, *satisfy the linear difference equation*

$$\mathscr{M}_n(w)\{y_n(w)\} = 0.$$

Moreover, if the integers r, s are chosen such that

$$|[\arg w + \pi[\beta - 1 - 2r]| < \pi[\beta + 2]/2,$$
$$|\arg w + \pi[\beta - 2s - 2h]| < \pi[\beta + 1], \qquad h = 1, \cdots, \beta,$$

then the functions $l_{n,j}(w\, e^{i\pi(\beta - 1 - 2r)})$, $j = 1, \cdots, p$, $g_n(w\, e^{i\pi(\beta - 2s - 2h)})$, $h = 1, \cdots, \beta$, *form a basis*, $\mathscr{B}_{r,s}$, *of the difference equation in the sector*

$$T_{r,s}: \pi \max(2r - 3\beta/2, 2s - 1) < \arg w < \pi \min(2r + 2 - \beta/2, 2s + 3).$$

Proof. From the Mellin–Barnes integral representation for $g_n(w\, e^{i\pi(\beta + 2m)})$, one obtains

$$\mathscr{M}_n(w)\{g_n(w\, e^{i\pi(\beta + 2m)})\}$$
$$= \frac{(-1)^q}{2\pi i}\left(\int_L - \int_{1+L}\right) \frac{\prod_{j=1}^q \Gamma(1 + b_j - s)(n + \lambda - q)_s}{\prod_{j=1}^p \Gamma(a_j - s)(n + 1)_{-s}} (w\, e^{i\pi(\beta + 2m)})^s\, ds,$$

where L is chosen to separate the poles of the $\Gamma(b_j - s)$, $j = 1, \cdots, q$, from those of $\Gamma(n + \lambda + s)$. By inspection, the integrand of this integral has no poles between L and $1 + L$, and hence the resulting integral is zero. The computation for $l_{n,j}(w\, e^{i\pi(\beta - 1 + 2m)})$ is similar. The last part of the theorem then follows from the fact that Theorems 1, 2 are applicable, and that the asymptotic behavior of the various functions in $\mathscr{B}_{r,s}$ clearly indicates that they are linearly independent as functions of n.

THEOREM 6. *Under the conditions of Theorem* 1 *and the restriction*

$$b_h - b_k \neq \text{a negative integer}; \qquad h, k = 1, \cdots, q, \quad h \neq k,$$

the functions $F_h(w\, e^{i\pi})$, $h = 1, \cdots, q$,

$$F_h(w) = \frac{(n + \lambda)_{b_h}}{(n + 1)_{-b_h}} w^{b_h}$$
$${}_{p+3}F_q\left(\begin{matrix} 1, b_h - n, b_h + n + \lambda, b_h + 1 - a_1, \cdots, b_h + 1 - a_p \\ b_h + 1 - b_1, \cdots, b_h + 1 - b_q \end{matrix}\middle| w\right),$$

also form a basis of

$$\mathscr{M}_n(w)\{y_n(w)\} = 0.$$

Proof. As

$$F_h(w) = \frac{\prod_{j=1}^q \Gamma(b_h + 1 - b_j)}{\prod_{j=1}^p \Gamma(b_h + 1 - a_j)} \frac{\Gamma(n + 1)}{\Gamma(n + \lambda)}$$
$$G^{1,p+2}_{p+3,q+1}\left(w \middle| \begin{matrix} 1 - n - \lambda, a_1, \cdots, a_p, b_h, n + 1 \\ b_h, b_1, \cdots, b_q \end{matrix}\right),$$

it follows as in Theorem 5, that $\mathscr{M}_n(w)\{F_h(w\,e^{i\pi})\} = 0$. Then from the explicit formulas

$$g_n(w) = \sum_{h=1}^{q} K_h\, e^{-i\pi(\beta-1)b_h} F_h(w\,e^{i\pi(\beta-1)}),$$

$$l_{n,j}(w) = \sum_{h=1}^{q} K_h \Gamma(1 + b_h - a_j)\Gamma(a_j - b_h)\, e^{-i\pi\beta_h} F_h(w\,e^{i\pi\beta}),$$

$$K_h = \prod_{\substack{k=1\\k\neq h}}^{q} \Gamma(b_k - b_h) \Big/ \prod_{k=1}^{p} \Gamma(a_k - b_h),$$

it follows that the q functions $F_h(w\,e^{i\pi})$ span the q-dimensional space spanned by $\mathscr{B}_{r,s}$, and are linearly independent as functions of n.

3. The extended Jacobi functions. The extended Jacobi functions are defined by

$$J_n(w) = {}_{p+2}F_{q-1}\left(\begin{matrix}-n, n+\lambda, \alpha_1, \cdots, \alpha_p\\ \beta_1, \cdots, \beta_{q-1}\end{matrix}\middle| w\right), \qquad \beta = q - p > 2,$$

which can be identified with the $F_q(w)$ in § 2, with

$$\text{(3.1)} \qquad \begin{aligned} \alpha_j &= 1 - a_j, \qquad j = 1, \cdots, p,\\ \beta_j &= 1 - b_j, \qquad j = 1, \cdots, q, \qquad b_q = 0. \end{aligned}$$

Under this identification, $g_n(w), l_{n,j}(w), \cdots$, will be denoted by $g_n^{\#}(w), l_{n,j}^{\#}(w), \cdots$.

As we are interested in linear combinations of the $g_n(w)$, the following characterization is convenient.

PROPOSITION 3. *With the hypothesis and notation of Theorem* 1, *let*

$$S(\Omega, w) = 1 + \sum_{j=1}^{m-1} S_j(w)\Omega^{-j} + O\left(\left[\frac{(1+|w|)^{2\Delta}}{\Omega}\right]^m\right),$$

$$T(\Omega, w) = 2\Omega P(v_0) + \log S(\Omega, w).$$

Then there exist polynomials $T_j(w)$ such that

$$\log S(\Omega, w) = \sum_{j=1}^{m-1} T_j(w)\Omega^{-j} + O\left(\left[\frac{(1+|w|)^{2\Delta}}{\Omega}\right]^m\right),$$

$$T_1(w) = S_1(w), \quad T_2(w) = S_2(w) - [S_1(w)]^2/2,$$

$$\partial_w T_j(w) \leqq 2\Delta j.$$

Moreover, as

$$2\Omega P(v_0) = \begin{cases} 3^{-1}w\Omega^{3-\beta}Q(w\Omega^{2-\beta}), & Q(0) = 1,\\ 3^{-1}w\Omega^{3-\beta} + o(1), & \beta > 3,\\ \dfrac{w}{3} - \dfrac{w^2}{15\Omega} + \dfrac{2w^3}{189\Omega^2} + o(1), & \beta = 3,\end{cases}$$

where $Q(W)$ is analytic at $W = 0$, there exist polynomials $\tilde{T}_j(w)$ such that

$$g_n(w) = \sqrt{\frac{(2\pi)^{\beta-1}}{\beta}}\Omega^{\beta\gamma}\exp\{-\beta\Omega + T(\Omega, w)\},$$

$$\Omega \to \infty, \quad |\arg \Omega^\beta| \leqq \pi[\beta + 1] - \varepsilon, \quad \varepsilon > 0,$$

$$T(\Omega, w) = \sum_{j=0}^{m-1} \tilde{T}_j(w)\Omega^{-j} + O\left(\left[\frac{(1 + |w|)^{2\Delta}}{\Omega}\right]^m\right) + O\left(\frac{w}{\Gamma(\beta - 2 - m)\Omega^{\beta-3}}\right),$$

$$\tilde{T}_0(w) = \frac{aw}{3}, \qquad \tilde{T}_1(w) = \frac{-aw^2}{15} + \frac{bw}{3} + S_1(w),$$

$$\tilde{T}_2(w) = \frac{a2w^3}{189} + \frac{cw}{3} - \frac{[S_1(w)]^2}{2} + S_2(w),$$

$$a = \frac{1}{\Gamma(4 - \beta)}, \qquad b = \frac{\beta - 3}{\Gamma(5 - \beta)}, \qquad c = \frac{(\beta - 3)(\beta - 4)}{2\Gamma(6 - \beta)}.$$

THEOREM 7. *Let the parameters $\alpha_j, \beta_j, \lambda$ be independent of the large parameter n, $\arg n \to 0$ as $n \to \infty$, and satisfy the conditions*

$$\beta_k \neq \text{a nonpositive integer}, \qquad k = 1, \cdots, q - 1,$$

$$\alpha_k - \alpha_j \neq \text{an integer}, \qquad j, k = 1, \cdots, p, \quad j \neq k.$$

Then for $\beta = q - p \geqq 3$, and

$$w = o(N^{2\mu}), \qquad (wN^2)^{-1} = o(1), \qquad n \to \infty,$$

$$N^2 = n(n + \lambda), \qquad 2\mu = \max\left(\tfrac{2}{3}, \tfrac{4}{5}\beta - 2\right) < \beta - 2,$$

$${}_{p+2}F_{q-1}\left(\begin{matrix}-n, n + \lambda, \alpha_1, \cdots, \alpha_p\\ \beta_1, \cdots, \beta_{q-1}\end{matrix}\middle| w\right)$$

$$\sim \sum_{j=1}^{p} K_j \frac{(n + \lambda)_{-\alpha_j}}{(n + 1)_{\alpha_j}} w^{-\alpha_j}$$

$${}_{q+1}F_{p+2}\left(\begin{matrix}\alpha_j, \alpha_j + 1 - \beta_1, \cdots, \alpha_j + 1 - \beta_{q-1}, 1\\ \alpha_j + n + 1, \alpha_j + 1 - n - \lambda, \alpha_j + 1 - \alpha_1, \cdots, \alpha_j + 1 - a_p\end{matrix}\middle| \frac{(-1)^\beta}{w}\right)$$

$$+ \frac{2}{\sqrt{\beta(2\pi)^{\beta-1}}} \frac{\prod_{k=1}^{q-1}\Gamma(\beta_k)}{\prod_{k=1}^{p}\Gamma(\alpha_k)} \cdot \Omega^{\beta\gamma}$$

$$\cdot \exp\left\{\beta\cos(\pi/\beta)\Omega + \frac{aw}{3} - \frac{\cos(\pi/\beta)V(w)}{\Omega} + O\left(\frac{(1 + |w|)^{4\Delta}}{\Omega^2}\right) + O\left(\frac{w}{\Gamma(\beta - 4)\Omega^{\beta-3}}\right)\right\}$$

$$\cdot \cos\left\{\beta\sin(\pi/\beta)\Omega + \pi\gamma + \frac{\sin(\pi/\beta)V(w)}{\Omega} + O\left(\frac{(1 + |w|)^{4\Delta}}{\Omega^2}\right) + O\left(\frac{w}{\Gamma(\beta - 4)\Omega^{\beta-3}}\right)\right\},$$

$$\Omega \to \infty, \quad |\arg \Omega^\beta| \leqq 2\pi - \varepsilon, \qquad \varepsilon > 0,$$

$$\Omega^\beta = wN^2 = wn(n + \lambda), \qquad \Delta = \min\left(\frac{2}{3}, \frac{1}{2\beta - 5}\right),$$

$$K_j = \prod_{\substack{k=1 \\ k \neq j}}^{p} \frac{\Gamma(\alpha_k - \alpha_j)}{\Gamma(\alpha_k)} \prod_{k=1}^{q-1} \frac{\Gamma(\beta_k)}{\Gamma(\beta_k - \alpha_j)}, \qquad 2\beta\gamma = \beta - 1 + 2A_1^{\#} - 2B_1^{\#},$$

$$V(w) = \frac{-aw^2}{15} - \frac{aw(1 + 3\lambda + 2A_1^{\#} - 2B_1^{\#})}{6} - \frac{bw}{3} + A_2^{\#} - B_2^{\#}$$

$$+ \frac{(B_1^{\#} - A_1^{\#})}{2\beta}[\beta(B_1^{\#} + A_1^{\#}) + A_1^{\#} - B_1^{\#} - 2] - \frac{\beta^2 - 12\beta + 11}{24\beta},$$

$$\prod_{j=1}^{p} (x + \alpha_j) = \sum_{j=0}^{p} A_j^{\#} x^{p-j}, \qquad \prod_{j=1}^{q-1} (x + \beta_j) = \sum_{j=0}^{q-1} B_j^{\#} x^{q-1-j},$$

where the parameters a, b are the same as in Proposition 3, *and Ω, N are positive when w, n and λ are positive.*

Proof. Using a partial fraction decomposition as in Theorem 3, it can be shown that there exist constants C_j, D_h such that

$$\prod_{k=1}^{p} \frac{\Gamma(s + \alpha_k)\Gamma(-s + 1 - \alpha_k)}{\Gamma(\alpha_k)} \prod_{k=1}^{q-1} \frac{\Gamma(\beta_k)}{\Gamma(s + \beta_k)\Gamma(-s + 1 - \beta_k)}$$

$$= \sum_{j=1}^{p} C_j \, e^{i\pi s(2\omega - \beta)}\Gamma(s + \alpha_j)\Gamma(-s + 1 - \alpha_j) + \sum_{h=1}^{\beta} D_h \, e^{i\pi s(\beta + 1 - 2h)},$$

where ω = the greatest integer $\leqq \beta/2$. In particular,

$$C_j = \frac{e^{i\pi(2\omega - \beta)\alpha_j}}{\Gamma(\alpha_j)} \prod_{\substack{k=1 \\ k \neq j}}^{p} \frac{\Gamma(\alpha_k - \alpha_j)\Gamma(1 + \alpha_j - \alpha_k)}{\Gamma(\alpha_k)} \prod_{k=1}^{q-1} \frac{\Gamma(\beta_k)}{\Gamma(\beta_k - \alpha_j)\Gamma(1 + \alpha_j - \beta_k)},$$

$$D_1 \prod_{k=1}^{p} \Gamma(\alpha_k) = e^{-i\pi\beta\gamma}(2\pi)^{1-\beta} \prod_{k=1}^{q-1} \Gamma(\beta_k), \qquad D_\beta = e^{i\pi 2\beta\gamma} D_1.$$

Substituting this expansion in the Mellin–Barnes integral representation for $J_n(w)$, it follows that

$$(3.2) \qquad J_n(w) = \sum_{j=1}^{p} C_j l_{n,j}^{\#}(w \, e^{i\pi(2\omega - \beta)}) + \sum_{h=1}^{\beta} D_h g_n^{\#}(w \, e^{i\pi(\beta + 1 - 2h)}).$$

For $-\pi \min(2, \beta/2) < \arg[wn(n + \lambda)] < 2\pi$, the $l_{n,j}^{\#}$ and $g_n^{\#}$ can be replaced by their asymptotic expansions as given in Theorems 2, 1 or Proposition 3. Clearly, the $\sum_{j=1}^{p}$ sum in (3.2) reduces to the $\sum_{j=1}^{p}$ sum in the Theorem for $-\pi[2 + 4\omega - \beta] < 2\arg(wN^2) < \pi[2 - 4\omega + 3\beta]$. Since

$$\cos\left\{\frac{\arg(wN^2) - \pi}{\beta}\right\} - \cos\left\{\frac{\arg(wN^2) + \pi(1 - 2h)}{\beta}\right\}$$

$$= 2\sin\left\{\frac{\pi(h - 1)}{\beta}\right\}\sin\left\{\frac{h\pi - \arg(wN^2)}{\beta}\right\} > 0,$$

$$h = 2, 3, \cdots, \beta - 2, \quad \beta > 3, \quad -2\pi < \arg(wN^2) < 2\pi,$$

the $g_n^{\#}(w\,e^{i\pi(\beta-1)})$ and $g_n^{\#}(w\,e^{-i\pi(\beta-1)})$ terms in (3.2) dominate. A straightforward computation shows that

$$D_\beta g_n^{\#}(w\,e^{i\pi(1-\beta)}) = \frac{\prod_{k=1}^{q-1}\Gamma(\beta_k)}{\prod_{k=1}^{p}\Gamma(\alpha_k)}\frac{\Omega^{\beta\gamma}}{\sqrt{\beta(2\pi)^{\beta-1}}}$$

$$\cdot\exp\left\{\beta\,e^{i\pi/\beta}\Omega + \frac{aw}{3} + i\pi\gamma - \frac{e^{-i\pi/\beta}\tilde{T}_1^{\#}((-1)^{\beta-1}w)}{\Omega} + O\left(\frac{(1+|w|)^{4\Delta}}{\Omega^2}\right)\right.$$

$$\left. + O\left(\frac{w}{\Gamma(\beta-4)\Omega^{\beta-3}}\right)\right\}.$$

Remember that for $w \to e^{i\pi(1-\beta)}w$, Ω is replaced by $e^{i\pi(1-\beta)/\beta}\Omega$. As $D_1 g_n^{\#}(w\,e^{-i\pi(1-\beta)})$ is essentially the complex conjugate of $D_\beta g_n^{\#}(w\,e^{i\pi(1-\beta)})$, equation (3.2) reduces to the statement of the theorem for

$$-\pi\min(2, \beta/2) < \arg(wN^2) < 2\pi.$$

It then follows from Theorem 3 and the parameter identifications (3.1) that

$$l_{n,j}^{\#}(W) = e^{i\pi 2\alpha_j} l_{n,j}^{\#}(W\,e^{i\pi 2}) + (-2\pi i)\,e^{i\pi\alpha_j} g_n^{\#}(W\,e^{i\pi}),$$

so that (3.2) can be rewritten as

$$J_n(w) = \sum_{j=1}^{p} C_j\,e^{i\pi 2\alpha_j} l_{n,j}^{\#}(w\,e^{i\pi(2+2\omega-\beta)}) + \sum_{h=1}^{\beta}\tilde{D}_h g_n^{\#}(w\,e^{i\pi(\beta+1-2h)}),$$

$$\tilde{D}_h = D_h, \quad h \neq \beta - \omega; \quad \tilde{D}_{\beta-\omega} = D_{\beta-\omega} + (-2\pi i)\sum_{j=1}^{p} e^{i\pi\alpha_j}C_j.$$

Just as above, this expansion implies the theorem for

$$-2\pi < \arg(wN^2) < \pi\min(2, \beta/2).$$

Combination of these results yields the theorem.

Remark 6. The asymptotic behavior of $J_n(w)$ outside of $|\arg(wn(n+\lambda))| \leqq 2\pi - \varepsilon, \varepsilon > 0$, can be found using Theorem 3 in a systematic manner.

Remark 7. With the parameter identifications (3.1), $J_n(w)$ satisfies the linear difference equation $\mathscr{M}_n^{\#}\{y_n(w)\} = 0$, which is described in more detail in [7].

REFERENCES

[1] B. L. J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes-integrals*, Compositio Math., 15 (1963), pp. 239–341.

[2] E. T. Copson, *Asymptotic Expansions*, Cambridge University Press, Cambridge, 1965.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vols. 1–3, McGraw-Hill, New York, 1953.

[4] J. L. Fields and Y. L. Luke, *Asymptotic expansions of a class of hypergeometric polynomials, with respect to the order*, J. Math. Anal. Appl., 6 (1963), pp. 394–403.

[5] ———, *Asymptotic expansions of a class of hypergeometric polynomials with respect to the order. II*, Ibid., 7 (1963), pp. 440–450.

[6] J. L. Fields, *A note on the asymptotic expansion of a ratio of gamma functions*, Proc. Edinburgh Math. Soc. (2), 15 (1966), pp. 43–45.

[7] J. L. Fields, Y. L. Luke and J. Wimp, *Recursion formulae for generalized hypergeometric functions*, J. Approximation Theory, 1 (1968), pp. 137–166.

[8] J. L. FIELDS AND J. WIMP, *On the factorization of a class of difference operators*, Bull. Amer. Math. Soc., 74 (1968), pp. 1068–1071.

[9] J. L. FIELDS, *A linear scheme for rational approximations*, J. Approximation Theory, 6 (1972), pp. 161–175.

[10] ———, *The asymptotic expansion of the Meijer G-function*, Math. Comp., 26 (1972), pp. 757–765.

[11] D. L. LANSING, *Solution to Problem #66-10*, SIAM Rev., 9 (1967), pp. 593–598.

[12] Y. L. LUKE, *The Special Functions and their Approximations*, vols. I and II, Academic Press, New York, 1969.

[13] M. B. MCNEIL, *Problem #66-10*, *Asymptotic expansion of a Laplace transform*, SIAM Rev., 8 (1966), p. 234.

[14] G. F. MILLER, *On the convergence of the Chebyshev series for functions possessing a singularity in the range of representation*, SIAM J. Numer. Anal., 3 (1966), pp. 390–409.

[15] G. NÉMETH, *Polynomial Approximations to the Function* $\psi(a, c; x)$, Reports of the Central Institute for Physics, Budapest, 1965.

[16] J. WIMP, *The asymptotic representation of a class of G-functions for large parameter*, Math. Comp., 21 (1967), pp. 639–646.

[17] ———, *Recursion formulae for hypergeometric functions*, Ibid., 22 (1968), pp. 363–373.

[18] E. M. WRIGHT, *The asymptotic expansion of the generalized Bessel function*, Proc. London Math. Soc. (2), 38 (1935), pp. 257–270.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

ON TWO CONJECTURES OF ASKEY CONCERNING NORMALIZED HANKEL DETERMINANTS FOR THE CLASSICAL POLYNOMIALS*

GEORGE GASPER†

Abstract. It is shown that the Jacobi polynomials $p_n(x) = P_n^{(\alpha,\beta)}(x)$ satisfy

$$\begin{vmatrix} \dfrac{p_n(x)}{p_n(1)} & \dfrac{p_n'(x)}{p_n'(1)} \\ \dfrac{p_n'(x)}{p_n'(1)} & \dfrac{p_n''(x)}{p_n''(1)} \end{vmatrix} < 0, \qquad -1 < x < 1,$$

when $\alpha \geqq \beta > -1$ and $n = 2, 3, \cdots$; thus proving a conjecture of Askey. Another conjecture of Askey concerning Laguerre polynomials is proved and direct proofs are given for some inequalities involving normalized Wronskian-type determinants.

1. Introduction. A few years ago Askey [1] posed the problem of determining the values of α, β for which the Jacobi polynomials $p_n(x) = P_n^{(\alpha,\beta)}(x)$ satisfy

$$A_n(x) = \begin{vmatrix} \dfrac{p_n(x)}{p_n(1)} & \dfrac{p_n'(x)}{p_n'(1)} \\ \dfrac{p_n'(x)}{p_n'(1)} & \dfrac{p_n''(x)}{p_n''(1)} \end{vmatrix} < 0, \qquad -1 < x < 1, \tag{1}$$

for $n = 2, 3, \cdots$. Equation (1) fails when $\beta > \alpha > -1$, since then $A_n(-1) > 0$. Askey conjectured that (1) holds for $\alpha \geqq \beta > -1$; but this turned out to be unexpectedly hard to prove, even for the Chebyshev polynomials

$$\cos n\theta = T_n(\cos\theta) = P_n^{(-1/2,-1/2)}(\cos\theta)/P_n^{(-1/2,-1/2)}(1).$$

For a while I even doubted (1), and it was not until Askey communicated to me a proof of (1) for the Chebyshev polynomials that I again attempted to prove this inequality. Askey's proof could not be extended to yield the complete conjecture since it depended on special properties of trigonometric functions. However, it did supply the motivation which eventually led to the proof given here. Another conjecture in [1] concerning Laguerre polynomials will also be proven and simpler proofs of two theorems in [1] will be given.

The main tools used in proving (1) for $\alpha \geqq \beta > -1$ are an identity (equation (4) below) and the simple observation (previously used in [2]) that if $q_n(x)$ is a polynomial of degree n with zeros $x_1, x_2, \cdots, x_n$, then

$$\frac{q_n'}{q_n} = \sum_{k=1}^{n} (x - x_k)^{-1}, \qquad -\left(\frac{q_n'}{q_n}\right)' = \sum_{k=1}^{n} (x - x_k)^{-2}.$$

In fact, it is found that these tools yield the following generalization of Askey's conjecture.

* Received by the editors March 6, 1972.

† Department of Mathematics, Northwestern University, Evanston, Illinois 60201. This research was supported in part by a grant from the National Science Foundation.

2. Main result.

THEOREM 1. *Let* $p_n(x) = P_n^{(\alpha,\beta)}(x)$, $A_n(x)$ *be defined as in* (1) *and* $n = 2, 3, \cdots$. *If* $\alpha \geqq \beta > -1$, *then*

$$B_n(x) \equiv A_n(x) + a_n(1 - x^2)(p_n''(x))^2 \begin{cases} < 0, & -1 < x < 1, \\ = 0, & x = 1, \\ > 0, & x > 1, \end{cases}$$

where

$$a_n = \frac{4\Gamma(\alpha + 1)\Gamma(\alpha + 2)\Gamma(n - 1)\Gamma(n)}{(n + \alpha + \beta + 1)^2(n + \alpha + \beta + 2)(\Gamma(n + \alpha + 1))^2}.$$

If $\alpha = \beta > -1$, *then* $B_n(-1) = 0$ *and* $B_n(x) > 0$ *for* $x < -1$.

Proof. From [3, Chap. IV] we have

$$P_n^{(\alpha,\beta)}(1) = \binom{n + \alpha}{n},$$

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$(1 - x)^2p'' + [\beta - \alpha - (\alpha + \beta + 2)x]p' + n(n + \alpha + \beta + 1)p = 0,$$

where $p = p_n(x) = P_n^{(\alpha,\beta)}(x)$. Thus

$$(2) \qquad (1 - x^2)p''' + [\beta - \alpha - (\alpha + \beta + 4)x]p'' + (n - 1)(n + \alpha + \beta + 2)p' = 0$$

and

$$(3) \quad A_n(x) = a_n[(\alpha + 2)n(n + \alpha + \beta + 1)pp'' - (\alpha + 1)(n - 1)(n + \alpha + \beta + 2)(p')^2],$$

where a_n is as defined above. Equation (2) and the differential equation for p yield

$$\begin{aligned}(\alpha + 2)[\alpha - \beta + (\alpha + \beta + 2)x]p'p'' - (\alpha + 1)[\alpha - \beta + (\alpha + \beta + 4)x]p'p''\\ = (\alpha + 2)n(n + \alpha + \beta + 1)pp'' - (\alpha + 1)(n - 1)(n + \alpha + \beta + 2)p'^2\\ + (1 - x^2)p''^2 + (\alpha + 1)(1 - x^2)[p''^2 - p'p'''];\end{aligned}$$

so that, by (3),

$$\begin{aligned} B_n(x) &= A_n(x) + a_n(1 - x^2)p''^2 \\ (4) \qquad &= a_n(\alpha - \beta)(1 - x)p'p'' + a_n(\alpha + 1)(1 - x^2)[p'p''' - p''^2] \\ &= a_n(1 - x)[(\alpha - \beta)(p''/p') + (\alpha + 1)(1 + x)(p''/p')']p'^2. \end{aligned}$$

Since

$$\begin{aligned}(\alpha - \beta)(p''/p') + (\alpha + 1)(1 + x)(p''/p')' &= \sum_{k=1}^{n-1} \frac{(\alpha - \beta)(x - x_k) - (\alpha + 1)(1 + x)}{(x - x_k)^2} \\ &= \sum_{k=1}^{n-1} \frac{(\beta - \alpha)(1 + x_k) - (\beta + 1)(1 + x)}{(x - x_k)^2},\end{aligned}$$

where $x_1, x_2, \cdots, x_{n-1}$ are the zeros of p', then to complete the proof it suffices to observe [3, § 3.3] that these zeros are real and simple and are located in the open interval $(-1, 1)$.

3. Laguerre polynomials. For Laguerre polynomials $\lambda_n(x) = L_n^\alpha(x)$, the natural analogue of (1) would be

$$\begin{vmatrix} \dfrac{\lambda_n(x)}{\lambda_n(0)} & \dfrac{\lambda_n'(x)}{\lambda_n'(0)} \\ \dfrac{\lambda_n'(x)}{\lambda_n'(0)} & \dfrac{\lambda_n''(x)}{\lambda_n''(0)} \end{vmatrix} < 0, \qquad x > 0.$$

However, this fails for large values of x, and so Askey conjectured in [1] that if one used the normalizations

$$\begin{aligned} l_n(x) &= (-1)^n n! L_n^\alpha(x) = x^n + \cdots, \\ l_n'(x)/n &= (-1)^{n-1}(n-1)! L_{n-1}^{\alpha+1}(x) = x^{n-1} + \cdots, \\ l_n''(x)/n(n-1) &= (-1)^{n-2}(n-2)! L_{n-2}^{\alpha+2}(x) = x^{n-2} + \cdots, \end{aligned}$$

then

$$\begin{vmatrix} l_n(x) & l_n'(x)/n \\ l_n'(x)/n & l_n''(x)/n(n-1) \end{vmatrix} < 0, \qquad x > 0. \tag{5}$$

This inequality actually holds for $-\infty < x < \infty$. In fact, it was a surprise to find that (5) is a special case of a general result which I have known for some time but never realized that it had this interpretation.

THEOREM 2. *Let $q_n(x)$ be a polynomial of degree n which has only real zeros. Then*

$$\begin{vmatrix} q_n(x) & q_n'(x)/n \\ q_n'(x)/n & q_n''(x)/n(n-1) \end{vmatrix} \leqq 0, \qquad -\infty < x < \infty, \quad n \geqq 2, \tag{6}$$

and equality holds if and only if either all zeros of $q_n(x)$ are equal or the point x is a zero of $q_n(x)$ of at least order two.

Proof. Let $C_n(x)$ denote the determinant in (6). If $x_1, x_2, \cdots, x_n$ are the zeros of $q = q_n(x)$, then

$$\begin{aligned} n^2(n-1)C_n(x) &= [nqq'' - (n-1)q'^2] \\ &= [(q'/q)^2 + n(q'/q)']q^2 \\ &= \left[\left(\sum_{k=1}^n (x - x_k)^{-1}\right)^2 - n \sum_{k=1}^n (x - x_k)^{-2}\right] q^2, \end{aligned}$$

so the result follows from Schwarz's inequality.

4. Normalized Wronksian-type determinants. Recall that if orthonormal polynomials $q_n(x)$ are normalized by $q_n(x) = k_n x^n + \cdots$ with $k_n > 0$, then it follows from the Christoffel–Darboux formula [3, p. 43] that

$$\begin{vmatrix} q_n(x) & q_{n+1}(x) \\ q_n'(x) & q_{n+1}'(x) \end{vmatrix} = \frac{k_{n+1}}{k_n}[q_0^2(x) + q_1^2(x) + \cdots + q_n^2(x)] > 0, \qquad -\infty < x < \infty. \tag{7}$$

For $p_n(x) = P_n^{(\alpha,\beta)}(x)$, Askey [1] observed that if the normalization is changed to

$$\Delta_n(x;\alpha,\beta) = \begin{vmatrix} \dfrac{p_n(x)}{p_n(1)} & \dfrac{p_{n+1}(x)}{p_{n+1}(1)} \\[2ex] \dfrac{p_n'(x)}{p_n'(1)} & \dfrac{p_{n+1}'(x)}{p_{n+1}'(1)} \end{vmatrix},$$

then the sign of $\Delta_n(x;\alpha,\beta)$ is sensitive to the spectral interval, i.e.,

$$\Delta_n(x;\alpha,\beta) \begin{cases} > 0, & -1 < x < 1, \\ = 0, & x^2 = 1, \\ < 0, & x^2 > 1, \end{cases} \tag{8}$$

provided that $\alpha, \beta > -1$ and $n \geqq 1$. For Laguerre polynomials $\lambda_n(x) = L_n^\alpha(x)$, he found that if $\alpha > -1$, $n \geqq 1$, and

$$\Delta_n(x;\alpha) = \begin{vmatrix} \dfrac{\lambda_n(x)}{\lambda_n(0)} & \dfrac{\lambda_{n+1}(x)}{\lambda_{n+1}(0)} \\[2ex] \dfrac{\lambda_n'(x)}{\lambda_n'(0)} & \dfrac{\lambda_{n+1}'(x)}{\lambda_{n+1}'(0)} \end{vmatrix},$$

then

$$\Delta_n(x;\alpha) \begin{cases} > 0, & x > 0, \\ = 0, & x = 0, \\ < 0, & x < 0. \end{cases} \tag{9}$$

His proof of (8) depended on the identity

$$\begin{aligned} &|1 - x|^{\alpha+1}|1 + x|^{\beta+1}\Delta_n(x;\alpha,\beta) \\ &\quad = h_n \int_x^1 |1 - y|^{\alpha+1}|1 + y|^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(y) P_n^{(\alpha+1,\beta+1)}(y)\, dy, \qquad h_n > 0, \end{aligned} \tag{10}$$

which he used to analyze the relative extrema of the left-hand side of (10). For Laguerre polynomials one can obtain (9) directly from (use the formulas in [3, § 5.1])

$$\begin{aligned}\Delta_n(x;\alpha) &= b_n[(n+1)\lambda_{n+1}(x)\lambda_n'(x) - n\lambda_n(x)\lambda_{n+1}'(x)]\\ &= -b_n x\lambda_n^2(x)(\lambda_n'(x)/\lambda_n(x))'\\ &= b_n x\lambda_n^2(x)\sum_{k=1}^{n}(x-x_k)^{-2},\end{aligned}$$

where

$$b_n = \frac{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)}$$

and $x_1, x_2, \cdots, x_n$ are the (real) zeros of $\lambda_n(x) = L_n^\alpha(x)$. There did not seem to be a direct proof for the Jacobi case (8) until, while reading an old paper of Szegö [4], I noticed that after a change in notation, equation (1″) on page 89 of Szegö's paper yields the identity

$$\Delta_n(x;\alpha,\beta) = c_n(1-x^2)[u_{n-1}(x)u_n'(x) - u_{n-1}'(x)u_n(x)], \tag{11}$$

where $u_n(x) = P_n^{(\alpha+1,\beta+1)}(x)$ and

$$c_n = \frac{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n)\Gamma(n+1)}{2\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)}.$$

Thus (8) follows from (7) and (11). Equation (11) can easily be proved by using

$$-2n(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = [(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)]'$$

(see [3, (4.10.1)]) and this formula with n replaced by $n+1$.

Similarly, for Laguerre polynomials the formula

$$n\,e^{-x}x^\alpha L_n^\alpha(x) = [e^{-x}x^{\alpha+1}L_{n-1}^{\alpha+1}(x)]'$$

can be used to show that

$$\Delta_n(x;\alpha) = 2c_n x[v_{n-1}'(x)v_n(x) - v_{n-1}(x)v_n'(x)], \tag{12}$$

where $v_n(x) = L_n^{\alpha+1}(x)$ and c_n is as defined above. In deriving (9) from (12) and (7) one needs to use the fact that $L_n^{\alpha+1}(x) = k_n(-x)^n + \cdots$ with $k_n > 0$. Another interesting identity which should be pointed out is

$$\Delta_n(x;\alpha) = -\frac{\alpha+1}{x}\begin{vmatrix}\dfrac{L_{n-1}^\alpha(x)}{L_{n-1}^\alpha(0)} & \dfrac{L_n^\alpha(x)}{L_n^\alpha(0)}\\[2ex] \dfrac{L_n^\alpha(x)}{L_n^\alpha(0)} & \dfrac{L_{n+1}^\alpha(x)}{L_{n+1}^\alpha(0)}\end{vmatrix}.$$

This shows the connection between $\Delta_n(x;\alpha)$ and a determinant of Turán type. The connection between (5) and an inequality of Turán type for Charlier polynomials [3, p. 34] and also analogous results for orthogonal polynomials of a discrete variable will be considered elsewhere.

Acknowledgment. I wish to thank Professor R. Askey for communicating his result to me and for several interesting discussions on this subject.

REFERENCES

[1] R. ASKEY, *An inequality for the classical polynomials*, Nederl. Akad. Wetensch. Proc. Ser. A, 73, and Indag. Math., 32 (1970), pp. 22–25.

[2] G. GASPER, *An inequality of Turán type for Jacobi polynomials*, Proc. Amer. Math. Soc., 32 (1972), pp. 435–439.

[3] G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, American Mathematical Society, Providence, 1967.

[4] ———, *Über die Entwicklung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems*, Math. Z., 12 (1922), pp. 61–94.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

AN INNER PRODUCT INEQUALITY*

MICHAEL H. MOORE†

Abstract. A fundamental inequality which provides lower bounds for real-valued inner products is stated and proved.

1. Introduction. Let V be an inner product space. We denote the inner product of two vectors x and y of V by "$(x|y)$".

An upper bound on the absolute value of an inner product $(x|y)$ is, of course, immediately available from the famous Schwarz inequality; we have, for all x and y in V,

$$|(x|y)| \leqq \|x\| \cdot \|y\|.$$

This inequality expresses the intuitively clear fact that the length of a projection is less than or equal to that of the vector projected. Thus, the Schwarz inequality is a statement about the lengths of vectors and has nothing to do with the angle between them.

Our purpose here is to obtain, by reversing the emphasis on lengths vis-à-vis angles, a companion inequality which gives a lower bound on the absolute value of real-valued inner products. Thus, our inequality will express an equally obvious geometric fact in which the angle between the two vectors in an inner product is the prominent feature, the lengths playing no role whatever.

The main result—expressed in the following theorem—would seem to be fundamental in character and therefore interesting. Another form of this result which may be more useful for various purposes is given in Theorem 2.

2. Main Theorem. *Let V be a real inner product space, and let $x, y, z \in V$ be nonzero vectors such that*

$$|(x|y)| \geqq (1 - \varepsilon)\|x\| \cdot \|y\|,$$

$$|(x|z)| \geqq (1 - \varepsilon)\|x\| \cdot \|z\|,$$

where ε is a positive real number (presumably small). Then

$$|(y|z)| \geqq \gamma(\varepsilon)\|y\| \cdot \|z\|,$$

where $\gamma(\varepsilon) = 1 - \varepsilon - \sqrt{2\varepsilon}$.

The statement of the theorem is the formalization of the following intuitively plausible fact: if the direction of the vector y is close to that of x, and if the direction of the vector z is close to that of x, then the direction of y is close to that of z. This idea is certainly intuitively correct and, indeed, scarcely needs proof. It is therefore remarkable that the result seems to have gone unnoticed.

Proof. It suffices to consider the case where x, y, z are unit vectors. Further, we may assume without loss that $(x|y) \geqq 0$ (for if not replace x by $-x$). Then

$$|(y|z)| = |(y - x + x|z)| = |(y - x|z) + (x|z)|$$

$$\geqq |(x|z)| - |(y - x|z)| \geqq 1 - \varepsilon - |(y - x|z)|.$$

* Received by the editors February 3, 1972, and in revised form June 22, 1972.

† Department of Mathematics, University of Florida, Gainesville, Florida 32601.

But, using the Schwarz inequality, we have

$$|(y - x|z)| \leqq \|y - x\| \cdot \|z\| = \|y - x\|,$$

and

$$\begin{aligned} \|y - x\|^2 = (y - x|y - x) &= \|x\|^2 + \|y\|^2 - 2(x|y) \\ &= \|x\|^2 + \|y\|^2 - 2|(x|y)| \\ &\leqq 1 + 1 - 2(1 - \varepsilon) = 2\varepsilon. \end{aligned}$$

This gives

$$|(y - x|z)| \leqq \sqrt{2\varepsilon}.$$

Hence we obtain

$$|(y|z)| \geqq 1 - \varepsilon - \sqrt{2\varepsilon} = \gamma(\varepsilon)$$

and the proof is complete.

For small ε, a somewhat better result can be obtained from another (slightly longer) proof based on the triangle inequality; one finds that the inequality of the theorem still holds when γ has the value $1 - 4\varepsilon$. Thus we may sharpen the conclusion of the theorem by taking

$$\gamma(\varepsilon) = \max(1 - \varepsilon - \sqrt{2\varepsilon}, 1 - 4\varepsilon, 0).$$

The inequality is, of course, trivial when $\gamma(\varepsilon) = 0$, and this is annoying since the other two terms in the formula for γ become negative for $\varepsilon > 2 - \sqrt{3} \simeq 0.268$.

3. A more useful form. The fame of the Schwarz inequality rests not only on its elegance and simplicity but also on its extraordinary utility everywhere in analysis. One is frequently required to estimate inner products of various sorts (as, for example, when $V = L_2$ with the usual integral inner product), and for this an upper bound seems usually to be sufficient. Occasionally one may wish to obtain a lower bound, and for this the inequality just developed may be useful. However, the statement of the main theorem concerns not just two vectors but rather refers to the interplay of three. This deficiency may be repaired somewhat as in the following theorem, which is really just a rephrasing of the previous one.

THEOREM 2. *Let V be a real inner product space, and let $y, z \in V$ be any nonzero vectors. Then for any nonzero vector $x \in V$,*

$$|(y|z)| \geqq \gamma_x \cdot \|y\| \cdot \|z\|,$$

where

$$\gamma_x = \max\left\{4 \cdot \min\left(\frac{|(x|y)|}{\|x\| \cdot \|y\|}, \frac{|(x|z)|}{\|x\| \cdot \|z\|}\right) - 3, 0\right\}.$$

Remark. The value of γ_x could be sharpened, as will be clear from the proof; we avoid this for the sake of simplicity.

Proof. Choose any nonzero $x \in V$. The main theorem then holds for x, y, z with $\gamma(\varepsilon) = 1 - 4\varepsilon$ (for simplicity), where

$$\varepsilon = 1 - \min\left(\frac{|(x|y)|}{\|x\| \cdot \|y\|}, \frac{|(x|z)|}{\|x\| \cdot \|z\|}\right).$$

Hence, $\max(1 - 4\varepsilon, 0) = \gamma_x$, and the proof is complete.

An example. Let us conclude with an example of how one might use Theorem 2.

Suppose that $V = L_2(0, \infty)$ with the usual integral inner product. Consider the two vectors y and z of V given by

$$y(t) = \exp[-(1/2)t^2] \cos[(1 + \alpha)t],$$

$$z(t) = \exp[-(1/2)t^2] \cos[(1 - \alpha)t],$$

where α is a parameter, small in absolute value. Suppose that we are interested in the inner product

$$I(\alpha) = (y|z) = \int_0^\infty y(t)z(t)\,dt$$

$$= \int_0^\infty \exp[-t^2] \cos[(1 + \alpha)t] \cos[(1 - \alpha)t]\,dt,$$

which we imagine to be difficult to evaluate exactly (this is, of course, not really the case) because of the presence of the two cosine terms in the integrand, and that we therefore desire a lower bound $L(\alpha)$ for $|I(\alpha)|$ in addition to the usual upper bound provided by the Schwarz inequality.

Choose

$$x(t) = \exp[-(1/2)t^2].$$

We find that

$$\|x\|^2 = \frac{\sqrt{\pi}}{2},$$

$$\|y\|^2 = \frac{\sqrt{\pi}}{2} \exp[-\tfrac{1}{2}(1 + \alpha)^2] \cosh[\tfrac{1}{2}(1 + \alpha)^2],$$

$$\|z\|^2 = \frac{\sqrt{\pi}}{2} \exp[-\tfrac{1}{2}(1 - \alpha)^2] \cosh[\tfrac{1}{2}(1 - \alpha)^2],$$

$$(x|y) = \frac{\sqrt{\pi}}{2} \exp[-\tfrac{1}{4}(1 + \alpha)^2],$$

$$(x|z) = \frac{\sqrt{\pi}}{2} \exp[-\tfrac{1}{4}(1 - \alpha)^2],$$

$$\frac{(x|y)}{\|x\| \cdot \|y\|} = \{\cosh[\tfrac{1}{2}(1 + \alpha)^2]\}^{-1/2},$$

$$\frac{(x|z)}{\|x\| \cdot \|z\|} = \{\cosh[\tfrac{1}{2}(1 - \alpha)^2]\}^{-1/2}.$$

Thus we get

$$\min\left(\frac{|(x|y)|}{\|x\|\cdot\|y\|},\frac{|(x|z)|}{\|x\|\cdot\|z\|}\right)=\begin{cases}\dfrac{|(x|y)|}{\|x\|\cdot\|y\|} & \text{if } \alpha \geqq 0,\\[2ex] \dfrac{|(x|z)|}{\|x\|\cdot\|z\|} & \text{if } \alpha < 0\end{cases}$$

$$= \{\cosh[\tfrac{1}{2}(1+|\alpha|)^2]\}^{-1/2},$$

and so

$$\gamma_x \equiv \gamma(\alpha) = 4\cdot\{\cosh[\tfrac{1}{2}(1+|\alpha|)^2]\}^{-1/2} - 3.$$

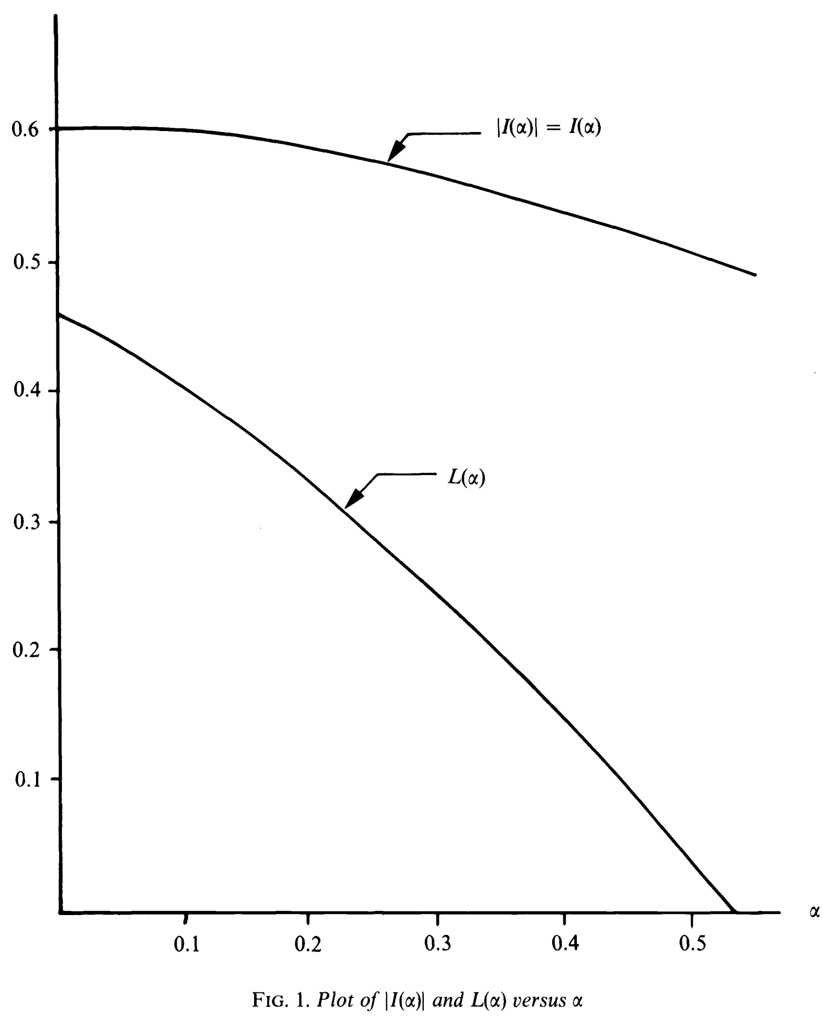

FIG. 1. *Plot of* $|I(\alpha)|$ *and* $L(\alpha)$ *versus* α

The quantity $\gamma(\alpha)$ above is positive for $|\alpha| < 0.536$. From Theorem 2, therefore, a lower bound $L(\alpha)$ for $|I(\alpha)|$ is given by

$$\begin{aligned} L(\alpha) &= \gamma(\alpha)\cdot\|y\|\cdot\|z\| \\ &= \frac{\sqrt{\pi}}{2}\exp\left[-\tfrac{1}{2}(1+\alpha^2)\right]\left(4\cdot\left\{\cosh\left[\tfrac{1}{2}(1-|\alpha|)^2\right]\right\}^{1/2}\right. \\ &\quad \left. - 3\cdot\left\{\cosh\left[\tfrac{1}{2}(1+\alpha)^2\right]\cosh\left[\tfrac{1}{2}(1-\alpha)^2\right]\right\}^{1/2}\right). \end{aligned}$$

On the other hand, the integral for $I(\alpha)$ is well known (see, for example, [1, p. 21, #20]); it is

$$\begin{aligned} I(\alpha) &= \int_0^\infty \exp\left[-t^2\right]\cos\left[(1+\alpha)t\right]\cos\left[(1-\alpha)t\right]dt \\ &= \frac{\sqrt{\pi}}{2}\exp\left[-\tfrac{1}{2}(1+\alpha^2)\right]\cosh\left[\tfrac{1}{2}(1-\alpha^2)\right]. \end{aligned}$$

Curves for comparison of $I(\alpha)$ and $L(\alpha)$ as functions of α are shown in Fig. 1.

REFERENCE

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, vol. 1, McGraw-Hill, New York, 1954.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

ASYMPTOTIC EXPANSIONS OF INTEGRAL TRANSFORMS WITH OSCILLATORY KERNELS: A GENERALIZATION OF THE METHOD OF STATIONARY PHASE*

R. A. HANDELSMAN† AND N. BLEISTEIN‡

Abstract. Integrals with integrands of the form $H(\lambda\phi(t))f(t)$ are considered for $\lambda \to \infty$ and $H(t)$ oscillatory for large argument. It is shown that the set of critical points for such integrals includes zeros of the phase function ϕ as well as all of those that arise in the analysis of the standard integrals of Fourier type, that is, for the special case where $H(t) = \exp\{it\}$. The contribution to the asymptotic expansion from each type of critical point is derived. In particular, a formula is obtained which generalizes the stationary phase formula associated with Fourier-type integrals.

1. Introduction. In the method of stationary phase, one is concerned with the asymptotic expansion, as $\lambda \to \infty$, of functions defined by integrals of the form

$$I(\lambda) = \int_a^b \exp\{i\lambda\varphi(t)\}g(t)\,dt. \tag{1.1}$$

We shall assume that the details of this method are familiar to the reader and need not be discussed.

Our concern here shall be with the asymptotic expansion of integrals of the form

$$I(\lambda) = \int_a^b H(\lambda\varphi(t))g(t)\,dt \tag{1.2}$$

in the case where the *kernel function* $H(t)$ is oscillatory for both large positive and large negative arguments. More precisely, we assume that, as $t \to +\infty$,

$$H(t) \sim \exp\left\{i\sum_{l=0}^{l<\nu/\delta} b_l t^{\nu-\delta l}\right\}\sum_{m=0}^{\infty}\sum_{n=0}^{N(m)} c_{mn}t^{-r_m}(\log t)^n. \tag{1.3}$$

Here each b_l is real, $N(m)$ is finite for all m, both ν and δ are positive, and $\{\mathrm{Re}\,(r_m)\}$ is a strictly increasing sequence with limit $+\infty$. In the limit $t \to -\infty$, we assume that an expansion of the form (1.3) holds with t replaced by $|t|$ and with, in general, different constants.

Clearly the Fourier kernel $\exp\{it\}$ is a special case of the general kernel we propose to study. Thus we should expect to recover from our asymptotic analysis of (1.2) the stationary phase results valid for (1.1).

We should point out that there are functions, such as the Airy function $Ai(t)$, which are oscillatory in one of the limits $t \to \pm\infty$ and exponential in the other. It will be apparent that integrals (1.2) with such functions as kernels can also be treated by the methods to be developed below.

* Received by the editors January 17, 1972, and in revised form July 12, 1972. This research was supported in part by the Air Force Office of Scientific Research under Grant AFOSR 70-1923 and in part by the National Science Foundation under Grant NSF GP-21052.

† Department of Mathematics, University of Illinois, Chicago Circle Campus, Chicago, Illinois 60680.

‡ University of Denver, Denver, Colorado 80210.

The method itself involves applications and generalizations of an asymptotic technique recently developed by Handelsman and Lew [1], [2], [3]. This technique makes heavy use of the Mellin transform whose relevant properties are discussed below. In references [1], [2] and [3] only the case $\varphi(t) = t$ is treated. Here, however, we shall consider more general φ and, in particular, shall allow φ to be nonmonotonic.

As is well known, to derive the asymptotic expansion of (1.1) one must first identify its set of critical points. These include endpoints of integration, stationary points of φ and points where either φ or g fails to be infinitely differentiable. We shall show that, in addition to all of the above, the set of critical points for (1.2) also includes the zeros of φ. Indeed, this is one of the main results of this paper.

To gain some insight into the critical nature of the zeros of φ, suppose that (1.3) is only an asymptotic result and, in particular, is not valid near $t = 0$. Then, no matter how large λ is, there always exists a neighborhood of each zero of φ throughout which $H(\lambda\varphi)$ is not asymptotically described by our assumed asymptotic forms. In other words, the asymptotic expansion of $H(\lambda\varphi)$, as $\lambda \to \infty$, undergoes a drastic change as t passes through any of these neighborhoods. For this reason we can think of these neighborhoods as "boundary layer regions". It is certainly reasonable that the rapid change in the asymptotic behavior of $H(\lambda\varphi)$ as t passes through a boundary layer region will affect the asymptotic expansion of I.

In the light of the above argument we can understand why the zeros of φ are *not* critical points for (1.1). Indeed, the Fourier kernel $H(t) = \exp(it)$ has an asymptotic expansion as $t \to +\infty$ of the form (1.3). This expansion, however, holds for all t so that there are no boundary layer regions of the type just described.

In the following section we reduce our problem to the study of certain integrals of canonical type. In § 3, we consider some results concerning Mellin transforms that are needed to implement our methods. Finally the desired asymptotic expansion of I is obtained in §§ 4 and 5.

2. Reduction to canonical integrals. Because there are many possible critical points for integrals of the form (1.2) with H an oscillatory kernel, it is convenient to have a means for isolating them so that their contributions to the asymptotic expansion of I can be studied separately. This can be accomplished by using neutralizer functions. These were first introduced by van der Corput [4], and we shall assume that their basic properties are familiar to the reader. The net effect of the neutralization process is to reduce the asymptotic analysis of (1.3) to the study of a sum of integrals each having exactly one critical point either as an upper or as a lower endpoint of integration.

Suppose first that $t = t_0$ is a critical point at which φ is nonzero. After neutralization, $t = t_0$ will appear as either an upper or lower endpoint of integration in at most two of the integrals to be asymptotically evaluated. To obtain the corresponding contributions to the asymptotic expansion of I, one need only replace $H(\lambda\varphi)$ by the appropriate asymptotic expansion and integrate the resulting series term by term. Thus, finding these contributions is reduced to the asymptotic evaluation of many integrals of the form (1.1). We have, therefore, that the only critical points which require nonstandard methods of analysis are the zeros of φ.

As a result of the above discussion, we shall focus our attention on obtaining the contribution to the asymptotic expansion of (1.2) corresponding to a given zero of φ. If we denote this zero by $t = c$ and the contribution by $I_c(\lambda)$, then after neutralization we find that

$$I_c(\lambda) = I_{c_+}(\lambda) + I_{c_-}(\lambda), \tag{2.1}$$

where

$$I_{c_-}(\lambda) = \int_a^c H(\lambda\varphi)g_{c_-}(t)\,dt \tag{2.2}$$

and

$$I_{c_+}(\lambda) = \int_c^b H(\lambda\varphi)g_{c_+}(t)\,dt. \tag{2.3}$$

Here $g_{c_-}(t)$ vanishes for $t < \alpha < c$ with α chosen so that neither φ nor φ' vanishes in $[\alpha, c)$. Furthermore, $g_{c_-}(t) = g$ in some small negative half neighborhood of $t = c$.

Similarly, $g_{c_+}(t)$ vanishes for $c < \beta < t$ with β chosen so that neither φ nor φ' vanishes in $(c, \beta]$. Finally, $g_{c_+} = g$ in some small positive half neighborhood to $t = c$. Of course if c coincides with one of the endpoints of integration in (1.2), then only one of the integrals I_{c_-}, I_{c_+} is nonzero.

Suppose now that as $t \to c - 0$, φ has an asymptotic expansion whose leading term is given by

$$\varphi \sim \gamma_0(c - t)^{\nu_0}, \qquad \nu_0 > 0. \tag{2.4}$$

If in (2.2) we introduce the new variable of integration

$$s = \mu_-\varphi(t), \qquad \mu_- = \operatorname{sgn} \gamma_0, \tag{2.5}$$

then we can write

$$I_{c_-}(\lambda) = \int_0^\infty H(\lambda\mu_- s)G_-(s)\,ds. \tag{2.6}$$

Here,

$$G_-(s) = \begin{cases} g_{c_-}(t(s))(dt/ds), & 0 \leqq s \leqq \mu_-\varphi(\beta), \\ 0, & \mu_-\varphi(\beta) < s. \end{cases} \tag{2.7}$$

Similarly, if we assume that as $t \to c + 0$,

$$\varphi(t) \sim \eta_0(t - c)^{\rho_0}, \qquad \rho_0 > 0, \tag{2.8}$$

and set

$$s = \mu_+\varphi, \qquad \mu_+ = \operatorname{sgn} \eta_0, \tag{2.9}$$

then I_{c_-} can be written

$$I_{c_+}(\lambda) = \int_0^\infty H(\lambda\mu_+ s)G_+(s)\,ds. \tag{2.10}$$

Here

$$G_+(s) = \begin{cases} g_+(t(s))(dt/ds), & 0 \leqq s \leqq \mu_+\varphi(\alpha), \\ 0, & \mu_+\varphi(\alpha) < s. \end{cases} \tag{2.11}$$

Thus we have reduced our problem to the study of the two canonical integrals

$$I_\pm(\lambda) = \int_0^\infty H(\pm\lambda s)G(s)\,ds, \tag{2.12}$$

where the kernel $H(t)$ is oscillatory in each of the limits $t \to \pm\infty$ and $G(s)$ vanishes for s outside of some finite interval.

3. Results on Mellin transforms. As we shall see, Mellin transforms play an important role in our asymptotic development. Indeed, one might anticipate this upon observing that each of the canonical integrals (2.12) can be expressed as a Mellin convolution [3].

The Mellin transform of a function $f(s)$ is defined by

$$M[f; z] = \int_0^\infty f(s)s^{z-1}\,ds, \qquad z = x + iy, \tag{3.1}$$

when this integral exists. Furthermore, if f is such that

$$f(s) = \begin{cases} O(s^p), & s \to 0+, \\ O(s^{-r}), & s \to +\infty, \end{cases} \tag{3.2}$$

then $M[f; z]$ converges and is holomorphic in the strip

$$-p < \operatorname{Re}(z) = x < r. \tag{3.3}$$

Also, within this strip $\lim_{|y|\to\infty} |M[f; x + iy]| = 0$.

From our point of view, there are two results concerning Mellin transforms that are of special significance. The first is the simple relation

$$M[f(\lambda s); z] = \lambda^{-z}\int_0^\infty f(s)s^{z-1}\,ds = \lambda^{-z}M[f; z]. \tag{3.4}$$

The second involves integrals of the form

$$J = \int_0^\infty f(s)h(s)\,ds. \tag{3.5}$$

Indeed, suppose that $M[f; 1 - z]$ and $M[h; z]$ exist and are holomorphic in overlapping vertical strips. (This will always be the case if J is absolutely convergent.) If $\operatorname{Re}(z) = c$ lies in the common strip of analyticity, then we have

$$\int_0^\infty f(s)h(s)\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[h; z]M[f; 1 - z]\,dz \tag{3.6}$$

which is Parseval's theorem for Mellin transforms.

Upon combining these last two results, we find that our canonical integrals (2.12) have the representations

$$I^{\pm}(\lambda) = \frac{1}{2\pi i}\int_{c_{\pm}-i\infty}^{c_{\pm}+i\infty} \lambda^{-z} M[H(\pm s); z] M[G(s); 1 - z]\, dz. \tag{3.7}$$

Here $\operatorname{Re}(z) = c_{\pm}$ lies in the strip of analyticity of the integrand. We note that the total dependence of the integrand on λ is contained in the factor λ^{-z}.

Our plan, which is standard, is to displace the contour of integration in (3.7) to the right, apply Cauchy's integral theorem, and derive thereby an asymptotic expansion of $I^{\pm}(\lambda)$ as a residue series. In order to accomplish this we must obtain certain information about the analytic continuations of the functions $M[H(\pm s); z]$ and $M[G(s); 1 - z]$ into the right half-plane. Specifically, we must locate and classify the singularities of these continuations, and we must estimate their behavior as $z \to \infty$ along vertical lines.

For oscillatory functions, the required information is contained in the following.

LEMMA 1. *Suppose that $H(s)$ is locally integrable on $(0, \infty)$, $O(s^p)$, as $s \to 0+$, and $O(s^{-r})$, as $s \to +\infty$, with $-p < r$. Suppose further that, as $s \to +\infty$, $H(s)$ has an asymptotic expansion of the form* (1.3) *in which event $-r \geqq r_0$. Then $M[H; z]$ can be continued into the right half-plane* $\operatorname{Re}(z) > -p$ *as a holomorphic function. Furthermore, in this right half-plane,*

$$|M[H; z]| = O(|y|^{(x - \operatorname{Re}(r_0))/\nu - 1/2 + \varepsilon}), \qquad |y| \to \infty, \tag{3.8}$$

for any $\varepsilon > 0$.[1]

Proof. The proof of this lemma is given in the Appendix.

As an example, and to illustrate the sharpness of the estimate (3.8), let us consider the function $H = \exp\{is^{\nu}\}$ whose Mellin transform is given by [5]

$$M[\exp(is^{\nu}); z] = \exp\left(\frac{i\pi z}{2\nu}\right)\Gamma\left(\frac{z}{\nu}\right), \qquad 0 < x < 1. \tag{3.9}$$

In this case $r_0 = 0$, and the analytic continuation into $\operatorname{Re}(z) > 1$ is explicit. We note that $\Gamma(z/\nu)$ is analytic in $\operatorname{Re}(z) > 0$. The estimate (3.8) follows from the known asymptotic expansion of the gamma function

$$\Gamma(z) \sim |y|^{x - 1/2} \exp[-\pi|y|/2], \qquad |y| \to \infty. \tag{3.10}$$

As we have indicated above, our plan is to displace the contour of integration in (3.7) to the right. In order to accomplish this we must of course determine the analytic continuation of $M[G; 1 - z]$ into the right half-plane. Let us assume for the present that this has been done and $M[G; 1 - z]$ is a meromorphic function. Then to justify the displacement of the contour to the line $\operatorname{Re}(z) = k > c_{\pm}$, one must still show that

$$\lim_{|y| \to \infty} M[H(\pm s); z] M[G(s); 1 - z] = 0, \qquad c_{\pm} \leqq x \leqq k. \tag{3.11}$$

[1] Note that here and in what follows the statement $|f(z)| = O(|y|^{-r})$, $|y| \to \infty$ implies $|f| < K|y|^{-r}$, $K < \infty$. K can depend on x but is independent of y.

The estimate (3.8) implies an algebraic growth of $M[H(\pm s); z]$ in this limit which worsens with increasing x. This growth must therefore be compensated by a commensurate decay of the analytic continuation of $M[G(s); 1 - z]$. In the following sequence of lemmas, we shall establish sufficient conditions for such decay.

LEMMA 2. *Let $G(s)$ be q times continuously differentiable on* $(0, \infty)$. *Let $G^{(q+1)}(s)$ be piecewise continuous on $[0, k]$ and continuous for $s \geqq k$. Finally suppose that there exists a real number x_0 such that for all $x > x_0$, $(s(d/ds))^p(s^x G(s))$ vanishes, as $s \to 0+$, for $p = 0, 1, 2, \cdots, q$ and, as $s \to \infty$, for $p = 0, 1, 2, \cdots, q + 1$. Then, as $|y| \to \infty$,*

$$M[G; z] = O(|y|^{-q-1}) \tag{3.12}$$

for all $x > x_0$.

Proof. The proof of this lemma is given in the Appendix.

Remarks. The hypotheses of Lemma 2 simply provide sufficient information to allow for the estimation of $M[G; z]$ via integration by parts. Furthermore, the assumptions on G imply that $M[G; z]$ is holomorphic in $-x_0 < \operatorname{Re}(z)$.

The next two lemmas follow from analogous results for Fourier transforms. Their proofs will be omitted here, but can be constructed from the corresponding proofs in Titchmarsh [6].

LEMMA 3. *Let $G(s)$ satisfy the conditions of Lemma 2, except now replace the condition on $G^{(q+1)}(s)$ by the assumption that $(s(d/ds))^q(s^x G(s))$ is of bounded total variation. Then*

$$M[G; z] = O(|y|^{-q-1}) \tag{3.13}$$

as $|y| \to \infty$, for all $x > x_0$.

LEMMA 4. *Let $G(s)$ satisfy the conditions of Lemma 2, except now replace the condition on $G^{(q+1)}(s)$ by the assumption that $(s(d/ds))^q(s^x G(s))$ is Hölder continuous of order γ on $[0, k]$ and of bounded total variation for $s > k$. Then*

$$M[G; z] = O(|y|^{-q-\gamma}) \tag{3.14}$$

as $|y| \to \infty$, for all $x > x_0$.

Lemmas 2–4 yield estimates on the decay of $M[G; z]$ in its region of absolute convergence or equivalently in its region of analyticity. We now wish to obtain analogous information outside of this region. As we shall soon see, the analytic continuation of $M[G; z]$ to the left (and hence of $M[G; 1 - z]$ to the right) depends to a large extent on the nature of $G(s)$ near $s = 0+$. Indeed we have the following result due to Handelsman and Lew [3].

LEMMA 5. *Suppose that $M[G; z]$ is holomorphic in the region $-\alpha < \operatorname{Re}(z) < \beta$, and that, as $s \to 0+$,*

$$G(s) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} d_{mn} s^{a_m} (\log s)^n, \tag{3.15}$$

with $\operatorname{Re}(a_m) \uparrow \infty$ and $N(m)$ finite for each m. Then, $\alpha = \operatorname{Re}(a_0)$ and $M[G; z]$ can be continued into $\operatorname{Re}(z) \leqq -\operatorname{Re}(a_0)$ as a meromorphic function with poles at the points $z = -a_m$. Moreover, about these points, $M[G; z]$ has a Laurent expansion *with singular part*

$$-\sum_{n=0}^{N(m)} d_{mn} \frac{\Gamma(n+1)}{(-z-a_m)^{n+1}}.$$

Remark. We note that when $N(m) = 0$ for each m, i.e., when no logarithms appear in the expansion (3.15), all of the poles in the analytic continuation of $M[G; z]$ to the left are simple.

If we combine the results of Lemmas 1 and 5, then we can conclude that when $G(s)$ has an expansion, as $s \to 0+$, of the form (3.15), all of the singularities of the analytic continuation of $M[H(\pm s); z]$, $M[G; 1 - z]$ into the right half-plane are determined by the exponents a_m. Moreover, these singularities are poles so that our proposed deformation of contour will indeed yield a residue series for $I^{\pm}(\lambda)$. We must still estimate $M[G; 1 - z]$ as $|y| \to \infty$ in order to justify the deformation. For this purpose we now state the following.

LEMMA 6. *Let $G(s)$ satisfy the smoothness conditions of Lemma 2. Also, let $(s(d/ds))^p(s^x G)$ vanish, as $s \to +\infty$, for $p = 0, 1, \cdots, q + 1$ and $x > 1 - \mathrm{Re}(a_0) = x_0$. Finally, suppose that (3.15) holds and that the asymptotic expansion of $G^{(m)}(s)$, $m = 0, \cdots, q + 1$, as $s \to 0+$, is obtained by successively differentiating (3.15) term by term. Then*

$$M[G; z] = O(|y|^{-q-1}), \qquad |y| \to \infty, \tag{3.16}$$

for all x. Here by $M[G; z]$ we mean the analytic continuation of this Mellin transform into the entire z-plane.

Proof. The proof of this lemma is given in the Appendix.

COROLLARY. *If, in Lemma 6, the stated conditions hold for all q, then $M[G; z] = O(|y|^{-r})$ for all r and all x.*

We remark that, if in Lemma 6, the smoothness conditions of Lemma 2 are replaced by those of either Lemma 3 or Lemma 4, then the corresponding changes must be made in the estimate (3.16). Nevertheless, one still finds that the results obtained are valid for the analytic continuation of the Mellin transform into the entire complex plane.

To illustrate some of the results obtained above let us consider an explicit example. Indeed, suppose that

$$G(s) = s^a e^{-s} \tag{3.17}$$

which satisfies the conditions of Lemma 6 with $a_0 = a$ and $q = \infty$. Then the corollary predicts that, as $|y| \to \infty$, $M[G; z]$ decays faster than any power of $|y|$. For this example we have the explicit result,

$$M[G; z] = \Gamma(a + z) = O\left(\exp\left(\frac{-\pi|y|}{2}\right)\right), \qquad |y| \to \infty,$$

which agrees with this prediction. Furthermore, we have from known properties of the gamma function that the analytic continuation of $\Gamma(a + z)$ into $\mathrm{Re}(z) < -a$ has simple poles at the points $z = -(a + m)$, $m = 0, 1, 2, \cdots$, with corresponding singular parts

$$(-1)^m/m!(z + a + m). \tag{3.18}$$

As is readily seen, this last result is in agreement with that predicted by Lemma 5.

4. Asymptotic expansion of $I^{\pm}(\lambda)$. By using the theory of Mellin transforms developed in the previous section, we shall now derive asymptotic expansions for the two canonical integrals (2.11). We first note that if $H(\pm s)$ has the asymptotic expansion (1.3) as $s \to \infty$, and $G(s)$ has the asymptotic expansion (3.15) as $s \to 0+$, then in (3.7)

$$c_{\pm} < \min(\operatorname{Re}(r_0), \operatorname{Re}(1 + a_0)) \tag{4.1}$$

since this is the right limit of the common strip of analyticity of the integrand in that equation.

We state the main result concerning the asymptotic behavior of $I^{\pm}(\lambda)$ in the following.

THEOREM 1. *Let $G(s)$ satisfy the conditions of Lemma 6 and $H(\pm s)$ satisfy the conditions of Lemma 1. Then*

$$\begin{aligned} I^{\pm}(\lambda) = &\sum_{\operatorname{Re}(a_m+1)<k} \sum_{n=0}^{N(m)} d_{mn}\lambda^{-(a_m+1)} \\ &\sum_{j=0}^{n} \binom{n}{j} (-\log\lambda)^j M^{(n-j)}[H(\pm s); 1 + a_m] + \mathscr{E}(\lambda; k). \end{aligned} \tag{4.2}$$

Here,

$$\begin{aligned} \mathscr{E}(\lambda; k) &= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \lambda^{-z} M[H(\pm s); z] M[G(s); 1 - z]\, dz \\ &= o(\lambda^{-k}), \qquad \lambda \to \infty, \end{aligned} \tag{4.3}$$

and

$$k < \nu(q + 1/2) + \operatorname{Re}(r_0), \tag{4.4}$$

where $k \neq \operatorname{Re}(a_m) + 1$ for any m.

Proof. In the exact representation (3.7) we displace the vertical contour of integration to the line $\operatorname{Re}(z) = k > c_{\pm}$. We note that by Lemmas 1 and 5 the analytic continuation of the integrand in (3.7) into the right half-plane is a meromorphic function with poles at the points $z = a_m + 1$, $m = 0, 1, \cdots$. Indeed, we find, upon formally applying Cauchy's integral theorem, that (4.2) is valid. Thus to complete the proof of the theorem we need only justify the displacement itself and establish the error estimate given by (4.3) and (4.4).

It follows from (3.8) and (3.16) that

$$M[H(\pm s); x + iy] M[G(s); 1 - x - iy] = O(|y|^{-\varepsilon(x)}), \qquad |y| \to \infty. \tag{4.5}$$

Here,

$$\varepsilon(x) = q + 3/2 - (x - \operatorname{Re}(r_0))/\nu. \tag{4.6}$$

Thus we can displace the contour to the line $\operatorname{Re}(z) = k$ so long as $\varepsilon(k) > 0$, i.e., so long as

$$k < \nu(q + 3/2) + \operatorname{Re}(r_0). \tag{4.7}$$

With (4.7) satisfied we have that $\mathscr{E}(\lambda; k)$ exists. The estimate (4.3) need not hold, however. We note that $\mathscr{E}(\lambda; k)$ can be viewed as a Fourier transform with respect to $\log \lambda$. Indeed, we have

(4.8)
$$\lambda^k \mathscr{E}(\lambda; k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp\{-iy\log\lambda\} M[H(\pm s); k + iy]M[G(s); 1 - k - iy]\,dy.$$

Suppose now that (4.4) is satisfied which, in turn, implies that $\varepsilon(k) > 1$. Then by applying the Riemann–Lebesgue lemma, we find that, as $\lambda \to \infty$, the right side of (4.8) is $o(1)$, and the estimate (4.3) follows.

COROLLARY. *Let $H(\pm s)$ satisfy the hypotheses of Lemma* 1 *and let $G(s)$ satisfy the hypotheses of Lemma* 6 *with $q = \infty$. Then the infinite expansion*

(4.9)
$$I^{\pm}(\lambda) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{N(m)} d_{mn}\lambda^{-(a_m+1)} \sum_{j=0}^{n}\binom{n}{j}(-\log\lambda)^j M^{(n-j)}[H(\pm s); 1 + a_m]$$

holds as $\lambda \to \infty$.

Proof. It follows from the corollary to Lemma 6 that, in this case,

(4.10)
$$M[H(\pm s); x + iy]M[G(s); 1 - x - iy] = o(|y|^{-r}), \qquad |y| \to \infty,$$

for all r and all x. Hence, we can let k go to $+\infty$ in (4.2) and (4.3) to obtain the desired result.

We note that when, in (3.15), $d_{mn} = 0$ for $n \geqq 1$ and all m, i.e., when no logarithms appear in the asymptotic expansion of $G(s)$, as $s \to 0+$, the asymptotic expansion (4.9) reduces to

(4.11)
$$I(\lambda) \sim \sum_{m=0}^{\infty} d_{m0}\lambda^{-(a_m+1)} M[H(\pm s); 1 + a_m].$$

In the proof of Theorem 1, the Riemann–Lebesgue lemma was applied to estimate $\lambda^k \mathscr{E}(\lambda; k)$ as $\lambda \to \infty$. Recently, Bleistein, Handelsman and Lew [7] have obtained a generalization of this lemma which, under slightly more restrictive assumptions, can be used to improve the error estimate found in Theorem 1. Indeed, we have the following.

THEOREM 2. *Let the hypotheses of Theorem* 1 *be satisfied so that* (4.5) *holds with $\varepsilon(x)$ defined by* (4.6). *Suppose further that*

(4.12)
$$M[H(\pm s); x + iy]M[G(s); 1 - x - iy] \sim c_0 \exp\{i\alpha y^{\mu}\}|y|^{-\varepsilon(x)},$$

as $|y| \to \infty$, where α and μ are any real numbers. Then (4.2) *and* (4.3) *hold with*

(4.13)
$$k < \nu(q + 1) + \operatorname{Re}(r_0), \qquad k \neq \operatorname{Re}(a_m + 1), \quad m = 0, 1, 2, \cdots.$$

Proof. It follows from Theorem 1 that all we need show is that (4.3) holds for all k satisfying (4.13). Thus consider $\lambda^k \mathscr{E}(\lambda; k)$ as given by (4.8). The results of reference [7] show that whenever $\varepsilon(k) > 0$,

(4.14)
$$\lim_{\lambda \to \infty} \lambda^k \mathscr{E}(\lambda; k) = 0,$$

except possibly when $1 < \mu < 2$ in (4.12). It is further shown that (4.14) still holds with $1 < \mu < 2$, so long as

$$0 < \varepsilon(k) - \left(1 - \frac{\mu}{2}\right) = q + 1 - \frac{(k - \operatorname{Re}(r_0))}{\nu} + \frac{1}{2} - \left(1 - \frac{\mu}{2}\right). \tag{4.15}$$

Since $1 < \mu$, it is clear (4.15) is satisfied whenever (4.13) holds. This completes the proof.

We wish to emphasize that the hypotheses of Theorem 2 differ from those of Theorem 1 only in that the former includes an additional assumption concerning the oscillatory behavior of $M[H(\pm s); z]M[G(s); 1 - z]$ in the limit $|y| \to \infty$. It is this more specific information that allows us to apply the results of reference [7] and thereby establish the validity of (4.3) with k restricted by (4.13).

Our concern, of course, is ultimately with the integrals $I_{c_\pm}(\lambda)$ from which the canonical integrals $I^\pm(\lambda)$ were directly derived. We recall that $s = 0$ in $I^\pm(\lambda)$ corresponds to $t = c$ in $I_{c_\pm}(\lambda)$, where c is a point in the original domain of integration at which the phase function φ vanishes. If we assume that the conditions of the corollary to Theorem 1 hold, then we find that $s = 0$ is the only critical point for $I^\pm(\lambda)$. Thus the infinite expansion (4.9) can be used to obtain the contribution to the asymptotic expansion of (1.2) corresponding to a zero of φ.

We must point out, however, that we have not, as yet, established the critical nature of $t = c$. This can be done by explicitly obtaining the expansions of $I_{c_\pm}(\lambda)$ and adding them. Only if the resulting sum is nontrivial can we conclude that $t = c$ is a critical point for $I(\lambda)$. We shall investigate this point further in the following section along with some illustrative examples.

5. Explicit results and examples. We wish now to determine the contribution to the asymptotic expansion of (1.2) corresponding to an interior zero of φ. Moreover, we want to express this result explicitly in terms of the original functions φ and g. In principle, we could, by using the results of the previous section, find as many terms of this contribution as desired. The computations, however, become exceedingly awkward as the number of terms increases and hence, for the most part, we shall be content here with obtaining expansions to leading order only. We shall assume throughout this section that the functions φ and g are such that either Theorem 1 or 2 can be applied to obtain the expansions below to the orders stated.

Let us suppose that in (1.2) $\varphi(c) = 0$ with $a < c < b$. If, as in § 2, we denote the contribution corresponding to $t = c$ by $I_c(\lambda)$, then we have

$$I_c = I_{c_+}(\lambda) + I_{c_-}(\lambda) \tag{5.1}$$

with $I_{c_\pm}(\lambda)$ defined by (2.10) and (2.6) respectively.

We now assume that, as $t \to c + 0$,

$$\begin{aligned} g(t) &\sim g_+(t - c)^{\omega_+ - 1}, \qquad \varphi(t) \sim \eta_0(t - c)^{\rho_0}, \\ \varphi'(t) &\sim \rho_0\eta_0(t - c)^{\rho_0 - 1}, \qquad\qquad \rho_0 > 0. \end{aligned} \tag{5.2}$$

In this event, the change of variable (2.9) is easily inverted to leading order. Indeed, we find that

$$G_+(s) \sim (g_+/\rho_0)|\eta_0|^{-\omega_+/\rho_0} s^{(\omega_+/\rho_0 - 1)}, \tag{5.3}$$

as $s \to 0+$. Hence it follows from (4.11) that, in this case,

$$(5.4) \qquad I_{c_+}(\lambda) = \frac{g_+}{\rho_0}\left(\frac{1}{\lambda|\eta_0|}\right)^{\omega_+/\rho_0} M\left[H(\mu_+ s); \frac{\omega_+}{\rho_0}\right] + o(\lambda^{-\omega_+/\rho_0}),$$

as $\lambda \to \infty$.[2] Here $\mu_+ = \operatorname{sgn} \eta_0$.

Similarly, if as $t \to c - 0$,

$$(5.5) \qquad g(t) \sim g_-(c-t)^{\omega_- - 1}, \qquad \varphi \sim \gamma_0(c-t)^{\nu_0},$$

$$(5.6) \qquad \varphi' \sim -\nu_0\gamma_0(c-t)^{\nu_0 - 1}, \qquad \nu_0 > 0,$$

then we find

$$(5.7) \qquad I_{c_-}(\lambda) = \frac{g_-}{\nu_0}\left(\frac{1}{\lambda|\gamma_0|}\right)^{\omega_-/\nu_0} M\left[H(\mu_- s); \frac{\omega_-}{\nu_0}\right] + o(\lambda^{-\omega_-/\nu_0}).$$

Here $\mu_- = \operatorname{sgn} \gamma_0$.

Upon adding (5.4) and (5.7) we obtain the desired contribution from $t = c$ to leading order. In most instances the constants in the assumed expansions (5.2) are closely related to the corresponding constants in (5.5) and (5.6). Two cases are worthy of special consideration. Suppose first that g is continuous and nonzero at $t = c$ so that

$$(5.8) \qquad g_+ = g_- = g(c), \qquad \omega_+ = \omega_- = 1.$$

Suppose further that φ is differentiable at $t = c$ with $\varphi'(c) \neq 0$. Then

$$(5.9) \qquad \eta_0 = -\gamma_0 = \varphi'(c), \qquad \rho_0 = \nu_0 = 1.$$

It follows from (5.5)–(5.9) that

$$(5.10) \qquad I_c(\lambda) \sim \frac{g(c)}{|\varphi'(c)|\lambda}\{M[H(s); 1] + M[H(-s); 1]\}.$$

Let us now suppose that relations (5.8) hold, but that φ has a simple stationary point at $t = c$. Then

$$(5.11) \qquad \eta_0 = \gamma_0 = \varphi''(c)/2, \qquad \rho_0 = \nu_0 = 1,$$

so that, in this case, we have

$$(5.12) \qquad I_c(\lambda) \sim g(c)\left(\frac{2}{\lambda|\varphi''(c)|}\right)^{1/2} M[H(\operatorname{sgn} \varphi''(c)s); \tfrac{1}{2}].$$

This last formula is a generalization of the standard stationary phase formula corresponding to $H(s) = \exp(\pm is)$. (See Example 1 below.)

To illustrate what happens when logarithms appear in the expansion of $G(s)$, as $s \to 0+$, let us suppose that $c = a$ in (1.2), φ is as in (5.2), and

$$(5.13) \qquad g(t) \sim g_{01}(t-c)^{\omega - 1}\log(t-c) + g_{00}(t-c)^{\omega - 1}, \qquad t \to c + 0.$$

[2] Note that if (5.4) is derived by applying Theorem 2, then $G_+(s)$ must have q continuous derivatives on $(0, \infty)$, where $q > (1/\nu)[\omega_+/\rho_0 - \operatorname{Re}(r_0)] - 1$.

After some calculation we find that, in (3.15), $N(0) = 1$ and

$$a_0 = \frac{\omega}{\rho_0} - 1, \qquad d_{01} = \frac{g_{01}}{\rho_0^2}|\eta_0|^{-\omega/\rho_0}, \tag{5.14}$$

$$d_{00} = \frac{|\eta_0|^{-\omega/\rho_0}}{\rho_0}\left[g_{00} - \frac{g_{01}\log|\eta_0|}{\rho_0}\right].$$

Thus, it follows from (4.2) and (5.14) that, in this case,

$$I_c(\lambda) = I_{c_+}(\lambda) \sim \frac{(\lambda|\eta_0|)^{-\omega/\eta_0}}{\rho_0}\left\{\frac{g_{01}}{\rho_0}\log\lambda M\left[H(\mu_+s);\frac{\omega}{\rho_0}\right]\right. \tag{5.15}$$

$$\left. + \left(\frac{g_{01}}{\rho_0}\log|\eta_0| - g_{00}\right)M\left[H(\mu_+s);\frac{\omega}{\rho_0}\right] - \frac{g_{01}}{\rho_0}\frac{d}{dz}M[H(\mu_+s);z]\Big|_{z=\omega/\rho_0}\right\}.$$

As a final general result, let us obtain an infinite asymptotic expansion of $I_c(\lambda)$ in the case where

$$\varphi(t) \equiv t - c \tag{5.16}$$

and $g(t)$ is infinitely differentiable at $t = c$. Then

$$g(t) \sim \sum_{m=0}^{\infty} \frac{g^{(m)}(c)}{m!}(t - c)^m, \qquad t \to c + 0, \tag{5.17}$$

$$g(t) \sim \sum_{m=0}^{\infty} \frac{(-1)^m g^{(m)}(c)}{m!}(c - t)^m, \qquad t \to c - 0.$$

Now applying (4.11) we obtain

$$I_c(\lambda) \sim \sum_{m=0}^{\infty} \frac{\lambda^{-(m+1)}g^{(m)}(c)}{m!}\{M[H(s); m + 1] + (-1)^m M[H(-s); m + 1]\}. \tag{5.18}$$

If any terms in this sum (5.18) are nonzero, then we must conclude that $t = c$ is a critical point for $I(\lambda)$. Alternatively, if the right-hand side of (5.18) is identically zero, then $t = c$ is not critical. The issue depends solely on the kernel and at that only through the quantities

$$M[H(s); m + 1] + (-1)^m M[H(-s); m + 1], \qquad m = 0, 1, 2, \cdots. \tag{5.19}$$

Thus, in general, $t = c$ is critical whenever H is such that at least one of the quantities (5.19) is nonzero. Furthermore, it is readily seen that the same conclusion holds when φ is any C^∞ function that vanishes at $t = c$.

We shall now consider two illustrative examples.

Example 1. Suppose $H(s)$ is the complex Fourier kernel $\exp(is)$. We have, by direct computation,

$$M[\exp(is); z] = \Gamma(z)\exp(\pi iz/2) \tag{5.20}$$

and

$$M[\exp(is); z] = e^{-\pi iz}M[\exp(-is); z]. \tag{5.21}$$

From this last relation we find

$$M[\exp(is); m+1] + (-1)^m M[\exp(-is); m+1] = 0, \qquad m = 0, 1, 2, \cdots, \tag{5.22}$$

and hence, as anticipated in the Introduction, the interior zeros of φ are *not* critical points for Fourier-type integrals.

Suppose now that g is continuous at $t = c$ and φ has a simple stationary point there. Then it follows from (5.12) and (5.20) that

$$I_c(\lambda) \sim g(c)\sqrt{\frac{2\pi}{\lambda|\varphi''(c)|}}\exp\left\{\operatorname{sgn}\varphi''(c)\frac{\pi i}{4}\right\}. \tag{5.23}$$

This will be recognized as the standard stationary phase formula in the case where $\varphi(c) = 0$. This last restriction is of course unnecessary and can be avoided quite simply. Indeed, suppose that at $t = c$, φ has a simple stationary point, but $\varphi(c) \neq 0$. Then we write

$$I(\lambda) = \exp(i\lambda\varphi(c))\int_a^b \exp[i\lambda(\varphi(t) - \varphi(c))]g(t)\,dt. \tag{5.24}$$

Since

$$\psi = \varphi(t) - \varphi(c) \tag{5.25}$$

has a simple stationary point at $t = c$ and $\psi(c) = 0$, we find that we need only multiply (5.23) by $\exp\{i\lambda\varphi(c)\}$ to obtain the valid result in this case. Furthermore, the contribution from any critical point at which $\varphi \neq 0$ can be recovered in an analogous manner from the corresponding contribution in the case where φ vanishes at the critical point.

Finally, suppose that $c = a$ and (5.13) holds. Suppose further that $\varphi(t) - \varphi(c)$ satisfies the relations satisfied by $\varphi(t)$ in (5.2). Then from (5.15), (5.20) and the remarks of the preceding paragraph, we find that now

$$\begin{aligned} I_c(\lambda) \sim \exp&\left\{i\lambda\varphi(c) + \frac{\mu_+\pi i\omega}{2\rho_0}\right\}\frac{(\lambda|\eta_0|)^{-\omega/\rho_0}}{\rho_0}\Gamma\left(\frac{\omega}{\rho_0}\right) \\ &\cdot\left\{\frac{g_{01}}{\rho_0}\log\lambda + \frac{g_{01}}{\rho_0}\log|\eta_0| - g_{00} - \frac{g_0}{\rho_0}\left(\psi\left[\frac{\omega}{\rho_0}\right] + \frac{\mu_+\pi i}{2}\right)\right\}. \end{aligned} \tag{5.26}$$

Here $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$. We might point out that the last case was considered in detail by Erdélyi [8] and by McKenna [9].

Example 2. Let us now suppose that in (1.2)

$$H(s) = J_n(s), \qquad n = 0, 1, 2, \cdots. \tag{5.27}$$

Here J_n is the Bessel function of the first kind of order n. We have [5]

$$M[J_n(s); z] = 2^{z-1}\frac{\Gamma((1/2)z + (1/2)n)}{\Gamma((1/2)n - (1/2)z + 1)}. \tag{5.28}$$

Since $J_n(s)$ is even about zero when n is even and odd about zero when n is odd, we have

$$M[J_n(s); z] = \begin{cases} -M[J_n(-s); z], & n \text{ odd}, \\ M[J_n(-s); z], & n \text{ even}. \end{cases} \tag{5.29}$$

From this it follows that

$$\begin{aligned} &M[J_n(s); m + 1] + (-1)^m M[J_n(-s); m + 1] \\ &\quad = \frac{2^m \Gamma((1/2)[m + n + 1])}{\Gamma((1/2)[n + 1 - m])}[1 + (-1)^{m+n}], \qquad m = 0, 1, 2, \cdots, \end{aligned} \tag{5.30}$$

and hence for any integer n one-half of the quantities (5.30) are not zero. Thus $t = c$ is a critical point in this case.

Appendix. In this Appendix, we shall prove Lemmas 1, 2 and 6 of the text.

Proof of Lemma 1. We introduce the functions

$$\sigma_k(s) = \exp\{-s^{-k} + is^{\nu}\omega(s)\} \sum_{m=0}^{M} \sum_{n=0}^{N(m)} c_{mn} s^{-r_m} (\log s)^n, \tag{A.1}$$

$$H_k(s) = H(s) - \sigma_k(s). \tag{A.2}$$

Here,

$$\omega(s) = \sum_{l=0}^{l<\nu/\delta} b_l s^{-\delta l}, \tag{A.3}$$

and for any positive k, we choose $M = M(k)$ to be the largest integer such that

$$\operatorname{Re}(r_M - r_0) < k. \tag{A.4}$$

We observe that $H(s)$ and σ_k have identical asymptotic expansions, as $s \to +\infty$, to order $s^{-r_M}(\log s)^{N(M)}$. As a result, $H_k(s) = O(s^{-k+\operatorname{Re}(r_0)})$ from which it follows that $M[H_k(s); z]$ is analytic in a strip with right limit $\operatorname{Re}(z) < k + \operatorname{Re}(r_0)$. The real exponential factor in σ_k assures us that $M[\sigma_k(s); z]$ is analytic in the left half-plane $\operatorname{Re}(z) < \operatorname{Re}(r_0)$. Thus the left limit of the strip of analyticity of $M[H_k; z]$ is the same as that of $M[H; z]$. Let us denote this limit by $\operatorname{Re}(z) = -\alpha$.

Below, we list the relevant Mellin transforms along with their strips of analyticity. We recall that in its strip of analyticity a Mellin transform decays to zero as $|y| \to \infty$.

Mellin transform	*Strip of analyticity*
$M[H; z]$	$-\alpha < \operatorname{Re}(z) < \operatorname{Re}(r_0)$
$M[\sigma_k; z]$	$-\infty < \operatorname{Re}(z) < \operatorname{Re}(r_0)$
$M[H_k; z]$	$-\alpha < \operatorname{Re}(z) < \operatorname{Re}(r_0) + k$

From the above list we see that in order to analytically continue $M[H; z]$ into the region $\operatorname{Re}(z) < k + \operatorname{Re}(r_0)$ we need only determine the analytic continuation of $M[\sigma_k; z]$ into the strip $\operatorname{Re}(r_0) \leqq \operatorname{Re}(z) < \operatorname{Re}(r_0) + k$. Furthermore, since $M[H_k; z]$ decays to zero as $|y| \to \infty$ in $-\alpha < \operatorname{Re}(z) < \operatorname{Re}(r_0) + k$, any algebraic growth, in this limit, of $M[H; z]$ must arise from $M[\sigma_k; z]$.

Since $\sigma_k(s)$ is a finite sum, so is its Mellin transform. A typical term in $M[\sigma_k; z]$ is given by

$$I[z; r] = \int_0^\infty \exp\{-s^{-k} + is^\nu \omega(s)\}(\log s)^n s^{z-r-1}\, ds, \qquad \mathrm{Re}(z) < \mathrm{Re}(r). \tag{A.5}$$

In (A.5) we rotate the path of integration onto the ray $\arg s = \theta$, where $0 < \theta \operatorname{sgn}(b_0) < \min(\pi/(2k), \pi/\nu)$. The effect of this rotation is to introduce sufficient decay at ∞ so that the integral in (A.5) converges for *all* z. Hence, $I[z; r]$ can be continued into the entire z-plane as a holomorphic function. It remains only to estimate the continuation as $|y| \to \infty$.

In (A.5) we stretch the integration variable s by the factor $|y|^{1/\nu}$ to obtain

$$\begin{aligned} I[z; r] = |y|^{(x-r)/\nu} \int_0^\infty & \exp\{(-|y|^{1/\nu}s)^{-k}\} \exp\{i|y|\psi(s; y)\} \\ & \cdot \left(\frac{1}{\nu}\log|y| + \log s\right)^n x^{x-r-1}\, ds. \end{aligned} \tag{A.6}$$

Here,

$$\psi(s; y) = s^\nu \omega(s|y|^{1/\nu}) + \operatorname{sgn}(y) \log s. \tag{A.7}$$

We note that when $-\operatorname{sgn}(y)/\nu b_0 < 0$, no stationary points of ψ (points at which $\psi_s = 0$) are near the positive real axis. When $-\operatorname{sgn}(y)/\nu b_0 > 0$, however, there are simple stationary points on or near the positive real axis. The results of reference [7] which justify the formal application of the ordinary method of stationary phase to (A.6) can be applied. Indeed, such a calculation yields

$$I[z; r] = \begin{cases} O(|y|^{-R}) \quad \text{for all } R, & \dfrac{-\operatorname{sgn}(y)}{\nu b_0} < 0, \\[2ex] O(|y|^{(x-r)/\nu - 1/2 + \varepsilon}), \quad \varepsilon > 0, & \dfrac{-\operatorname{sgn}(y)}{\nu b_0} > 0, \end{cases} \tag{A.8}$$

as $|y| \to \infty$. This completes the proof.

Proof of Lemma 2. By hypothesis,

$$M[G; z] = \int_0^\infty s^{iy-1}(s^x G(s))\, ds \tag{A.9}$$

is absolutely convergent for all $\mathrm{Re}(z) = x > x_0$.[3] Upon integrating by parts q times and using the stated properties of $(s(d/ds))^p(s^x G)$, we obtain

$$M[G; z] = \left(\frac{1}{-iy}\right)^q \int_0^\infty s^{iy-1}\left(s\frac{d}{ds}\right)^q (s^x G(s))\, ds. \tag{A.10}$$

We now break the interval of integration at the points of discontinuity of $G^{(q+1)}(s)$ and integrate by parts once more in the resulting finite sum of integrals. In this

[3] Note that the assumptions made imply that $G(s) = o(s^{-r})$, as $s \to \infty$, for all r and $G(s) = o(s^{-x})$, as $s \to 0+$, for any $x > x_0$.

manner we obtain

$$\int_0^\infty s^{iy-1}\left(s\frac{d}{ds}\right)^q (s^x G(s))\,ds = O(|y|^{-1}), \tag{A.11}$$

which, when combined with (A.10), completes the proof.

Proof of Lemma 6. If $\operatorname{Re}(z) = x > -\operatorname{Re}(a_0)$, then the result follows from Lemma 2 when we note that the conditions $(s(d/ds))^p(s^x G) = 0$, $p = 0, 1, \cdots, q$, as $s \to 0+$ are implied by the assumed differentiability properties of the expansion (3.15). Now suppose that ρ is any real number greater than $\operatorname{Re}(a_0)$. Also let $\mu(\rho)$ be a positive integer satisfying

$$\operatorname{Re}(a_{\mu-1}) < \rho \leqq \operatorname{Re}(a_\mu), \tag{A.12}$$

and let $\nu(\rho)$ be any integer such that $\operatorname{Re}(a_0) + \nu > \operatorname{Re}(a_\mu)$.

We now consider the functions

$$\sigma_\rho(s) = \left(\sum_{m=0}^{\mu-1}\sum_{n=0}^{N(m)} d_{mn} s^{a_m}(\log s)^n\right) e^{-s^\nu}, \tag{A.13}$$

$$G_\rho(s) = G(s) - \sigma_\rho(s) \tag{A.14}$$

and note that $\nu(\rho)$ has been chosen so that, as $s \to 0+$,

$$G_\rho = O(s^{\operatorname{Re}(a_\mu)}(\log s)^{N(\mu)}). \tag{A.15}$$

We also note that $G_\rho(s)$ has all of the properties attributed to $G(s)$ in the statement of the lemma. Hence, upon applying Lemma 2, we immediately find that, for $x > -\operatorname{Re}(a_\mu)$ and $|y| \to \infty$,

$$M[G_\rho(s); z] = O(|y|^{-q-1}). \tag{A.16}$$

By direct calculation we have that

$$\begin{aligned} M[\sigma_\rho; z] &= \sum_{m=0}^{\mu-1}\sum_{n=0}^{N(m)} d_{mn}\int_0^\infty s^{a_m+z-1}(\log s)^n e^{-s^\nu}\,ds \\ &= \sum_{m=0}^{\mu-1}\sum_{n=0}^{N(m)} d_{mn}\frac{d^n}{dz^n}\left(\int_0^\infty s^{a_m+z-1} e^{-s^\nu}\,ds\right) \\ &= \sum_{m=0}^{\mu-1}\sum_{n=0}^{N(m)} \frac{d_{mn}}{\nu}\frac{d^n}{dz^n}\left(\Gamma\left[\frac{a_m+z}{\nu}\right]\right) \end{aligned} \tag{A.17}$$

in the region $\operatorname{Re}(z) > -\operatorname{Re}(a_0)$ and by analytic continuation in the entire z-plane. We know, moreover, that each term in (A.17) decays exponentially as $|y| \to \infty$ for *all* x. Finally, since

$$M[G; z] = M[\sigma_\rho; z] + M[G_\rho; z], \tag{A.18}$$

we have that (3.16) holds for $\operatorname{Re}(z) > -\operatorname{Re}(a_{\mu(\rho)})$. However, ρ is arbitrary and $\lim_{\rho\to\infty} \operatorname{Re}[a_{\mu(\rho)}] = \infty$, so that upon letting $\rho \to \infty$ we obtain the desired result.

REFERENCES

[1] R. A. HANDELSMAN AND J. S. LEW, *Asymptotic expansion of a class of integral transforms via Mellin transforms*, Arch. Rational Mech. Anal., 35 (1969), pp. 382–396.

[2] ———, *Asymptotic expansion of Laplace transforms near the origin*, this Journal, 1 (1970), pp. 118–130.

[3] ———, *Asymptotic expansion of a class of integral transforms with algebraically dominated kernels*, J. Math. Anal. Appl., 35 (1971), pp. 405–433.

[4] J. G. VAN DER CORPUT, *On the method of critical points. I*, Nederl. Akad. Wetensch., Proc., 51 (1948), pp. 650–658.

[5] A. ERDÉLYI, ET AL., *Table of Integral Transforms*, vol. 1, McGraw-Hill, New York, 1954.

[6] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Oxford University Press, London, 1948.

[7] N. BLEISTEIN, R. A. HANDELSMAN AND J. S. LEW, *Functions whose Fourier transforms decay at infinity: An extension of the Riemann–Lebesque lemma*, this Journal, 3 (1972), pp. 485–495.

[8] A. ERDÉLYI, *Asymptotic expansions of Fourier integrals involving logarithmic singularities*, SIAM J. Appl. Math., 4 (1956), pp. 38–47.

[9] J. MCKENNA, *Note on asymptotic expansions of Fourier integrals with logarithmic singularities*, Ibid., 15 (1965), pp. 108–112.

SIAM J. MATH. ANAL.
Vol. 4, No. 3, August 1973

THE APPROXIMATE SOLUTION OF CONVOLUTION-TYPE INTEGRAL EQUATIONS*

FRANK STENGER†

Abstract. A result on the convergence of an approximate factorization of $\exp \phi$, where $\phi \in L^2(-\infty, \infty) \cap L^\infty(-\infty, \infty)$, is obtained and then used to get an explicit approximate solution $f^{(h)}$ of the problem $f(x) = \int_0^\infty K(x - t) f(t)\, dt + g(x), x > 0$, where K and g are in $L^2(-\infty, \infty)$. The approximation $f^{(h)}$ depends on a parameter h and satisfies $\| f - f^{(h)} \|_2 \to 0$ as $h \to 0$. A computationally more accessible explicit approximation $f_k^{(h)}$ is also obtained, which depends on a parameter k, and satisfies $|f^{(h)}(x) - f_k^{(h)}(x)| \to 0$ as $k \to 0$ for all $x \geqq 0$. Explicit bounds are obtained, for $|f(x) - f^{(h)}(x)|$ and also for $|f^{(h)}(x) - f_k^{(h)}(x)|$.

1. Introduction. Let $L^p(-\infty, \infty)(1 \leqq p < \infty)(p = \infty)$ denote the set of all complex-valued Lebesgue measurable functions f defined on $(-\infty, \infty)$ such that $\int_{-\infty}^\infty |f(t)|^p\, dt < \infty$ $(\text{ess sup}_{t \in (-\infty,\infty)} |f(t)| < \infty)$. The set $L^p(-\infty, \infty)$ is normed by

$$\| f \|_p = \left(\int_{-\infty}^\infty |f(t)|^p\, dt \right)^{1/p}, \qquad 1 \leqq p < \infty,$$

(1.1)

$$\| f \|_\infty = \operatorname*{ess\,sup}_{t \in (-\infty,\infty)} |f(t)|, \qquad p = \infty.$$

Let l^p denote the set of all functions q, where $q(t) = \sum_{-\infty}^\infty a_k t^k$, such that $\sum_k |a_k|^p < \infty$ if $1 \leqq p < \infty$.

Let $\mathscr{R}$ denote the family of all functions F given by

$$F(x) = \int_{-\infty}^\infty e^{ixt} f(t)\, dt, \tag{1.2}$$

where $f \in L^2(-\infty, \infty)$. Let $\mathscr{R}_+(\mathscr{R}_-)$ denote the subset of $\mathscr{R}$ for which $f(t) = 0$ if $t < 0\ (t > 0)$. It is known that $\mathscr{R} = L^2(R)$. Given $F \in \mathscr{R}$, we can recover f by means of the formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ixt} F(x)\, dx. \tag{1.3}$$

Let l_+^p denote the set of all functions $q \in l^p$ for which $a_k = 0$ if $k < 0$, and let l_-^p denote the set of all functions $q \in l^p$ for which $a_k = 0$ if $k > 0$.

In the present paper we give a constructive proof of the result stated in the abstract above. This proof enables us to obtain an explicit approximate solution $f^{(h)}$ of the equation

$$f(t) = \int_0^\infty k(t - \tau) f(\tau)\, d\tau + g(t), \qquad t > 0, \tag{1.4}$$

where k and g are functions in $L^2(-\infty, \infty)$ whose respective Fourier transforms $K \in \mathscr{R}$ and $G_+ \in \mathscr{R}_+$ can be explicitly expressed. The solution of (1.4) by the

* Received by the editors August 20, 1971, and in revised form April 25, 1972.

† Centre de Recherches Mathématiques, Université de Montréal, Montréal 101, Canada. This research was partially supported by the National Science Foundation under Grant GP 20293 at the University of Utah.

classical Wiener–Hopf technique depends upon finding the Fourier transform $F_+ \in \mathscr{R}_+$ of the function f in (1.4). The function F_+ satisfies the equation

$$F_+(1 - K) = G_+ + H_-, \tag{1.5}$$

where under suitable conditions on k, $H_- \in \mathscr{R}_-$. The discrete Wiener–Hopf problem related to (1.5) is to find a function $p_+ \in l^2_+$ which satisfies

$$p_+ q = r_+ + s_-, \tag{1.6}$$

where $q \in l^2$, $r_+ \in l^2_+$, $s_- \in l^2_-$, and where q and r_+ are given. The equation (1.6) may be obtained from (1.5) if we replace the independent variable x in (1.5) by $i(1 - z)/(1 + z)$; this transformation is a conformal map of the interior of the unit circle in the z-plane onto the upper half of the x-plane.

The use of the Whittaker cardinal function representation enables us to obtain an approximate solution $f^{(h)}$ of (1.4) by first obtaining an approximate solution $F^{(h)}_+$ of (1.5), which has the property $\|F_+ - F^{(h)}_+\|_2 \to 0$ as the parameter $h \to 0$. We assume that $K \in \mathscr{R} \cap L^\infty(-\infty, \infty)$, and that we can approximate $\log(1 - K)$ by $\log(1 - K^{(h)})$ such that $\|\log(1 - K) - \log(1 - K^{(h)})\|_q \to 0$ as $h \to 0$, for $q = 2$ and $q = \infty$. We can thus explicitly construct factors $1 + K^{(h)}_+$ and $1 + K^{(h)}_-$ such that $K^{(h)}_+ \in \mathscr{R}_+$, $K^{(h)}_- \in \mathscr{R}_-$, and such that

$$(1 - K^{(h)})^{-1} = (1 + K^{(h)}_+)(1 + K^{(h)}_-) \tag{1.7}$$

a.e. on $(-\infty, \infty)$. This leads to an explicit expression for $F^{(h)}_+$, and, a fortiori, an explicit expression for $f^{(h)}$, by use of (1.3).

In the case when K is real on $(-\infty, \infty)$, and $K(x + iy)$ is analytic in the region $\{x + iy : |y| \leqq d\}$ of the complex $x + iy$ plane, we obtain an explicit bound on $\|f - f^{(h)}\|_\infty$; this bound is $O(e^{-\pi d/h})$ as $h \to 0$.

Obtaining $f^{(h)}$ from $F^{(h)}_+$ is not a trivial problem, since it involves the evaluation of a repeated integral. We overcome this difficulty by obtaining an explicit Fourier series approximation $f^{(h)}_k$ to $f^{(h)}$. We obtain an explicit bound on $|f^{(h)}(t) - f^{(h)}_k(t)|$, which is valid when K is real on $(-\infty, \infty)$, and $K(x + iy)$ and $G_+(x + iy)$ are analytic in the region $\{x + iy : |y| \leqq d\}$. For fixed h and t we find in this case that $|f^{(h)}(t) - f^{(h)}_k(t)| = O(e^{-\pi d/k})$ as $k \to 0$.

The solution of (1.5) and (1.6) has been carried out by others.[1] In [1] Noble, and in [2], Carrier, Krook and Pearson solve special cases of (1.5) for which the factorization

$$(1 - K)^{-1} = (1 + K_+)(1 + K_-) \tag{1.8}$$

a.e. on $(-\infty, \infty)$ can be explicitly expressed, where $K_+ \in \mathscr{R}_+$ and $K_- \in \mathscr{R}_-$. In addition, approximate methods have been used on particular problems—see for example Noble [1], Carrier, Krook and Pearson [2], Folias [3] and Carrier [23] —although these authors gave no proofs of convergence. In [4] Dombrovskaia obtains an approximate solution[1] of (1.5) by approximating[2] $1 - K$ by a ratio of

---

[1] The method of the present paper is the only method known to the author, which establishes the convergence of an approximate method based on the classical Wiener–Hopf method.

[2] In [4] K is the Fourier transform of a function $K \in L^1(-\infty, \infty)$.

two polynomials, and then finding the roots of these polynomials. In [5], Baxter, and in [6], Orth obtain an approximate factorization of (1.6), where p_+, q, r_+ and s_- are respectively in l^1_+, l^1, l^1_+ and l^1_-, by solving a system of linear equations. This approximate factorization converges in l^1 to the exact factorization as the order of the system of linear equations to be solved becomes infinite.

We believe that the approximate method given in this paper has several advantages over methods given in [4], [5], [6] since, in obtaining an explicit factorization, we require neither the solution of a system of linear equations, nor the solution of a nonlinear problem that involves finding the roots of polynomials. In addition, we have given an explicit approximate solution $f_k^{(h)}$ of (1.4). Finally we mention the paper [15] in which an approximate solution of (1.4) is obtained under the assumption that k and g are in $L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$. The assumptions of the present paper are more general and the approximation method converges more rapidly than the corresponding ones in [15].

We also mention some results of a more abstract nature. In [7], Widon considers the problem (1.6), where p_+, q, r_+, and s_- are in l^2. He proves that conditions similar (though somewhat less explicit and less general) to those of the conditions (a), (b) and (c) in the abstract of this paper are necessary and sufficient for the existence of a solution p_+ of (1.6). In [8] Wik proves that if the upper limit ∞ in (1.4) is replaced by T, where $k, g \in L^1(-\infty, \infty)$, then the solution of the perturbed problem approaches the solution of (1.4) as $T \to \infty$. Recently Douglas and Taylor [9] found necessary and sufficient conditions such that the equation

$$\int_0^\infty f(t)\, d\mu(x - t) = g(x), \qquad x > 0, \tag{1.9}$$

has a solution $f \in L^p(-\infty, \infty)$ given $g \in L^p(0, \infty)$, where μ is an arbitrary Borel measure defined on $[-\infty, \infty]$. In a future paper we hope to study the applicability of this result, using the results in [17].

From the point of view of the numerical solution of more general types of equations we mention the work of Shinbrot [19], [20] who considers equations of the form

$$g(x) = \int_E k(x - t) f(t)\, dt, \qquad x \in E, \tag{1.10}$$

where E is a subset of $[-\infty, \infty]$, the papers of McNabb and Schumitzky [22], [24] who consider the transformation of Fredholm-type integral equations to Volterra-type equations, and the paper of Atkinson [26], in which he derives a numerical method for solving (1.4), but with $k(t - \tau)$ replaced by $k(t, \tau)$. The papers [19], [20], [22], [24], [26] describe direct methods which do not involve Fourier transforms. The method of Shinbrot resembles the Galerkin approximation method; the method of McNabb and Schumitzky extends the factorization method of Krein [13] to integral equations that are more general than convolution type, while Atkinson applies a quadrature scheme to reduce the integral equation to a system of linear algebraic equations.

2. Stability of factorizations

2.1. Results from Fourier transforms. Let us briefly state some facts relating $L^2(-\infty, \infty)$ and Fourier transforms (see, e.g., [11], [12]). We start with a function $f \in L^2(-\infty, \infty)$, and its Fourier transform $F \in \mathscr{R}$, which are connected by the formulas (1.2) and (1.3). By Parseval's theorem, we have $\|f\|_2 = (2\pi)^{-1/2}\|F\|_2$. If $F, G \in \mathscr{R}$ are the Fourier transforms of f and g respectively, and if $FG \in L^2(-\infty, \infty)$, then FG is the Fourier transform of h, where $h \in L^2(-\infty, \infty)$ is given by

$$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau. \tag{2.1}$$

Let $\mathscr{R}_+$ and $\mathscr{R}_-$ be defined as in the introduction of this paper. Since

$$F(x) = \int_{-\infty}^{0} e^{ixt}f(t)\, dt + \int_{0}^{\infty} e^{ixt}f(t)\, dt, \tag{2.2}$$

every $F \in \mathscr{R}$ has a unique representation[3]

$$F = F_- + F_+, \tag{2.3}$$

where $F_+ \in \mathscr{R}_+$, $F_- \in \mathscr{R}_-$. We may thus define a projection $\mathscr{P}$ from $\mathscr{R}$ to $\mathscr{R}_+$, i.e., for every $F \in \mathscr{R}$, $\mathscr{P}F = F_+$, where F and F_+ are as in (2.3).

Let Ω denote the entire complex plane, and let

$$\Omega_+ = \{z = x + iy : y > 0\}, \qquad \Omega_- = \{z = x + iy : y < 0\}.$$

Let $F \in \mathscr{R}$ and $z \in \Omega_+$. The function $F_+(z)$ defined by

$$F_+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t - z}\, dt \tag{2.4}$$

has the property, that if we let $y \to 0^+$, then[4] $F_+(z) \to F_+(x)$ a.e. on $(-\infty, \infty)$. Similarly, for $z \in \Omega_-$, we have

$$F_-(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t - z}\, dt, \tag{2.5}$$

where $\lim_{(y \to 0^+)} F_-(x - iy) = F_-(x)$ a.e. on $(-\infty, \infty)$.

If we now set $z = x + iy$ in (2.4), $z = x - iy$ in (2.5), where $y > 0$, and consider the limits $\lim_{(y \to 0^+)} F_+(x + iy)$, $\lim_{(y \to 0^+)} F_-(x - iy)$, we find that

$$\begin{aligned} F_+ &= (1/2)[F + \mathscr{H}F], \\ F_- &= (1/2)[F - \mathscr{H}F], \end{aligned} \tag{2.6}$$

where $\mathscr{H}F$ denotes the Hilbert transform

$$(\mathscr{H}F)(x) = \frac{\text{P.V.}}{\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t - x}\, dt, \tag{2.7}$$

and where P.V. denotes the principal value.

[3] That is, $\mathscr{R} = \{F : F = F_+ + F_-, F_+ \in \mathscr{R}_+, F_- \in \mathscr{R}_-\}$.

[4] We have chosen not to distinguish between $F_+(z)$ and its restriction to the boundary, the line $-\infty \leqq x \leqq \infty$.

In [2, p. 23] it is shown that if $p \in (1, \infty)$, and if $F \in L^p(-\infty, \infty)$, then

$$\|\mathscr{H}F\|_p \leqq A_p \|F\|_p, \tag{2.8}$$

where [2, p. 24]

$$A_p = \int_0^\infty \frac{4At^{-1/p}}{\sqrt{A^2 + t^2}}\, dt, \tag{2.9}$$

and where A is a constant which is independent of F and p. Upon evaluating (2.9) we find that

$$A_p = p\left[\frac{2^{3+1/p}A^{1-1/p}\Gamma(1 + 1/(2p))\Gamma(1 - 1/(2p))}{\Gamma(2 - 1/(2p))}\right]. \tag{2.10}$$

We thus note that if $p \geqq 2$, then the quantity in square brackets is bounded by a constant independent of p. Hence if $p \in [2, \infty]$ and if $F \in L^p(-\infty, \infty)$, then

$$\|\mathscr{H}F\|_p \leqq pB\|F\|_p, \tag{2.11}$$

where B is a constant independent of F and p. The minimum value of the constant B such that (2.11) is true for all $F \in L^p(-\infty, \infty)$, for $2 \leqq p \leqq \infty$, does not appear to be known[5]. However, by [2, p. 18] the constant A in (2.10) is bounded by $32/\pi$. Also, if we take F to be the function which is 1 on the interval (0, 1) and zero on $(-\infty, \infty) - (0, 1)$, then

$$(\mathscr{H}F)(x) = \frac{1}{\pi i}\log\left|\frac{x}{x-1}\right|. \tag{2.12}$$

For this function we have $\|F\|_p = 1$, and

$$\|\mathscr{H}F\|_p = \left(\frac{2\Gamma(p+1)}{\pi^p}\sum_{n=1}^\infty \frac{1}{n^p}\right)^{1/p}. \tag{2.13}$$

Hence, using Stirling's formula, we get

$$\limsup_{p\to\infty} \frac{\|\mathscr{H}F\|_p}{p\|F\|_p} = \frac{1}{\pi e}. \tag{2.14}$$

Therefore,

$$\frac{1}{\pi e} \leqq B \leqq \frac{2^{7/2}32^{1/2}\Gamma(5/4)\Gamma(3/4)}{\pi^{1/2}\Gamma(7/4)} = \frac{256\Gamma(5/9)}{3\pi^{1/2}}. \tag{2.15}$$

Let $\phi \in L^2(-\infty, \infty) \cap L^\infty(-\infty, \infty)$. Then clearly $\phi \in L^p(-\infty, \infty)$, $p \geqq 2$, since

$$\begin{aligned} \|\phi\|_p &= \left(\int_{-\infty}^\infty |\phi(t)|^p\, dt\right)^{1/p} \\ &\leqq \operatorname*{ess\,sup}_{x\in(-\infty,\infty)} \left(|\phi(x)|^{p-2}\int_{-\infty}^\infty |\phi(t)|^2\, dt\right)^{1/p} \\ &= \|\phi\|_\infty^{(p-2)/p}\|\phi\|_2^{2/p}. \end{aligned} \tag{2.16}$$

[5] Recently S. K. Pichorides [28] has shown that the minimum value of B is $2/\pi$.

If $F, G \in \mathscr{R}_+(\mathscr{R}_-)$ then $F \pm G \in \mathscr{R}_+(\mathscr{R}_-)$. Furthermore if $F, G \in \mathscr{R}_+(\mathscr{R}_-)$, and if $F\,G \in L^2(-\infty, \infty)$, then $F\,G \in \mathscr{R}_+(\mathscr{R}_-)$. We also have the following.

THEOREM 2.1. *Let* $F \in \mathscr{R}_+(\mathscr{R}_-)$. (a) *If* $\exp(F) - 1 \in L^2(-\infty, \infty)$, *then* $\exp(F) - 1 \in \mathscr{R}_+(\mathscr{R}_-)$. (b) *If* $F, G \in \mathscr{R}_+(\mathscr{R}_-)$ *and if* $F \exp G \in L^2(-\infty, \infty)$, *then* $F \exp G \in \mathscr{R}_+(\mathscr{R}_-)$.

Proof. (a) By [16, p. 436], F is uniquely represented on Ω_+ by its "Fourier series,"

$$F(z) \equiv \sum_{n=1}^{\infty} F_n \frac{(z - ia)^{n-1}}{(z + ia)^n}, \qquad a > 0, \tag{2.17}$$

which converges absolutely on Ω_+ and a.e. on the boundary of Ω_+. The series representation

$$G(z) \equiv \exp(F(z)) - 1 = \sum_{n=1}^{\infty} G_n \frac{(z - ia)^{n-1}}{(z + ia)^n} \tag{2.18}$$

is obtained by substituting the series in (2.17) into $\exp(F) - 1$, and collecting coefficients of equal powers of $\zeta = (z - ia)/(z + ia)$, noting that $1/(z + ia) = (1 - \zeta)/(2ia)$. If $\int_{-\infty}^{\infty} |G(x)|^2\, dx < \infty$, then G also has a "Fourier series" representation

$$G(x) = \sum_{n=-\infty}^{\infty} G'_n \frac{(x - ia)^{n-1}}{(x + ia)^n}, \tag{2.19}$$

which converges a.e. on $(-\infty, \infty)$. By the uniqueness of such a representation (see [16]) it follows that we must have $G'_n = 0$ for $n \leqq 0$, and $G'_n = G_n$ for $n > 0$. That is [16, p. 436], $G \in \mathscr{R}_+$.

The proof for the case $G \in \mathscr{R}_-$ is similar, and is omitted. The proof of the (b)-part of Theorem 2.1 is also similar to that of the (a)-part, and we omit it.

2.2. Stability of factorizations. Let ϕ be a given function in $L^2(-\infty, \infty) \cap L^\infty(-\infty, \infty)$, and let the following assumption be satisfied.

ASSUMPTION 2.2. *For some* $\sigma > 2$ *there exist unique functions* ψ *and* ω *such that*

(a) $\psi \in \mathscr{R}_+ \cap L^\sigma(-\infty, \infty)$, $\omega \in \mathscr{R}_- \cap L^\sigma(-\infty, \infty)$;

(b) $\log(1 + \psi) \in \mathscr{R}_+$, $\log(1 + \omega) \in \mathscr{R}_-$;

(c) $\exp \phi = (1 + \psi)(1 + \omega)$ *a.e. on* $(-\infty, \infty)$.

THEOREM 2.3. *Let the Assumption 2.2 be satisfied, and let* $\{\phi_\mu\}_{\mu=1}^{\infty}$ *be a sequence of functions such that* $\|\phi - \phi_\mu\|_p \to 0$ *as* $\mu \to \infty$ *for* $p = 2$ *and* $p = \infty$. *Define* ψ_μ *and* ω_μ *by*

$$\log(1 + \psi_\mu) = \mathscr{P}\phi_\mu, \qquad \log(1 + \omega_\mu) = (1 - \mathscr{P})\phi_\mu. \tag{2.20}$$

If $\varepsilon > 0$ *and* $\tau \in [2, \sigma)$ *are given, then there exists an integer* $\mu_0 > 0$ *such that whenever* $\mu > \mu_0$, *then* $\psi_\mu \in \mathscr{R}_+ \cap L^\tau(-\infty, \infty)$, $\omega_\mu \in \mathscr{R}_- \cap L^\tau(-\infty, \infty)$,

$$\exp \phi_\mu = (1 + \psi_\mu)(1 + \omega_\mu) \tag{2.21}$$

a.e. on $(-\infty, \infty)$, *and*

$$\|\psi - \psi_\mu\|_p < \varepsilon, \qquad \|\omega - \omega_\mu\|_p < \varepsilon \tag{2.22}$$

for $p = 2$ *and* $p = \tau$.

Proof. Let $\tau \in [2, \sigma)$ be fixed, and let us consider the identity

$$\psi - \psi_\mu = [e^{(1/2)(\phi + \mathscr{H}\phi)} - 1][1 - e^{-(1/2)(\eta + \mathscr{H}\eta)}] + 1 - e^{-(1/2)(\eta + \mathscr{H}\eta)}, \tag{2.23}$$

where $\eta = \phi - \phi_\mu$. Letting $p \in [2, \tau]$, taking p norms of each side, and using Hölder's inequality on the first term on the right of (2.23), we get

$$\begin{aligned} \|\psi - \psi_\mu\|_p &\leqq \|e^{(1/2)(\phi + \mathscr{H}\phi)} - 1\|_{\alpha p} \|1 - e^{-(1/2)(\eta + \mathscr{H}\eta)}\|_{\beta p} \\ &\quad + \|1 - e^{-(1/2)(\eta + \mathscr{H}\eta)}\|_p, \end{aligned} \tag{2.24}$$

where $\alpha > 1$ is chosen so that $\alpha p < \sigma$, and $\beta = \alpha/(\alpha - 1)$.

Since by (2.6) $e^{(1/2)(\phi + \mathscr{H}\phi)} - 1 = \psi$, the first term on the right of (2.24) is clearly bounded. We use the well-known inequality $|e^x - 1| \leqq |x| e^{|x|}$, to find that for any $q \geqq 2$,

$$\|1 - e^{-(1/2)(\eta + \mathscr{H}\eta)}\|_q \leqq \|e^{-(1/2)\eta}\|_\infty (\|\tfrac{1}{2}\eta\|_q + \|(\tfrac{1}{2}\mathscr{H}\eta)\, e^{-(1/2)\mathscr{H}\eta}\|_q). \tag{2.25}$$

Since η converges to zero in $L^2(-\infty, \infty) \cap L^\infty(-\infty, \infty)$ as $\mu \to \infty$, the term $e^{|(1/2)\eta|}$ is clearly uniformly bounded. For the term $\|\frac{1}{2}\mathscr{H}\eta\, e^{-(1/2)\mathscr{H}\eta}\|_q$ we have by (2.11) that

$$\begin{aligned} \|(\tfrac{1}{2}\mathscr{H}\eta)\, e^{|(1/2)\mathscr{H}\eta|}\|_q^q &\leqq \sum_{n=0}^{\infty} \frac{q^n \|\mathscr{H}\eta\|_{n+q}^{n+q}}{2^{n+q} n!} \\ &\leqq \sum_{n=0}^{\infty} \frac{q^n B^{n+q} (n+q)^{n+q} \|\eta\|_{n+q}^{n+q}}{n! 2^{n+q}}. \end{aligned} \tag{2.26}$$

Now using (2.16) we get

$$\|\eta\|_{n+q}^{n+q} \leqq \|\eta\|_2^2 \|\eta\|_\infty^{n+q-2}, \tag{2.27}$$

from which it follows that

$$\|(\tfrac{1}{2}\mathscr{H}\eta)\, e^{|(1/2)\mathscr{H}\eta|}\|_q^q \leqq \frac{B^q \|\eta\|_2^2 \|\eta\|_\infty^{q-2}}{2^q} \sum_{n=0}^{\infty} \frac{B^n q^n (n+q)^{n+q} \|\eta\|_\infty^n}{2^n n!}. \tag{2.28}$$

Now if

$$\limsup_{n \to \infty} \left(\frac{q^n (n+q)^{n+q} B^n}{2^n n!} \|\eta\|_\infty^n \right)^{1/n} = \frac{qeB}{2} \|\eta\|_\infty < 1, \tag{2.29}$$

then the series on the right of (2.28) converges. Since $\|\eta\|_\infty \to 0$ as $\mu \to \infty$, the series on the right of (2.28) converges and is uniformly bounded for all μ sufficiently large. Furthermore, since $q \geqq 2$, the term multiplying the series in (2.28) approaches zero as $\|\eta\|_p \to 0$, $p = 2, \infty$. Hence, $\|\psi - \psi_\mu\|_p \to 0$ as $\mu \to \infty$ for all $p \in [2, \tau]$, where $\tau \in [2, \sigma)$ is arbitrary.

Since $\log(1 + \psi_\mu) \in \mathscr{R}_+$, and since $\psi_\mu \in L^2(-\infty, \infty)$ for all μ sufficiently large, it follows by Theorem 2.1 that $\psi \in \mathscr{R}_+$.

The proof concerning the functions ω and ω_μ is similar, and we omit it. Since $\phi_\mu = \log(1 + \psi_\mu) + \log(1 + \omega_\mu)$, the equation (2.21) follows.

This completes the proof of Theorem 2.3.

3. Application to Wiener–Hopf equations.

3.1. The Wiener–Hopf technique. Consider the solution of the integral equation

$$f(x) = \int_0^\infty k(x - t)f(t)\,dt + g(x), \qquad x > 0. \tag{3.1}$$

The procedure of solving (3.1) by the standard Wiener–Hopf technique (see Noble [1], or Carrier [23]) is to add a function $s(x)$, which is at this point unknown, to the right of (3.1), so that (3.1) can be replaced by the equation

$$f(x) = \int_{-\infty}^\infty k(x - t)f(t)\,dt + g(x) = s(x), \qquad x \in (-\infty, \infty). \tag{3.2}$$

Here $f(x) = g(x) = 0$ if $x < 0$, $s(x) = 0$ if $x > 0$. One then takes the Fourier transform of each side of (3.2) to get

$$F_+(1 - K) = G_+ + S_-. \tag{3.3}$$

Let us now suppose that the following assumption is satisfied.

ASSUMPTION 3.1.

(a) $K \in \mathscr{R} \cap L^\infty(-\infty, \infty)$.

(b) $G_+ \in \mathscr{R}_+ \cap L^\rho(-\infty, \infty)$.

(c) *There exist unique functions* $K_+ \in \mathscr{R}_+ \cap L^\sigma(-\infty, \infty)$, $K_- \in \mathscr{R}_- \cap L^\sigma(-\infty, \infty)$, *such that* $\log(1 + K_+) \in \mathscr{R}_+$, $\log(1 + K_-) \in \mathscr{R}_-$, *and such that*

$$-\log(1 - K) = \log(1 + K_+) + \log(1 + K_-) \tag{3.4}$$

a.e. on $(-\infty, \infty)$.

(d) *The numbers* ρ *and* σ *satisfy*

$$\begin{aligned} &\rho > 2, \quad \sigma > 4, \\ &(\rho - 2)(\sigma - 4) \geqq 8. \end{aligned} \tag{3.5}$$

It is shown in [15] that if for fixed $k \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$ the equation (3.1) has a unique solution $f \in L^2(0, \infty)$ for all $g \in L^1(0, \infty) \cap L^2(0, \infty)$, then Assumption 3.1 is satisfied.

THEOREM 3.2. *If Assumption* 3.1 *is satisfied, then* (3.3) *has a unique solution* $F_+ \in \mathscr{R}_+$, *and* (3.1) *has a unique solution* $f \in L^2(0, \infty)$.

Proof. Let us seek a function $F_+ \in \mathscr{R}_+ \cap L^\rho(-\infty, \infty)$ which satisfies (3.3). Since $K \in L^\infty(-\infty, \infty)$, it follows by taking L^2 and L^ρ norms of each side of (3.3), that $S_- \in L^2(-\infty, \infty) \cap L^\rho(-\infty, \infty)$, i.e., that $S_- \in \mathscr{R}_- \cap L^\rho(-\infty, \infty)$. By (3.4) it follows that

$$(1 + K_+)(1 + K_-) = (1 - K)^{-1} \tag{3.6}$$

a.e. on $(-\infty, \infty)$. Hence,

$$F_+/(1 + K_+) = G_+ + K_-G_+ + (1 + K_-)S_-. \tag{3.7}$$

From (3.7) we obtain

$$\begin{aligned}\|F_+/(1+K_+)\|_2 &\leqq \|G_+\|_2 + \|K_-G_+\|_2 + \|S_-\|_2 + \|K_-S_-\|_2 \\ &\leqq \|G_+\|_2 + \|K_-\|_{2\alpha}\|G_+\|_{2\beta} + \|S_-\|_2 + \|K_-\|_{2\alpha}\|S_-\|_{2\beta} \\ &< \infty\end{aligned} \tag{3.8}$$

by Assumption 3.1, where the numbers α and β are chosen so that $1/\alpha + 1/\beta = 1$, $\rho/(\rho - 2) \leqq \alpha \leqq \sigma/4$, and $\sigma/(\sigma - 4) \leqq \beta \leqq \rho/2$. Since $F_+ \in \mathscr{R}_+$, and since $(1 + K_+)^{-1} = \exp(\mathscr{P}\log(1 - K))$, it follows by the (b)-part of Theorem 2.1 that $F_+/(1 + K_+) \in \mathscr{R}_+$.

Similarly it can be shown that $(1 + K_-)S_- \in \mathscr{R}_-$, and that $K_-G_+ \in \mathscr{R}$.

We now write (3.7) in the form

$$F_+/(1 + K_+) - G_+ - \mathscr{P}(K_-G_+) = (1 - \mathscr{P})(K_-G_+) + (1 + K_-)S_-. \tag{3.9}$$

On the left of (3.9) we have a function in R_+, while on the right of (3.9) we have a function in $\mathscr{R}_-$. From the unique representation of a function in $\mathscr{R}$ by a function in $\mathscr{R}_+$ plus a function in $\mathscr{R}_-$, both sides of (3.9) must vanish a.e. on $(-\infty, \infty)$, so that

$$F_+ = (1 + K_+)[G_+ + \mathscr{P}(K_-G_+)]. \tag{3.10}$$

Taking L^2 norms of each side of (3.10), we get

$$\|F_+\|_2 \leqq \|G_+\|_2 + \|K_+G_+\|_2 + \|\mathscr{P}(K_-G_+)\|_2 + \|K_+\mathscr{P}(K_-G_+)\|_2. \tag{3.11}$$

If we proceed as in (3.8), we find that the first three terms on the right of (3.11) are bounded. For the fourth term we have by Hölder's inequality and (2.11) that

$$\begin{aligned}\|K_+\mathscr{P}(K_-G_+)\|_2 &\leqq \|K_+\|_{4\alpha}\|\mathscr{P}(K_-G_+)\|_{2\delta} \\ &\leqq \|K_+\|_{4\alpha}\cdot\tfrac{1}{2}\|K_-G_+ + \mathscr{H}(K_-G_+)\|_{2\delta} \\ &\leqq \tfrac{1}{2}\|K_+\|_{4\alpha}(\|K_-G_+\|_{2\delta} + 2\delta B\|K_-G_+\|_{2\delta}) \\ &\leqq \tfrac{1}{2}(1 + 2\delta B)\|K_+\|_{4\alpha}\|K_-G_+\|_{2\delta} \\ &\leqq \tfrac{1}{2}(1 + 2\delta B)\|K_+\|_{4\alpha}\|K_-\|_{4\alpha}\|G_+\|_{2\beta},\end{aligned} \tag{3.12}$$

where $1 < \alpha < \infty$, $1 < \delta < \infty$, $1/\alpha + 1/\beta = 1$, $1/(2\alpha) + 1/\delta = 1$, and where α and β are again chosen as in (3.8). By Assumption 3.1 (b), (c) and (d) it follows that we clearly have $\|K_+\|_{4\alpha} < \infty$, $\|K_-\|_{4\alpha} < \infty$, and $\|G_+\|_{2\beta} < \infty$. Hence, $\|F_+\|_2 < \infty$. Similarly we can prove that $F_+ \in L^p(-\infty, \infty)$.

By returning to (3.3) it now follows that $S_- \in \mathscr{R}_- \cap L^p(-\infty, \infty)$.

Because of the unique representation (3.4) it follows that the solution F_+ given by (3.10) is unique. We thus obtain f by taking the inverse Fourier transform of F_+.

3.2. An approximate Wiener–Hopf method. Let $K(x)$ be defined as in § 3.1, and let us make the approximation

$$-\log[1 - K(x)] \cong \phi^{(h)}(x) \equiv \sum_j \phi_j S_j(h, x), \tag{3.13}$$

where for some fixed $h > 0$,

$$\phi_j = -\log[1 - K(jh + \tfrac{1}{2}h)], \tag{3.14}$$

$$S_j(h, x) = \frac{\sin[(\pi/h)(x - jh - \frac{1}{2}h)]}{(\pi/h)(x - jh - \frac{1}{2}h)}.$$

Now,

$$S_j(h, x) = \int_{-\pi/h}^{\pi/h} \left(\frac{h}{2\pi} e^{-i(j+1/2)ht}\right) e^{ixt}\, dt, \tag{3.15}$$

and therefore,

$$\begin{aligned} \mathscr{P}S_j(h, x) &= \int_0^{\pi/h} \left(\frac{h}{2\pi} e^{-i(j+1/2)ht}\right) e^{ixt}\, dt \\ &= T_j(h, x), \\ (1 - \mathscr{P})S_j(h, x) &= \overline{T_j(h, x)}, \end{aligned} \tag{3.16}$$

where

$$T_j(h, x) = \frac{1}{2}\exp\left[\frac{i\pi}{2h}\left(x - jh - \frac{1}{2}h\right)\right] S_j\left(\frac{h}{2}, \frac{x}{2}\right), \tag{3.17}$$

and where $\overline{T}_j$ denotes the complex conjugate of T_j.

Using the functions S_j and T_j we define $K_+^{(h)}$ and $K_-^{(h)}$ by

$$\begin{aligned} K_+^{(h)}(x) &= \exp\left\{\sum_j \phi_j T_j(h, x)\right\} - 1, \\ K_-^{(h)}(x) &= \exp\left\{\sum_j \phi_j \overline{T_j(h, x)}\right\} - 1. \end{aligned} \tag{3.18}$$

ASSUMPTION 3.3. *Let Assumption* 3.1 *be satisfied. Let*

$$\| -\log(1 - K) - \phi^{(h)} \|_p \to 0 \tag{3.19}$$

as $h \to 0$, *for* $p = 2$ *and* $p = \infty$, *where* $\phi^{(h)}$ *is defined in* (3.13).

Sufficient conditions under which (3.19) is satisfied can be found, for example, in [14].

THEOREM 3.4. *Let* $K_+^{(h)}$ *and* $K_-^{(h)}$ *be defined by* (3.20), *and let Assumption* 3.3 *be satisfied. If* $f \in L^2(0, \infty)$ *denotes the unique solution of* (3.1), *and if* $f^{(h)}$ *is defined by*

$$\begin{aligned} f^{(h)}(t) = {} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G_+(x)}{1 - K(x)} e^{-ixt}\, dx \\ & - \frac{1}{4\pi^2} \int_{-\infty}^{0} \left(\int_{-\infty}^{\infty} e^{-iy(t-u)} K_+^{(h)}(y)\, dy\right)\left(\int_{-\infty}^{\infty} e^{-ixu} K_-^{(h)}(x) G_+(x)\, dx\right) du \end{aligned} \tag{3.20}$$

if $t > 0$, *and by* $f^{(h)}(t) = 0$ *if* $t < 0$, *then*

$$\| f - f^{(h)} \|_2 \to 0 \tag{3.21}$$

as $h \to 0$.

Proof. Let $\omega < \sigma$ be chosen such that Assumption 2.2 is still satisfied, with σ replaced by ω. It follows then by Theorem 2.3 that $K_+^{(h)} \in \mathscr{R}_+ \cap L^\infty(-\infty, \infty)$, $K_-^{(h)} \in \mathscr{R}_- \cap L^\infty(-\infty, \infty)$ for all $h > 0$ sufficiently small, and that

$$\|K_+ - K_+^{(h)}\|_\rho \to 0, \qquad \|K_- - K_-^{(h)}\|_\rho \to 0 \tag{3.22}$$

as $h \to 0$ for all $\rho \in [2, \omega]$.

The function $f^{(h)}$ defined in Theorem 3.4 is readily seen to be the inverse Fourier transform of the function

$$\Phi_+^{(h)} = \mathscr{P}((1 - K)^{-1}G_+ - K_+^{(h)}(1 - \mathscr{P})(K_-^{(h)}G_+)). \tag{3.23}$$

We also have by (3.10) that

$$\begin{aligned} F_+ &= (1 + K_+)[G_+ + \mathscr{P}(K_-G_+)] \\ &= (1 + K_+)(1 + K_-)G_+ - (1 + K_+)(1 - \mathscr{P})(K_-G_+) \\ &= (1 - K)^{-1}G_+ - (1 + K_+)(1 - \mathscr{P})(K_-G_+) \\ &= \mathscr{P}[(1 - K)^{-1}G_+ - K_+(1 - \mathscr{P})(K_-G_+)], \end{aligned} \tag{3.24}$$

since $\mathscr{P}(1 - \mathscr{P}) = 0$.

Hence, we find that

$$\|F_+ - \Phi_+^{(h)}\|_2 = \|\mathscr{P}(K_+(1 - \mathscr{P})(K_-G_+) - K_+^{(h)}(1 - \mathscr{P})(K_-^{(h)}G_+))\|_2. \tag{3.25}$$

Now by Parseval's theorem it follows that if $H \in \mathscr{R}$, then $\|\mathscr{P}H\|_2 \leqq \|H\|_2$, $\|(1 - \mathscr{P})H\|_2 \leqq \|H\|_2$, and so

$$\begin{aligned} \|F_+ - \Phi_+^{(h)}\|_2 &\leqq \|K_+(1 - \mathscr{P})(K_-G_+) - K_+^{(h)}(1 - \mathscr{P})(K_-^{(h)}G_+)\|_2 \\ &\leqq \|(K_+ - K_+^{(h)})(1 - \mathscr{P})(K_-^{(h)}G_+)\|_2 \\ &\quad + \|K_+(1 - \mathscr{P})((K_- - K_-^{(h)})G_+)\|_2 \\ &\leqq \tfrac{1}{2}(1 + 2\delta B)(\|K_+ - K_+^{(h)}\|_{4\alpha}\|K_-^{(h)}\|_{4\alpha} \\ &\quad + \|K_- - K_-^{(h)}\|_{4\alpha}\|K_+\|_{4\alpha})\|G_+\|_{2\beta}, \end{aligned} \tag{3.26}$$

where α, β and δ are the same as in (3.12).

Since $\|K_+ - K_+^{(h)}\|_{4\alpha} \to 0$, $\|K_- - K_-^{(h)}\|_{4\alpha} \to 0$ as $h \to 0$, it follows that $\|F_+ - \Phi_+^{(h)}\|_2 \to 0$ as $h \to 0$. Since $f^{(h)}$ is the inverse Fourier transform of $\Phi_+^{(h)}$, (3.21) is a consequence of $\|F_+ - \Phi_+^{(h)}\|_2 \to 0$ as $h \to 0$, by Parseval's theorem.

This completes the proof of Theorem 3.4.

3.3. The evaluation of $f^{(h)}(t)$. We evaluate the integrals on the right of (3.20) by use of the Euler–Maclaurin formula, with step size k, where $0 < k < h$. In order to simplify the notation, we set

$$\begin{aligned} a_j &= K(jk), & j &= 0, \pm 1, \pm 2, \cdots, \\ b_j &= G_+(jk), & j &= 0, \pm 1, \pm 2, \cdots. \\ c_j &= K_+^{(h)}(jk), & j &= 0, \pm 1, \pm 2, \cdots, \\ d_j &= K_-^{(h)}(jk), & j &= 0, \pm 1, \pm 2, \cdots. \end{aligned} \tag{3.27}$$

Here we have chosen $0 < k < h$ in order to achieve some accuracy in the approximation of the integrals in (3.20). We have chosen the evaluation points jk, $j = 0, \pm 1, \pm 2, \cdots$, as opposed to $(j + \frac{1}{2})k$, $j = 0, \pm 1, \pm 2, \cdots$, in the previous section, in order to achieve a simpler program at the final stage of computation; the numerical evaluation of $K_+^{(h)}(x)$, $x = (j + \frac{1}{2})h$ would require a different procedure than that for the case when x does not have the form $(j + \frac{1}{2})h$.

We thus use the approximations

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K(x)G_+(x)}{1 - K(x)} e^{-ixt}\,dx \cong \begin{cases} \dfrac{k}{2\pi}\sum_j \dfrac{a_j b_j}{1 - a_j} e^{-ijkt}, & t \in [0, \pi/k], \\ 0, & t \notin [0, \pi/k] \end{cases}$$

$$(3.28)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} K_+^{(h)}(x)\, e^{-ixt}\,dx \cong \begin{cases} \dfrac{k}{2\pi}\sum_j c_j\, e^{-ijkt}, & t \in [0, \pi/k], \\ 0, & t \notin [0, \pi/k], \end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} K_-^{(h)}(x)G_+(x)\, e^{-ixt}\,dx \cong \begin{cases} \dfrac{k}{2\pi}\sum_j b_j d_j\, e^{-ijkt}, & t \in [-\pi/k, \pi/k], \\ 0, & t \notin [-\pi/k, \pi/k], \end{cases}$$

to get

$$(3.29)\qquad f^{(h)}(t) \cong f_k^{(h)}(t),$$

where

$$(3.30)\qquad \begin{aligned} f_k^{(h)}(t) = g(t) &+ \frac{k}{2\pi}\sum_j \frac{a_j b_j}{1 - a_j} e^{-ijkt} \\ &- \frac{k^2}{4\pi^2}\left(\frac{\pi}{k} - t\right)\sum_j b_j c_j d_j\, e^{-ijkt} \\ &- \frac{ik}{4\pi^2}\sum_{r,s} b_s c_r d_s \frac{e^{-irkt} - (-1)^{r-s}\, e^{-iskt}}{r - s}, \\ &r \neq s \quad \text{if} \quad 0 \leqq t \leqq \pi/k, \end{aligned}$$

and

$$f_k^{(h)}(t) = \begin{cases} g(t) & \text{if } t > \pi/k, \\ 0 & \text{if } t < 0. \end{cases}$$

3.4. Error bounds. Let $\mathscr{A}$ denote the family of all functions $H = H(x + iy)$ that are analytic in the region $\{x + iy : |y| \leqq d\}$, such that $H(x + iy) \to 0$ as $x \to \pm\infty$ for all $|y| \leqq d$, and such that

$$(3.31)\qquad M(H, d) \equiv \max_{y = \pm d}\left(\int_{-\infty}^{\infty} |H(x + iy)|^2\,dx\right)^{1/2} < \infty.$$

Let $H^{(h)}$ be defined by

$$H^{(h)}(x) = \sum_{j=-\infty}^{\infty} H((j+\tfrac{1}{2})h)S_j(h,x), \tag{3.32}$$

where S_j is defined in (3.14).

LEMMA 3.5. *Let* $H \in \mathscr{A}$. *Then*

$$\|H - H^{(h)}\|_2 \leqq \frac{2M(H,d)}{\sinh(\pi d/h)} \tag{3.33}$$

and

$$\|H - H^{(h)}\|_\infty \leqq \frac{M(H,d)}{(\pi d)^{1/2}\sinh(\pi d/h)}. \tag{3.34}$$

Proof. By proceeding as in [14, p. 148], we obtain the identity

$$\begin{aligned} H(x) - H^{(h)}(x) &= \frac{\cos(\pi/h)x}{2\pi i} \\ &\cdot \int_{-\infty}^{\infty}\left[\frac{H(t-id)}{(t-id-x)\cos(\pi/h)(t-id)} - \frac{H(t+id)}{(t+id-x)\cos(\pi/h)(t+id)}\right]dt. \end{aligned} \tag{3.35}$$

Now if $E \in \mathscr{R}$, $E_+ = \mathscr{P}E$, then for $y > 0$ it follows by (2.4) that

$$E_+(x+iy) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{E_+(t)}{t-x-iy}\,dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{E(t)}{t-x-iy}\,dt. \tag{3.36}$$

By a result of Hille [16, p. 440] we thus get

$$\left(\int_{-\infty}^{\infty}|E_+(x+iy)|^2\,dx\right)^{1/2} \leqq \left(\int_{-\infty}^{\infty}|E_+(x)|^2\,dx\right)^{1/2} = \|E_+\|_2, \tag{3.37}$$

and by Parseval's theorem, $\|E_+\|_2 \leqq \|E\|_2$. Now by successively taking

$$E(t) = \frac{H(t-id)}{\cos(\pi/h)(t-id)}, \qquad E(t) = \frac{H(t+id)}{\cos(\pi/h)(t+id)} \tag{3.38}$$

and using the inequality $|\cos(\pi/h)(t \pm id)| \geqq \sinh(\pi d/h)$, in (3.35), we obtain (3.33). The inequality (3.34) is obtained directly from (3.35) using Schwarz's inequality, i.e.,

$$\begin{aligned} \|H - H^{(h)}\|_\infty &\leqq \sup_{x\in(-\infty,\infty)} \frac{1}{2\pi\sinh(\pi d/h)}\int_{-\infty}^{\infty}\left\{\left|\frac{H(t-id)}{t-x-id}\right| + \left|\frac{H(t+id)}{t-x+id}\right|\right\}dt \\ &\leqq \frac{1}{2\pi\sinh(\pi d/h)}\left[\left(\int_{-\infty}^{\infty}|H(t-id)|^2\,dt\right)^{1/2} + \left(\int_{-\infty}^{\infty}|H(t+id)|^2\,dt\right)^{1/2}\right] \\ &\quad\cdot\left(\int_{-\infty}^{\infty}\frac{dt}{(t-x)^2+d^2}\right)^{1/2} \leqq \frac{M(h,d)}{(\pi d)^{1/2}\sinh(\pi d/h)}. \end{aligned} \tag{3.39}$$

This completes the proof.

THEOREM 3.6. *Let* $\phi \equiv -\log(1-K)$ *be real on* $(-\infty, \infty)$, *and let the conditions of Lemma* 3.5 *be satisfied, with* $H = \phi$. *If* $\eta = \phi - \phi^{(h)}$, *where* $\phi^{(h)}$ *is defined by* (3.13), *and if* $\|G_+\|_\infty < \infty$, *then*

$$\|f - f^{(h)}\|_\infty \leqq \frac{(\|\phi\|_2 + \|\eta\|_2)\|G_+\|_\infty \exp(\|\phi\|_\infty + \|\eta\|_\infty)}{\pi}\|\eta\|_2. \tag{3.40}$$

Remark 3.7. We note by (3.40) and Lemma 3.5 that $\|f - f^{(h)}\|_\infty = O(e^{-\pi d/h})$ as $h \to 0$.

Proof. Using (3.23) and (3.24) it follows for $t > 0$ that

$$\begin{aligned} f(t) - f^{(h)}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}[F_+(x) - \phi_+^{(h)}(x)]\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\mathscr{P}[K_+(1-\mathscr{P})(K_-G_+) - K_+^{(h)}(1-\mathscr{P})(K_-^{(h)}G_+)](x)\,dx \qquad (3.41)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}[K_+(1-\mathscr{P})(K_-G_+) - K_+^{(h)}(1-\mathscr{P})(K_-^{(h)}G_+)](x)\,dx, \end{aligned}$$

since dropping the operator $\mathscr{P}$ does not change the extreme left-hand side of (3.41) for $t > 0$.

Now writing the quantity in square brackets on the extreme right of (3.41) in the form $(K_+ - K_+^{(h)})(1-\mathscr{P})(K_-G_+) + K_+^{(h)}(1-\mathscr{P})(K_-^{(h)}G_+)$ and using Schwarz's inequality and the triangle inequality, we get

$$\begin{aligned} \|f - f^{(h)}\|_\infty &\leqq (1/(2\pi))[\|K_+ - K_+^{(h)}\|_2\|(1-\mathscr{P})(K_-G_+)\|_2 \\ &\qquad + \|K_+^{(h)}\|_2\|(1-\mathscr{P})((K_- - K_-^{(h)})G_+)\|_2] \qquad (3.42)\\ &\leqq (1/(2\pi))[\|K_+ - K_+^{(h)}\|_2\|K_-G_+\|_2 + \|K_+^{(h)}\|_2(K_- - K_-^{(h)})G_+\|_2] \\ &\leqq (\|G_+\|_\infty/2\pi)[\|K_-\|_2\|K_+ - K_+^{(h)}\|_2 + \|K_+^{(h)}\|_2\|K_- - K_-^{(h)}\|_2], \end{aligned}$$

where, in getting the second inequality on the right of (3.42), we have used the results that for any $H \in \mathscr{R}$, $\|\mathscr{P}H\|_2 \leqq \|H\|_2$ and $\|(1-\mathscr{P})H\|_2 \leqq \|H\|_2$.

Upon setting $\eta = \phi - \phi^{(h)}$, we get the identity

$$K_+ - K_+^{(h)} = e^{(1/2)(\phi + \mathscr{H}\phi)}[1 - e^{-(1/2)\eta} + e^{-(1/2)\eta}(1 - e^{-(1/2)\mathscr{H}\eta})]. \tag{3.43}$$

By assumption, ϕ is real on $(-\infty, \infty)$ which implies by (3.10) that η is real, and therefore $\mathscr{H}\phi$ and $\mathscr{H}\eta$ are purely imaginary. Hence,

$$\begin{aligned} &\|e^{(1/2)(\phi+\mathscr{H}\phi)}\|_\infty = \|e^{(1/2)\phi}\|_\infty \leqq e^{(1/2)\|\phi\|_\infty}, \qquad \|e^{-(1/2)\eta}\|_\infty \leqq e^{(1/2)\|\eta\|_\infty}, \\ &\|1 - e^{-(1/2)\eta}\|_2 \leqq \|\tfrac{1}{2}\eta\|_2\|e^{-(1/2)\eta}\|_\infty \leqq \tfrac{1}{2}\|\eta\|_2 e^{(1/2)\|\eta\|_\infty}, \\ &\|1 - e^{-(1/2)\mathscr{H}\eta}\|_2 \leqq \|e^{-(1/4)\mathscr{H}\eta}\|_\infty\|e^{(1/4)\mathscr{H}\eta} - e^{-(1/4)\mathscr{H}\eta}\|_2 \\ &\qquad = \|2i\sin((i/4)\mathscr{H}\eta)\|_2 \leqq \|\tfrac{1}{2}\mathscr{H}\eta\|_2 = \tfrac{1}{2}\|\mathscr{H}\eta\|_2 \leqq \tfrac{1}{2}\|\eta\|_2, \end{aligned}$$

where we have used the inequality $|\sin\theta| \leqq |\theta|$ for θ real, and where the very last inequality, $\|\mathscr{H}\eta\|_2 \leqq \|\eta\|_2$, was obtained by combining the representations (2.6) with Parseval's theorem. Therefore,

$$\|K_+ - K_+^{(h)}\|_2 \leqq e^{(1/2)\|\phi\|_\infty}[e^{(1/2)\|\eta\|_\infty} \cdot \tfrac{1}{2}\|\eta\|_2 + e^{(1/2)\|\eta\|_\infty} \cdot \tfrac{1}{2}\|\eta\|_2] = \|\eta\|_2\, e^{(1/2)(\|\phi\|_\infty + \|\eta\|_\infty)}. \tag{3.44}$$

Similarly, we obtain

$$\begin{aligned} \|K_- - K_-^{(h)}\|_2 &\leqq \|\eta\|_2\, e^{(1/2)(\|\phi\|_\infty + \|\eta\|_\infty)}, \\ \|K_-\|_2 &= \|e^{(1/2)\phi}(e^{-(1/2)\mathscr{H}\phi} - 1) + e^{(1/2)\phi} - 1\|_2 \\ &\leqq \tfrac{1}{2}\|\mathscr{H}\phi\|_2\, e^{(1/2)\|\phi\|_\infty} + \tfrac{1}{2}\|\phi\|_2\, e^{(1/2)\|\phi\|_\infty} \\ &\leqq \|\phi\|_2\, e^{(1/2)\|\phi\|_\infty} \leqq (\|\phi\|_2 + \|\eta\|_2)\, e^{(1/2)(\|\phi\|_\infty + \|\eta\|_\infty)}, \\ \|K_+^{(h)}\|_2 &\leqq \|\phi^{(h)}\|_2\, e^{(1/2)\|\phi^{(h)}\|_\infty} \leqq (\|\phi\|_2 + \|\eta\|_2)\, e^{(1/2)(\|\phi\|_\infty + \|\eta\|_\infty)}. \end{aligned} \tag{3.45}$$

Combining (3.44) and (3.45) with (3.42), we get (3.40).

This completes the proof of Theorem 3.6.

LEMMA 3.8. *Let the conditions of Theorem* 3.6 *be satisfied. If* $M(\cdot, d)$ *is defined as in* (3.31), *then*

$$M(\phi^{(h)}, d), M(\mathscr{P}\phi^{(h)}, d) \leqq 2^{1/2}\|\phi^{(h)}\|_2 \cosh(\pi d/h), \tag{3.46}$$

where

$$\|\phi^{(h)}\|_2^2 = \int_{-\infty}^{\infty} |\phi^{(h)}(x)|^2\, dx = h \sum_{j=-\infty}^{\infty} \phi_j^2, \tag{3.47}$$

and where $\phi_j = \phi(jh + \tfrac{1}{2}h)$.

Proof. Let S_j be defined by (3.14). The identity

$$S_j(h, x + iy) = \int_{-\pi/h}^{\pi/h} \left(\frac{h}{2\pi} e^{-i((j+1/2)h - iy)t}\right) e^{ixt}\, dt \tag{3.48}$$

and Parseval's theorem together yield

$$\begin{aligned} \alpha_{j,l} &\equiv \int_{-\infty}^{\infty} S_j(h, x + iy)\overline{S_l(h, x + iy)}\, dx \\ &= 2\pi\left(\frac{h}{2\pi}\right)^2 \int_{-\pi/h}^{\pi/h} e^{i(l-j)ht - 2yt}\, dt \\ &= \frac{h^2}{\pi} \frac{1}{(l-j)h + 2iy} \sin\left[(l - j)\pi + \frac{2\pi i y}{h}\right]. \end{aligned} \tag{3.49}$$

Consequently,

$$\begin{aligned} \int_{-\infty}^{\infty} |\phi^{(h)}(x + iy)|^2\, dx &= \int_{-\infty}^{\infty} \sum_{j,l} \phi_j \phi_l S_j(h, x + iy)\overline{S_l(h, x + iy)}\, dx \\ &= \sum_{j,l} \alpha_{j,l}\phi_j\phi_l. \end{aligned} \tag{3.50}$$

By assumption, ϕ is real on $(-\infty, \infty)$, and therefore each ϕ_j is real; since the sum on the extreme right of (3.50) is also real, we may replace $\alpha_{j,l}$ in (3.50) by its real part, which is given by

$$\beta_{j-l} \equiv \mathscr{R}l\alpha_{j,l} = \frac{2y}{\pi}\frac{(-1)^{j-l}}{(j-l)^2 + 4y^2/h^2}\sinh\left(\frac{2\pi y}{h}\right). \tag{3.51}$$

Hence in the notation of (3.47),

$$\begin{aligned}\int_{-\infty}^{\infty} |\phi^{(h)}(x+iy)|^2\,dx &= \sum_{j,l}\beta_{j-l}\phi_j\phi_l \\ &= \sum_j \phi_j \sum_l \beta_{j-l}\phi_l \\ &\leqq \frac{1}{h^{1/2}}\|\phi^{(h)}\|_2\left(\sum_j \left|\sum_l \beta_{j-l}\phi_l\right|^2\right)^{1/2} \\ &\leqq \frac{1}{h}\|\phi^{(h)}\|_2^2 \sum_l |\beta_l|.\end{aligned} \tag{3.52}$$

Now using the identity

$$\sum_{n=-\infty}^{\infty}\frac{1}{z^2+n^2} = \frac{\pi}{z}\frac{\cosh \pi z}{\sinh \pi z}, \tag{3.53}$$

which is readily obtained by means of the Fourier series expansion of e^{-zt} over $-\pi < t < \pi$, and also, using the extreme right of (3.51), we get

$$M(\phi^{(h)}, d)^2 \leqq \|\phi^{(h)}\|_2^2 \cosh\frac{2\pi d}{h}. \tag{3.54}$$

Since $\cosh(2\pi d/h) \leqq 2\cosh^2(\pi d/h)$, we get the first part of (3.46). The second part of (3.46) follows as a consequence of the inequality

$$\int_{-\infty}^{\infty} |\phi^{(h)}(x+iy)|^2\,dx \geqq \int_{-\infty}^{\infty} |\mathscr{P}\phi^{(h)}(x+iy)|^2\,dx,$$

which is valid for all real y.

This completes the proof.

Let $H \in \mathscr{A}$, and define $N(H, d)$ by

$$N(H,d) \equiv \sup_{\substack{y=\pm d \\ -\infty<x<\infty}} |H(x+iy)|. \tag{3.55}$$

LEMMA 3.9. *Let $\phi^{(h)}$ be defined as in Lemma* 3.8. *Then*

$$N(\phi^{(h)}, d),\ N(\mathscr{P}\phi^{(h)}, d) \leqq \frac{\|\phi^{(h)}\|_2}{(4\pi d)^{1/2}}e^{\pi d/h}. \tag{3.56}$$

Proof. By (3.13), (3.14) and (3.15), and the fact that $\|\phi^{(h)}\|_2 < \infty$, it follows that there exists a function $\Phi \in L^2(-\pi/h, \pi/h)$, such that

$$\phi^{(h)}(x) = \int_{-\pi/h}^{\pi/h} \Phi(t)\,e^{ixt}\,dt. \tag{3.57}$$

Hence,

$$\mathscr{P}\phi^{(h)}(x) = \int_0^{\pi/h} \Phi(t)\, e^{ixt}\, dt. \tag{3.58}$$

We now replace x by $x + iy$ in (3.57) and (3.58), apply Schwarz's inequality to each integral on the right, and note by Parseval's theorem that $\|\Phi\|_2 = (2\pi)^{-1/2}\|\phi^{(h)}\|_2$, to get (3.56).

This completes the proof.

THEOREM 3.10. *Let the conditions of Theorem* 3.6 *be satisfied, and let* $f_k^{(h)}$ *be defined by* (3.30). *If* $G_+(x + iy)$ *is an analytic function of* $x + iy$ *for all* $|y| \leqq d$, *if*

$$E = \max_{y = \pm d} \int_{-\infty}^{\infty} \left| \frac{K(x + iy)G_+(x + iy)}{1 - K(x + iy)} \right| dx < \infty, \tag{3.59}$$

and if $0 \leqq t < \pi/k$, *then*

$$\begin{aligned} |f^{(h)}(t) - f_k^{(h)}(t)| &\leqq \frac{2\, e^{dt} E}{e^{\pi d/k} \sinh(\pi d/k)} + \frac{\|G_+\|_\infty}{2\pi} \\ &\cdot [\|K_-^{(h)}\|_2 \|K_+^{(h)} - K_+^{(k)}\|_2 + \|K_+^{(k)}\|_2 \|K_-^{(h)} - K_-^{(k)}\|_2], \end{aligned} \tag{3.60}$$

where

$$\begin{aligned} \|K_\pm^{(h)} - K_\pm^{(k)}\|_2 &\leqq \frac{2\|\phi^{(h)}\|_2^2\, e^{\pi d/h} \cosh(\pi d/h)}{(2\pi d)^{1/2} \sinh(\pi d/k)}, \\ \|K_+^{(h)}\|_2, \|K_-^{(h)}\|_2 &\leqq \|\phi^{(h)}\|_2\, e^{(1/2)\|\phi^{(h)}\|_\infty}, \\ \|K_+^{(k)}\|_2 &\leqq \|K_+^{(h)}\|_2 + \|K_+^{(h)} - K_+^{(k)}\|_2. \end{aligned} \tag{3.61}$$

Remark 3.11. We note that $\|\phi^{(h)}\|_p \leqq \|\phi\|_p + \|\eta\|_p$ for $p = 2, \infty$, where $\eta = \phi - \phi^{(h)}$, and therefore a bound on $\|\eta\|_p$ is readily obtained by use of Lemma 3.5. We note in particular that for fixed h and t, $|f^{(h)}(t) - f_k^{(h)}(t)| = O(e^{-\pi d/k})$ as $k \to 0$. Thus a bound on $|f(t) - f_k^{(h)}(t)|$ may be obtained by combining the results of Theorem 3.6 and Theorem 3.10.

Proof of Theorem 3.10. The function $f_k^{(h)}$ defined in (3.30) is the inverse Fourier transform of a function $\Psi_k^{(h)}$, which is given by

$$\Psi_k^{(h)} = G_+ + \mathscr{P}[\omega_k - K_+^{(k)}(1 - \mathscr{P})\tau_k], \tag{3.62}$$

where

$$\begin{aligned} \omega_k(x) &= \sum_j \frac{a_j b_j}{1 - a_j} S_j(k, x), \\ K_+^{(k)}(x) &= \sum_j c_j S_j(k, x), \\ \tau_k(x) &= \sum_j b_j d_j S_j(k, x), \end{aligned} \tag{3.63}$$

and where a_j, b_j, c_j, d_j and S_j are defined as in (3.28) and (3.14). By (3.23), $f^{(h)}$ is the Fourier transform of the function

$$\Phi_+^{(h)} = G_+ + \mathscr{P}\left[\frac{KG_+}{1-K} - K_+^{(h)}(1-\mathscr{P})(K_-^{(h)}G_+)\right]. \tag{3.64}$$

If we take the inverse Fourier transform of $\Phi_+^{(h)} - \Psi_k^{(h)}$, and note, as in (3.41), that for $t > 0$, we can drop the operator $\mathscr{P}$ on the left, we obtain

$$\begin{aligned} |f^{(h)}(t) - f_k^{(h)}(t)| = \Bigg|\frac{1}{2\pi}\int_{-\infty}^{\infty} \Bigg[\frac{KG}{1-K} - \omega_k - (K_+^{(h)} - K_+^{(k)})(1-\mathscr{P})(K_-^{(h)}G_+) \\ - K_+^{(k)}(1-\mathscr{P})((K_-^{(h)} - K_-^{(k)})G_+)\Bigg](x)\, e^{-ixt}\, dx\Bigg|. \end{aligned} \tag{3.65}$$

By the definition of ω_k, it follows that for $0 \leqq t < \pi/k$, the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{KG}{1-K} - \omega_k\right](x)\, e^{-ixt}\, dx \tag{3.66}$$

is just the error of the approximation of $\int_{-\infty}^{\infty} [KG/(1-K)](x)\, e^{-ixt}\, dx$ by the Euler–Maclaurin formula, with step size k. This error is bounded in [14, p. 151]; using the results of [14, p. 151] we obtain the first term on the right of (3.60). We get the remaining part of (3.60) by following the steps taken on the right-hand side of (3.42). The method of bounding $\|K_-\|_2$ on the right-hand side of (3.45) is then used to get the second set of inequalities in (3.61). By Lemma 3.5 and (3.31) we thus get

$$\begin{aligned} \|K_+^{(h)} - K_+^{(k)}\|_2 &\leqq \frac{2M(\exp(\mathscr{P}\phi^{(h)}) - 1, d)}{\sinh(\pi d/k)} \\ &\leqq \frac{2M(\mathscr{P}\phi^{(h)}, d)N(\mathscr{P}\phi^{(h)}, d)}{\sinh(\pi d/k)} \\ &\leqq \frac{2\|\phi^{(h)}\|_2^2\, e^{\pi d/h}\cosh(\pi d/h)}{(2\pi d)^{1/2}\sinh(\pi d/k)}, \end{aligned} \tag{3.67}$$

where the second and third inequalities are obtained using Lemma 3.8 and Lemma 3.9. The term $\|K_-^{(h)} - K_-^{(k)}\|_2$ is similarly also bounded by the extreme right of (3.67).

This completes the proof of Theorem 3.10.

3.5. An example. The above described procedure has been applied by the author to the Picard equation [25, p. 95],

$$f(t) = \frac{1}{2\pi}\int_0^{\infty} e^{-|t-\tau|}f(\tau)\, d\tau + t^4 e^{-3t}, \qquad t > 0, \tag{3.68}$$

which has the explicit solution

$$
\begin{aligned}
f(t) = {} & \frac{12}{\beta}\left[\frac{1-\beta^2}{(3-\beta)^5} - \frac{(1-\beta)^2}{(3+\beta)^5}\right] e^{-\beta t} \\
& - \frac{1-\beta^2}{2\beta}\left\{24\left[\left(\frac{1}{3-\beta}\right)^5 - \left(\frac{1}{3+\beta}\right)^5\right] + 24t\left[\left(\frac{1}{3-\beta}\right)^4 - \left(\frac{1}{3+\beta}\right)^4\right]\right. \\
& + 12t^2\left[\left(\frac{1}{3-\beta}\right)^3 - \left(\frac{1}{3+\beta}\right)^3\right] + 4t^3\left[\left(\frac{1}{3-\beta}\right)^2 - \left(\frac{1}{3+\beta}\right)^2\right] \\
& \left. + t^4\left[\left(\frac{1}{3-\beta}\right) - \left(\frac{1}{3+\beta}\right)\right]\right\} e^{-3t} + t^4 e^{-3t},
\end{aligned}
\tag{3.69}
$$

where $\beta = \sqrt{1 - 1/\pi}$. Taking $h = \pi/10$ in (3.13) and using a step size $k = h/5$ in (3.28), we found that the approximation which we obtained was accurate to five significant figures.

REFERENCES

[1] B. NOBLE, *Methods Based on the Wiener–Hopf Technique*, Pergamon Press, New York, 1958.

[2] G. CARRIER, M. KROOK AND C. E. PEARSON, *Functions of a Complex Variable, Theory and Technique*, McGraw-Hill, New York, 1966.

[3] E. FOLIAS, *The stresses in a cracked spherical shell*, J. Math. and Phys., 44 (1965), pp. 164–176.

[4] I. N. DOMBROVSKIA, *Approximate solution of singular integral equations*, Ural. Gos. Univ. Math. Zap., 4 tetrad 2 (1963), pp. 38–45. (In Russian.)

[5] R. BAXTER, *A norm-inequality of a "finite section" Wiener–Hopf equation*, Illinois J. Math., 7 (1963), pp. 97–103.

[6] D. L. ORTH, *An algorithm for discrete Wiener–Hopf equations*, to appear.

[7] H. WIDOM, *Inversion of Toeplitz matrices II*, Illinois J. Math., 4 (1960), pp. 88–99.

[8] A. WIK, *Some results on convolution-type equations*, Preprint of the Department of Mathematics, Uppsala University.

[9] R. G. DOUGLAS AND J. L. TAYLOR, *Wiener–Hopf operators with measure kernels*, to appear.

[10] A. ZYGMUND, *Integrales Singuliers*, Publication du Séminaire de Mathématiques d'Orsay, 1965.

[11] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London, 1948.

[12] W. RUDIN, *Fourier Analysis on Groups*, Interscience, New York, 1962.

[13] M. G. KREIN, *Integral equations on the half-line with kernel depending upon the difference of arguments*, Amer. Math. Soc. Transl., 22 (1963), pp. 163–288.

[14] J. MCNAMEE, F. STENGER AND E. L. WHITNEY, *Whittaker's cardinal function in retrospect*, Math. Comp., 25 (1970), pp. 141–154.

[15] F. STENGER, *The approximate solution of Wiener–Hopf equations*, J. Math. Anal. Appl., 37 (1972).

[16] E. HILLE, *Analytic Function Theory*, vol. II, Blaisdell, Waltham, Mass., 1962.

[17] F. STENGER. *Constructive proofs of approximation by inner functions*, J. Approximation Theory, 4 (1971), pp. 373–386.

[18] C. CARATHÉODORY, *Theory of Functions of a Complex Variable*, vol. 1, Chelsea, New York, 1958.

[19] M. SHINBROT, *An inversion formula for certain general Wiener–Hopf operators*, to appear.

[20] ———, *The solution of some integral equations of Wiener–Hopf type*, Quart. Appl. Math., to appear.

[21] H. H. KAGIWADA, R. E. KALABA AND A. SCHUMITZKY, *A representation for the solution of Fredholm integral equations*, Proc. Amer. Math. Soc., 23 (1969), pp. 37–40.

[22] A. MCNABB AND A. SCHUMITZKY, *Factorization of operators II: A non-linear Volterra method for numerical solution of linear Fredholm equations*, J. Comput. System Sci., 4 (1970), pp. 103–128.

[23] G. F. CARRIER, *Analytic approximation techniques in applied mathematics*, SIAM J. Appl. Math., 13 (1965), pp. 68–95.

[24] A. McNabb and A. Schumitzky, *Factorization of operators III: Initial value methods for linear two-point boundary value problems*, J. Math. Anal. Appl., 31 (1970), pp. 391–405.

[25] W. Pogorzelski, *Integral Equations and Their Applications*, vol. 1, Pergamon Press, New York, 1966.

[26] K. Atkinson, *The numerical solution of integral equations on the half line*, SIAM J. Numer. Anal., 6 (1959), pp. 375–397.

[27] H. C. Kranzer, *Asymptotic factorization in nondissipative Wiener–Hopf problems*, J. Math. Mech., 17 (1967), pp. 577–600.

[28] S. K. Pichorides, *On the best values of the constants in the theorems of Riesz, Zygmund, and Kolmogorov*, Ph.D. thesis, University of Chicago, Chicago, 1971.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

A RESULT ON DIFFERENTIAL INEQUALITIES AND ITS APPLICATION TO HIGHER ORDER TRAJECTORY DERIVATIVES*

ROBERT W. GUNDERSON†

Abstract. A result on differential inequalities is obtained by considering the adjoint differential equation of the variational equation of the right side of the inequality. The main theorem is proved using basic results on differentiability of solutions with respect to initial conditions. The result is then applied to the problem of determining solution behavior using comparison techniques.

1. Introduction. In the following let I denote a compact interval and let f be continuous on the open set $D_0 \supset I \times R^n$. Let the vector functions x and u satisfy

$$x' = f(t, x), \tag{1}$$

$$u' \leqq f(t, u) \tag{2}$$

for $t \in I$, where the inequality of (2) implies the corresponding inequality for each of the component equations. Then, if f is additionally restricted by requiring a certain monotonic property on D_0 (called type K by Coppel [1]), a standard theorem from the theory of differential inequalities (Kamke [2]) asserts that if $x(t)$ is a right maximal solution on $[a, b] \subset I$ and $u(a) \leqq x(a)$, then $u(t) \leqq x(t)$ for all $t \in [a, b]$.

The above result has been used extensively to develop comparison type (vector Lyapunov function) stability theorems. It is also possible to rephrase the result in terms of a scalar nth order differential equation (Szarski [3, Chap. 2]) and to obtain comparison theorems in terms of higher order trajectory derivatives of v-functions (Gunderson [4]). However, these last theorems suffer from an overly restrictive condition in that the right side of the comparison equation must be of type K. This leads to the requirement that all $n - 1$ derivatives of the v-function satisfy comparison inequalities, along with the v-function itself, and to difficulties in the application.

In the following, a less restrictive condition of the required type is obtained by making use of some basic results on the differentiability of solutions with respect to their initial values. The condition agrees with a necessary condition for the solution of Chaplygin's problem obtained by Averbuch [5] through use of the Pontryagin maximal principle. Finally, the result is applied to the problem of determining solution behavior from higher order trajectory derivatives.

2. Component differential inequalities. In the following, $x(t, t_0, x_0)$ will denote a solution of (1) satisfying $x(t_0, t_0, x_0) = x_0$. It will be assumed that the solutions $x(t, s, u(s))$ exist on the compact interval I for each $s \geqq t_0, t_0 \in I$, where $u(t)$ satisfies inequality (2). Let f_x denote the matrix with $\partial f_i/\partial x_k$ at the intersection of the ith row and kth column and assume f_x continuous on the open region D_0.

* Received by the editors May 21, 1971, and in final revised form August 8, 1972.

† Department of Mathematics, Utah State University, Logan, Utah 84321. This work was supported by the National Aeronautics and Space Administration under Grant NGR-45-002-016.

The notation e_k will be used to denote the real n-tuple with unity as its kth entry and zeros elsewhere. The notation $f(x) \leqq g(x)$ for vector functions will be used if and only if $f_i(x) \leqq g_i(x)$ for all $i = 1, 2, \cdots, n$.

THEOREM. *Consider the adjoint differential equation of the variational equation*

$$y' = f_x(t, x(t, s, u(s)))y \tag{3}$$

for each $s \geqq t_0$, $t \in I$. *Suppose for some* k *every solution* $z(t, \bar{t}, e_k)$ *of the adjoint to* (3) *satisfies*

$$z(s, \bar{t}, e_k) \geqq 0$$

for each s *and* $\bar{t} \in I$, *with* $t_0 \leqq s \leqq \bar{t}$. *Then*

$$u_k(t) \leqq x_k(t, t_0, u(t_0))$$

for all $t \geqq t_0$, $t \in I$.

COROLLARY. *Let the* $n \times n$ *real matrix,* $A(t)$, *be continuous on* I *and suppose*

$$x' = A(t)x, \tag{1'}$$

$$u' \leqq A(t)u \tag{2'}$$

for $t \in I$. *Suppose the solution* $z(t, \bar{t}, e_k)$ *of the adjoint equation of* (1′) *is nonnegative on the interval* $[t_0, \bar{t}]$ *for each* $\bar{t} \geqq t_0$, $\bar{t} \in I$. *Then* (1′) *and* (2′) *imply*

$$u_k(t) \leqq x_k(t, t_0, u(t_0))$$

for all $t \geqq t_0$, $t \in I$.

Proof of theorem. For fixed k, $k = 1, 2, \cdots, n$, consider

$$v(s) = x_k(\bar{t}, s, u(s)),$$

where $\bar{t}$ is fixed and $t_0 \leqq s \leqq \bar{t}$. Under the assumptions placed on f, the unique solution $x(t, t_0, \xi)$ is of class C^1 on its open domain of definition $\omega- < t < \omega+$, $(t_0, \xi) \in D_0$, where $(\omega-, \omega+)$ denotes the maximal interval of existence (Hartman [6, p. 95]). Consequently,

$$v'(s) = \left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial s}\right|_{\xi = u(s)} + \sum_{i=1}^{n} \left[\left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial \xi_i}\right|_{\xi = u(s)}\right] u_i'(s).$$

Now suppose

$$\left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial \xi_i}\right|_{\xi = u(s)} \geqq 0, \qquad i = 1, 2, \cdots, n, \tag{A}$$

for all s, $t_0 \leqq s \leqq \bar{t}$. Then, since

$$u_i'(s) \leqq f_i(s, u(s)), \qquad i = 1, 2, \cdots, n,$$

it would follow that

$$\sum_{i=1}^{n} \left[\left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial \xi_i}\right|_{\xi = u(s)}\right] u_i'(s) \leqq \sum_{i=1}^{n} \left[\left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial \xi_i}\right|_{\xi = u(s)}\right] f_i(s, u(s))$$

for $t_0 \leqq s \leqq \bar{t}$. According to the theorem on differentiability of solutions with respect to their initial values mentioned above (Hartman [6, Thm. 3.1]),

$$\left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial s}\right|_{\xi = u(s)} = -\sum_{i=1}^{n} \left[\left.\frac{\partial x_k(\bar{t}, s, \xi)}{\partial \xi_i}\right|_{\xi = u(s)}\right] f_i(s, u(s)) \tag{4}$$

and

$$x_\xi(t, s, \xi)|_{\xi = u(s)} = Y(t, s), \tag{5}$$

where $Y(t, s)$ is the fundamental matrix of the variational equation

$$y' = f_x(t, x(t, s, u(s)))y \tag{6}$$

satisfying $Y(s, s) = I$, the identity matrix. From (4), it follows that if inequality (A) holds, then $v'(s) \leqq 0$ for $t_0 \leqq s \leqq \bar{t}$, so that

$$v(\bar{t}) = x_k(\bar{t}, \bar{t}, u(\bar{t})) = u_k(\bar{t}) \leqq x_k(\bar{t}, t_0, u(t_0)) = v(t_0).$$

Since $\bar{t} \geqq t_0$ was arbitrary, the theorem would follow.

It remains to show then that condition (A) is satisfied. From (5), this will be the case if the kth row of $Y(\bar{t}, s)$ is nonnegative for each s, $t_0 \leqq s \leqq \bar{t}$. Instead of (3), consider its adjoint

$$z' = -f_x^T(t, x(t, s, u(s)))z \tag{7}$$

and the fundamental matrix $Z(t, s)$ of (7) which satisfies $Z(s, s) = I$. Note that

$$Y^T(\bar{t}, s) = Z^{-1}(\bar{t}, s).$$

By assumption, every solution

$$z(t, \bar{t}, e_k) = Z(t, s)Z^{-1}(\bar{t}, s)e_k$$

satisfies

$$z(s, \bar{t}, e_k) = Z^{-1}(\bar{t}, s)e_k \geqq 0$$

for each s and $\bar{t} \in I$, with $t_0 \leqq s \leqq \bar{t}$. That is,

$$Y^T(\bar{t}, s)e_k \geqq 0,$$

which verifies inequality (A) and the theorem.

Remark. It can be shown that the conditions of the theorem imply that each component function $f_k(t, x)$ of $f(t, x)$ is monotone nondecreasing in each x_i for $i \neq k$. In the case that the conditions hold for all k, the usual type K condition of Kamke follows. However, for the problem of interest here, i.e., component inequalities, the monotone condition is not sufficient, as can be seen by considering the system of equations $x_1' = x_2$, $x_2' = -x_1$ with solution $x_1(t) = \cos t$, $x_2(t) = -\sin t$ and $u_1(t) \equiv -1/2$, $u_2(t) \equiv 0$ over the interval $[0, \pi]$.

3. Application to higher order trajectory derivatives. Consider the system of first order equations

$$x' = g(t, x), \tag{8}$$

where x and g are n-vectors, t is a scalar and $g \in C^{m-1}$ on $[t_0, \infty) \times R^n$. Let $v(x) = x^T H x$ be a positive definite quadratic form and suppose the trajectory derivatives of v formed relative to (8) satisfy

$$v^{(m)} + a_{m-1} v^{(m-1)} + \cdots + a_0 v \leqq 0 \tag{9}$$

for $(t, x) \in [t_0, \infty) \times R^n$. Consider the comparison system

$$r' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{m-1} \end{pmatrix} r, \tag{10}$$

where r is an m-vector. If the coefficients a_i are constants, then it is reasonably convenient to determine whether (10) satisfies the condition of the corollary. The results on component differential inequalities can then be used to obtain estimates on solution behavior for (8), such as the following theorem.

THEOREM. *Suppose the comparison system* (10) *has the solution property given in the corollary for* $k = 1$. *Then there exist constants* $C_1, \cdots, C_m$ *such that*

$$|x(t, t_0, x_0)|^2 \leqq \frac{1}{\gamma} \sum_{j=1}^{m} C_j u_j(t)$$

for $t \geqq t_0$, *where* γ *is the minimal eigenvalue of* H *and the functions* $u_j(t)$ *are linearly independent solutions of*

$$u^{(m)} + a_{m-1} u^{(m-1)} + \cdots + a_0 u = 0. \tag{11}$$

The proof follows immediately from the corollary upon writing the scalar inequality (9) as a system of first order inequalities and from observing that, since $v = x^T H x$ is positive definite, it will satisfy an estimate of the form $\gamma x^T x \leqq v(x)$.

REFERENCES

[1] W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath, Boston, 1965.

[2] E. KAMKE, *Zur Theorie der Systeme gewöhnlichen Differentialgleichungen, II*, Acta Math., 58 (1932), pp. 57–85.

[3] J. SZARSKI, *Differential Inequalities*, Monogratce Mat., Tom 43, PWN, Warsaw, 1965.

[4] R. W. GUNDERSON, *A comparison lemma for higher order trajectory derivatives*, Proc. Amer. Math. Soc., 27 (1971), pp. 543–548.

[5] A. I. AVERBUCH, *The connection between S. A. Chaplygin's theorem and the theory of optimal processes*, Automat. Remote Control, 22 (1961), pp. 1177–1181.

[6] P. HARTMAN, *Ordinary Differential Equations*, John Wiley, New York, 1964.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

ON THE SOLUTION OF A VOLTERRA INTEGRAL EQUATION WITH A WEAKLY SINGULAR KERNEL*

FRANK DE HOOG AND RICHARD WEISS†

Abstract. The solution $x(t)$ of the Volterra integral equation of the second kind $x(t) = f_1(t) + \sqrt{t}f_2(t) + \int_0^t g(t, s, x(s))(t - s)^{-1/2}\, ds$ is examined. It is shown that $x(t) = u(t) + \sqrt{t}v(t)$, where $u(t)$ and $v(t)$ are smooth under appropriate smoothness conditions on $f_1(t), f_2(t)$ and $g(t, s, x)$ and satisfy a system of Volterra integral equations of the second kind.

1. Introduction. A number of problems in mathematical physics can be formulated in terms of the Volterra integral equation

$$x(t) = f_1(t) + \sqrt{t}f_2(t) + \int_0^t \frac{g(t, s, x(s))}{\sqrt{t - s}}\, ds, \qquad 0 \leqq t \leqq T \tag{1.1}$$

(see, for instance, Chambre [1] and Levinson [21]).

Finite difference schemes, based on product integration, for Volterra integral equations of the second kind with weakly singular kernels have been investigated by Linz [3]. These schemes can be applied to (1.1) and are convergent if $g(t, s, x(s))$ is continuous with respect to s and t on $0 \leqq s \leqq t \leqq T$. To estimate the rate of convergence, information about the smoothness of $x(t)$ is required. An investigation of the smoothness of solutions of Volterra integral equations with weakly singular kernels has recently been made by Miller and Feldstein [4]. One of their results is that, if $f_2(t) = 0$ and $f_1(t)$ and $g(t, s, x)$ are sufficiently smooth, then

$$x'(t) = o(t^{-1/2}) \quad \text{as } t \to 0.$$

In this paper we extend this result and show that $x(t) = u(t) + \sqrt{t}v(t)$, where $u(t), v(t)$ are smooth under appropriate smoothness conditions on $f_1(t), f_2(t)$ and $g(t, s, x)$ and satisfy the system of equations

$$\begin{aligned} u(t) &= f_1(t) + \int_0^t \frac{\sqrt{s}}{\sqrt{t - s}} g_1(t, s, u(s), v(s))\, ds, \qquad 0 \leqq t \leqq T, \\ v(t) &= f_2(t) + \frac{1}{\sqrt{t}} \int_0^t \frac{g_2(t, s, u(s), v(s))}{\sqrt{t - s}}\, ds, \qquad 0 \leqq t \leqq T, \end{aligned} \tag{1.2}$$

where

$$g_1(t, s, u, v) = \frac{g(t, s, u + \sqrt{s}v) - g(t, s, u - \sqrt{s}v)}{2\sqrt{s}}, \tag{1.3a}$$

$$g_2(t, s, u, v) = \frac{g(t, s, u + \sqrt{s}v) + g(t, s, u - \sqrt{s}v)}{2}. \tag{1.3b}$$

* Received by the editors June 22, 1972.

† Computer Centre, The Australian National University, Canberra City, A.C.T., Australia.

The system (1.2) provides an alternative for the numerical computation of $x(t)$ in a neighborhood of the origin. This will be examined in a subsequent paper.

In § 2, we establish a number of basic lemmas. The equivalence of (1.1) and (1.2) and the smoothness of $u(t)$ and $v(t)$ are examined in § 3. In § 4, the results of § 3 are used to justify a procedure for the numerical solution of (1.1) suggested by Noble [5].

Since some of the arguments used are similar to those in [4], we have used the notation of [4] whenever possible.

2. Preliminaries. In this section we shall establish some lemmas which will be required in the subsequent analysis.

LEMMA 2.1. *Let* $f_1(t), f_2(t) \in C[0, T]$ *and* $g(t, s, x)$ *be continuous with respect to* t, s *on* $0 \leqq s \leqq t \leqq T$ *and globally Lipschitz continuous with respect to* x. *Then* (1.1) *has a unique solution* $x(t) \in C[0, T]$.

Proof. The result follows from the usual contraction mapping and translation argument on $C[0, T]$.

LEMMA 2.2. *Let*

(i) $f_1(t), f_2(t) \in C[0, T]$,

(ii) $g_1(t, s, u, v), g_2(t, s, u, v)$ *be continuous with respect to* t, s, u *and* v *on* $0 \leqq s \leqq t \leqq T, -\infty < u, v < \infty$ *and*

(iii)

$$\begin{aligned} |g_1(t, s, u, v_1) - g_1(t, s, u, v_2)| &\leqq L|v_1 - v_2|, \\ |g_1(t, s, u_1, v) - g_1(t, s, u_2, v)| &\leqq \frac{L}{\sqrt{s}}|u_1 - u_2|, \\ |g_2(t, s, u, v_1) - g_2(t, s, u, v_2)| &\leqq L\sqrt{s}|v_1 - v_2|, \\ |g_2(t, s, u_1, v) - g_2(t, s, u_2, v)| &\leqq L|u_1 - u_2| \end{aligned} \tag{2.1}$$

for some constant L *and all* u, u_1, u_2, v, v_1 *and* v_2. *Then the system of equations*

$$\begin{aligned} u(t) &= f_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+1/2}}{\sqrt{t-s}} g_1(t, s, u(s), v(s))\,ds, \\ v(t) &= f_2(t) + \frac{1}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}} g_2(t, s, u(s), v(s))\,ds, \end{aligned} \tag{2.2}$$

$$0 \leqq t \leqq T, \quad r = 0, 1, \cdots$$

has a unique solution $u(t), v(t) \in C[0, T]$.

Proof. Define

$$w(t) = Kv(t), \quad K > 0,$$

such that

$$\frac{2KL}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}}\,ds < 1, \qquad 0 \leqq t \leqq T.$$

The result follows by the application of a contraction mapping and translation argument on $C[0, T]$ to the corresponding system of equations for $u(t)$ and $w(t)$. This completes the proof.

Let $a(t) \in L^1(0, T)$. The resolvent $R(t)$ associated with a given kernel function $a(t)$ is defined as the unique L^1 solution of the linear equation

$$R(t) = a(t) + \int_0^t a(t-s)R(s)\,ds, \qquad 0 \leqq t \leqq T.$$

If $a(t)$ is nonnegative, then $R(t)$ is nonnegative a.e. (see [4, Lemma 1]).

LEMMA 2.3 (Tricomi [6, Chap. 1]). *If $X(t)$ is the solution of the linear equation*

$$X(t) = f(t) + \int_0^t a(t-s)X(s)\,ds, \qquad 0 \leqq t \leqq T,$$

then

$$X(t) = f(t) + \int_0^t R(t-s)f(s)\,ds, \qquad 0 \leqq t \leqq T.$$

LEMMA 2.4. *Let $f_1(t), f_2(t) \in C[0, T]$ and $u(t), v(t)$ be the solution of the system*

$$\begin{aligned} u(t) &= f_1(t) + \frac{L}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}(u(s) + \sqrt{s}v(s))\,ds, \\ v(t) &= f_2(t) + \frac{L}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}}(u(s) + \sqrt{s}v(s))\,ds, \end{aligned} \tag{2.3}$$

$$0 \leqq t \leqq T, \quad r = 0, 1, \cdots.$$

Then

$$\begin{aligned} u(t) &= f_1(t) + \frac{1}{2t^r}\int_0^t R(t-s)s^r(f_1(s) + \sqrt{s}f_2(s))\,ds, \\ v(t) &= f_2(t) + \frac{1}{2t^{r+1/2}}\int_0^t R(t-s)s^r(f_1(s) + \sqrt{s}f_2(s))\,ds, \end{aligned} \tag{2.4}$$

$$0 \leqq t \leqq T, \quad r = 0, 1, \cdots,$$

where $R(t)$ is the resolvent associated with the kernel function $a(t) = 2Lt^{-1/2}$.

Proof. Clearly from (2.3),

$$u(t) - \sqrt{t}v(t) = f_1(t) - \sqrt{t}f_2(t). \tag{2.5}$$

Let

$$w(t) = t^r(u(t) + \sqrt{t}v(t)).$$

Then from (2.3), $w(t)$ satisfies

$$w(t) = t^r(f_1(t) + \sqrt{t}f_2(t)) + 2L\int_0^t \frac{w(s)}{\sqrt{t-s}}\,ds, \qquad 0 \leqq t \leqq T.$$

Hence from Lemma 2.3,

$$w(t) = t^r(f_1(t) + \sqrt{t}f_2(t)) + \int_0^t R(t-s)s^r(f_1(s) + \sqrt{s}f_2(s))\,ds. \tag{2.6}$$

The result for $0 < t \leqq T$ follows from (2.5) and (2.6). From Lemma 2.1, $u(t), v(t) \in C[0, T]$. The result follows.

LEMMA 2.5. *Let $u_1(t)$, $u_2(t)$, $v_1(t)$ and $v_2(t)$ be the unique continuous solutions of the systems*

$$(2.7)\qquad \begin{aligned} u_1(t) &= f_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+1/2}}{\sqrt{t-s}} g_1(t,s,u_1(s),v_1(s))\,ds,\\ v_1(t) &= f_2(t) + \frac{1}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}} g_2(t,s,u_1(s),v_1(s))\,ds \end{aligned}$$

and

$$(2.8)\qquad \begin{aligned} u_2(t) &= q_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+1/2}}{\sqrt{t-s}} k_1(t,s,u_2(s),v_2(s))\,ds,\\ v_2(t) &= q_2(t) + \frac{1}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}} k_2(t,s,u_2(s),v_2(s))\,ds, \end{aligned}$$

where $f_1(t), f_2(t), q_1(t), q_2(t) \in C[0,T]$, $g_1(t,s,u,v)$, $g_2(t,s,u,v)$, $k_1(t,s,u,v)$, $k_2(t,s,u,v)$ *are continuous with respect to t, s, u and v on* $0 \leqq s \leqq t \leqq T$, $-\infty \leqq u, v \leqq \infty$, *and* $g_1(t,s,u,v)$, $g_2(t,s,u,v)$ *satisfy* (2.1). *Then*

$$|u_1(t) - u_2(t)| \leqq |\tilde{f}_1(t)| + \frac{1}{2t^r}\int_0^t R(t-s)s^r(|\tilde{f}_1(s)| + \sqrt{s}|\tilde{f}_2(s)|)\,ds$$

and

$$|v_1(t) - v_2(t)| \leqq |\tilde{f}_2(t)| + \frac{1}{2t^{r+1/2}}\int_0^t R(t-s)s^r(|\tilde{f}_1(s)| + \sqrt{s}|\tilde{f}_2(s)|)\,ds,$$

where

$$\begin{aligned} \tilde{f}_1(t) &= f_1(t) - q_1(t)\\ &\quad + \frac{1}{t^r}\int_0^t \frac{s^{r+1/2}}{\sqrt{t-s}}(g_1(t,s,u_2(s),v_2(s)) - k_1(t,s,u_2(s),v_2(s)))\,ds,\\ \tilde{f}_2(t) &= f_2(t) - q_2(t)\\ &\quad + \frac{1}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}}(g_2(t,s,u_2(s),v_2(s)) - k_2(t,s,u_2(s),v_2(s)))\,ds, \end{aligned}$$

and $R(t)$ is the resolvent associated with the kernel $2Lt^{-1/2}$.

Proof. Define

$$z(t) = u_1(t) - u_2(t),$$
$$w(t) = v_1(t) - v_2(t),$$

$$C_1(t,s) = \begin{cases} \dfrac{g_1(t,s,u_1(s),v_1(s)) - g_1(t,s,u_1(s),v_2(s))}{w(s)}, & w(s) \neq 0,\\ 0, & w(s) = 0; \end{cases}$$

$$D_1(t,s) = \begin{cases} \dfrac{\sqrt{s}(g_1(t,s,u_1(s),v_2(s)) - g_1(t,s,u_2(s),v_2(s)))}{z(s)}, & z(s) \neq 0, \\ 0, & z(s) = 0; \end{cases}$$

$$C_2(t,s) = \begin{cases} \dfrac{g_2(t,s,u_1(s),v_1(s)) - g_2(t,s,u_1(s),v_2(s))}{\sqrt{s}w(s)}, & \sqrt{s}w(s) \neq 0, \\ 0, & \sqrt{s}w(s) = 0; \end{cases}$$

$$D_2(t,s) = \begin{cases} \dfrac{g_2(t,s,u_1(s),v_2(s)) - g_2(t,s,u_2(s),v_2(s))}{z(s)}, & z(s) \neq 0, \\ 0, & z(s) = 0. \end{cases}$$

Clearly from (2.1),

$$|C_1(t,s)|, |C_2(t,s)|, |D_1(t,s)|, |D_2(t,s)| \leqq L. \tag{2.9}$$

Subtraction of (2.8) from (2.7) yields

$$z(t) = \tilde{f}_1(t) + \frac{1}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}(\sqrt{s}C_1(t,s)w(s) + D_1(t,s)z(s))\,ds,$$

$$w(t) = \tilde{f}_2(t) + \frac{1}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}}(\sqrt{s}C_2(t,s)w(s) + D_2(t,s)z(s))\,ds,$$

and it follows from (2.9) that

$$|z(t)| \leqq |\tilde{f}_1(t)| + \frac{L}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}(\sqrt{s}|w(s)| + |z(s)|)\,ds,$$

$$|w(t)| \leqq |\tilde{f}_2(t)| + \frac{L}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}}(\sqrt{s}|w(s)| + |z(s)|)\,ds.$$

Let $y_1(t), y_2(t) \in C[0,T]$ be two nonnegative functions such that

$$|z(t)| = |\tilde{f}_1(t)| - y_1(t) + \frac{L}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}(\sqrt{s}|w(s)| + |z(s)|)\,ds,$$

$$|w(t)| = |\tilde{f}_2(t)| - y_2(t) + \frac{L}{t^{r+1/2}}\int_0^t \frac{s^r}{\sqrt{t-s}}(\sqrt{s}|w(s)| + |z(s)|)\,ds.$$

Then from Lemma 2.4,

$$\begin{aligned} |z(t)| &= |\tilde{f}_1(t)| - y_1(t) \\ &\quad + \frac{1}{2t^r}\int_0^t R(t-s)s^r(|\tilde{f}_1(s)| - y_1(s) + \sqrt{s}(|\tilde{f}_2(s)| - y_2(s)))\,ds, \\ |w(t)| &= |\tilde{f}_2(t)| - y_2(t) \\ &\quad + \frac{1}{2t^{r+1/2}}\int_0^t R(t-s)s^r(|\tilde{f}_1(s)| - y_1(s) + \sqrt{s}(|\tilde{f}_2(s)| - y_2(s)))\,ds. \end{aligned}$$

The result follows since $R(t)$, $y_1(t)$ and $y_2(t)$ are nonnegative.

LEMMA 2.6. *Let*

$$f(t), g(t) \in C^{2n}[-b, +b], \quad b > 0,$$
$$F(t) = \sqrt{t}\{f(\sqrt{t}) - f(-\sqrt{t})\},$$
$$G(t) = g(\sqrt{t}) + g(-\sqrt{t}).$$

Then $F(t), G(t) \in C^n[0, b^2]$.

Proof. The result is clearly true for $n = 0$. Assume the result is true for $n = r$. Consider the case $n = r + 1$. Clearly

$$F'(t) = \frac{1}{2\sqrt{t}}\{f(\sqrt{t}) - f(-\sqrt{t})\} + \tfrac{1}{2}\{f'(\sqrt{t}) + f'(-\sqrt{t})\}.$$

From Taylor's theorem with integral remainder,

$$\frac{1}{2\sqrt{t}}(f(\sqrt{t}) - f(-\sqrt{t})) = f'(0) + \frac{\sqrt{t}}{2}\left\{\int_0^1 (1 - s)f''(s\sqrt{t})\,ds - \int_0^1 (1 - s)f''(-s\sqrt{t})\,ds\right\}.$$

Hence,

$$F'(t) = f'(0) + \{\tilde{g}(\sqrt{t}) + \tilde{g}(-\sqrt{t})\} + \sqrt{t}\{\tilde{f}(\sqrt{t}) - \tilde{f}(-\sqrt{t})\},$$

where $\tilde{f}(t)$, $\tilde{g}(t)$ satisfy the hypothesis with $n = r$. It follows that $F(t) \in C^{r+1}[0, b^2]$. Similarly $G(t) \in C^{r+1}[0, b^2]$. The result follows by induction.

COROLLARY 2.1. *Let* $g(t, s, x)$ *be* n *times continuously differentiable with respect to* t *and* s *on* $0 \leqq s \leqq t \leqq T$ *and* $2n$ *times continuously differentiable with respect to* x *for all* x. *Then* $sg_1(t, s, u, v)$, $g_2(t, s, u, v)$ *defined by* (1.3) *are* n *times continuously differentiable with respect to* t, s, u *and* v *on* $0 \leqq s \leqq t \leqq T$, $-\infty < u, v < \infty$.

3. Smoothness results. We first consider the relation between $x(t)$ defined by (1.1) and $u(t)$, $v(t)$ defined by (1.2).

THEOREM 3.1. *If*

(i) $f_1(t), f_2(t) \in C[0, T]$,

(ii) $g(t, s, x)$ *is continuous with respect to* t *and* s *on* $0 \leqq s \leqq t \leqq T$ *and globally Lipschitz continuous with respect to* x, *then* (1.1) *and* (1.2) *have unique continuous solutions* $x(t)$ *and* $u(t)$, $v(t)$. *Furthermore,*

$$x(t) = u(t) + \sqrt{t}v(t).$$

Proof. Existence and uniqueness of $x(t)$, $u(t)$ and $v(t)$ follow from Lemmas 2.1 and 2.2. The result follows since $u(t) + \sqrt{t}v(t)$ satisfies (1.1).

We shall now examine the smoothness of $u(t)$, $v(t)$. Since inductive arguments will be used, it is convenient to consider the more general problem

$$u(t) = f_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+1/2}g_1(t, s, u(s), v(s))}{\sqrt{t - s}}\,ds, \qquad r = 0, 1, \cdots,$$
(3.1)
$$v(t) = f_2(t) + \frac{1}{t^{r+1/2}}\int_0^t \frac{s^r g_2(t, s, u(s), v(s))}{\sqrt{t - s}}\,ds, \qquad r = 0, 1, \cdots,$$

or equivalently,

$$
\begin{aligned}
u(t) &= f_1(t) + \int_0^1 \frac{ts^{r+1/2} g_1(t, ts, u(ts), v(ts))}{\sqrt{1-s}}\, ds, \qquad r = 0, 1, \cdots, \\
v(t) &= f_2(t) + \int_0^1 \frac{s^r g_2(t, ts, u(ts), v(ts))}{\sqrt{1-s}}\, ds, \qquad r = 0, 1, \cdots .
\end{aligned} \tag{3.2}
$$

Formally differentiating (3.2), we obtain

$$
\begin{aligned}
u'(t) = F_1(t) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+3/2}}{\sqrt{t-s}} \Big\{ &\frac{\partial g_1}{\partial u}(t, s, u(s), v(s)) u'(s) \\
&+ \frac{\partial g_1}{\partial v}(t, s, u(s), v(s)) v'(s) \Big\}\, ds, \\
v'(t) = F_2(t) + \frac{1}{t^{r+3/2}} \int_0^t \frac{s^{r+1}}{\sqrt{t-s}} \Big\{ &\frac{\partial g_2}{\partial u}(t, s, u(s), v(s)) u'(s) \\
&+ \frac{\partial g_2}{\partial v}(t, s, u(s), v(s)) v'(s) \Big\}\, ds,
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(t) = f'_1(t) + \int_0^1 \frac{s^{r+1/2}}{\sqrt{1-s}} \Big\{ &g_1(t, ts, u(ts), v(ts)) \\
&+ t \frac{\partial g_1}{\partial t}(t, ts, u(ts), v(ts)) + ts \frac{\partial g_1}{\partial s}(t, ts, u(ts), v(ts)) \Big\}\, ds, \\
F_2(t) = f'_2(t) + \int_0^1 \frac{s^r}{\sqrt{1-s}} \Big\{ &\frac{\partial g_2}{\partial t}(t, s, u(ts), v(ts)) \\
&+ s \frac{\partial g_2}{\partial s}(t, ts, u(ts), v(ts)) \Big\}\, ds.
\end{aligned}
$$

We now consider the system

$$
\begin{aligned}
U(t) = F_1(t) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+3/2}}{\sqrt{t-s}} \Big\{ &\frac{\partial g_1}{\partial u}(t, s, u(s), v(s)) U(s) \\
&+ \frac{\partial g_1}{\partial v}(t, s, u(s), v(s)) V(s) \Big\}\, ds, \\
V(t) = F_2(t) + \frac{1}{t^{r+3/2}} \int_0^t \frac{s^{r+1}}{\sqrt{t-s}} \Big\{ &\frac{\partial g_2}{\partial u}(t, s, u(s), v(s)) U(s) \\
&+ \frac{\partial g_2}{\partial v}(t, s, u(s), v(s)) V(s) \Big\}\, ds.
\end{aligned} \tag{3.3}
$$

In the following lemma we prove that under appropriate assumptions on $f_1(t)$, $f_2(t)$, $g_1(t, s, u, v)$ and $g_2(t, s, u, v)$, (3.3) has a unique continuous solution $U(t)$, $V(t)$, which coincides with $u'(t)$, $v'(t)$, where $u(t)$, $v(t)$ is the solution of (3.1).

LEMMA 3.1. *Let*

(i) *$f_1(t), f_2(t)$, $sg_1(t, s, u, v)$ and $g_2(t, s, u, v)$ be continuously differentiable with respect to t and t, s, u, v respectively for $0 \leqq s \leqq t \leqq T$ and all u, v, and*

(ii)

$$\left|\frac{\partial g_1}{\partial u}\right| < \frac{L}{\sqrt{s}}, \qquad \left|\frac{\partial g_1}{\partial v}\right| < L, \qquad \left|\frac{\partial g_2}{\partial u}\right| < L, \qquad \left|\frac{\partial g_2}{\partial v}\right| < \sqrt{s}L, \qquad L = \text{const}.$$

Then the solution of (3.1) *is continuously differentiable and satisfies*

$$u'(t) = U(t), \quad v'(t) = V(t), \qquad 0 \leqq t \leqq T,$$

where $U(t)$, $V(t)$ is the unique continuous solution of (3.3).

Proof. It follows from Lemma 2.2 that (3.3) has a unique continuous solution $U(t)$, $V(t)$. Using an argument similar to Miller and Feldstein [4, Theorem 1], we may assume that $sg_1(t, s, u, v)$ and $g_2(t, s, u, v)$ have compact support.

Let δ be a real number in the range $0 < \delta < T/2$. For $0 < h \leqq \delta$ and $0 < t \leqq T - \bar{\delta}$, $\bar{\delta} = \delta/T$, define

$$z(t, h) = \frac{u(t(1 + h)) - u(t)}{th}, \qquad t > 0,$$

and

$$w(t, h) = \frac{v(t(1 + h)) - v(t)}{th}, \qquad t > 0.$$

Then

$$z(t, h) = \frac{f_1(t(1 + h)) - f_1(t)}{th} + \frac{1}{th}\int_0^1 \frac{s^{r+1/2}t}{\sqrt{1 - s}}\{(1 + h)g_1(t(1 + h),$$
$$ts(1 + h), u(ts(1 + h)), v(ts(1 + h))) - g_1(t, ts, u(ts), v(ts))\}\, ds.$$

By the mean value theorem,

$$\begin{aligned}(1 + h)&g_1(t(1 + h), ts(1 + h), u(ts(1 + h)), v(ts(1 + h))) - g_1(t, ts, u(ts), v(ts))\\ &= hg_1(t(1 + h), ts(1 + h), u(ts(1 + h)), v(ts(1 + h)))\\ &\quad + th\frac{\partial g_1}{\partial t}(t + \theta(th), ts(1 + h), u(ts(1 + h)), v(ts(1 + h)))\\ &\quad + tsh\frac{\partial g_1}{\partial s}(t, ts + \eta(tsh), u(ts(1 + h)), v(ts(1 + h)))\\ &\quad + (u(ts(1 + h)) - u(ts))\frac{\partial g_1}{\partial u}(t, ts, \hat{u}(ts), v(ts(1 + h)))\\ &\quad + (v(ts(1 + h)) - v(ts))\frac{\partial g_1}{\partial v}(t, ts, u(ts), \hat{v}(ts)),\end{aligned}$$

where $0 < \theta(th) < th$, $0 < \eta(tsh) < tsh$, $\hat{u}(ts)$ lies between $u(ts)$ and $u(ts(1 + h))$ and $\hat{v}(ts)$ lies between $v(ts)$ and $v(ts(1 + h))$. Hence

$$
\begin{aligned}
z(t, h) = F_1(t, h) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+3/2}}{\sqrt{t - s}} \Big\{ &\frac{\partial g_1}{\partial u}(t, s, \hat{u}(s), v(s + sh)) z(s, h) \\
&+ \frac{\partial g_1}{\partial v}(t, s, u(s), \hat{v}(s)) w(s, h) \Big\}\, ds,
\end{aligned} \tag{3.4}
$$

where

$$
\begin{aligned}
F_1(t, h) = \frac{f_1(t(1 + h)) - f_1(t)}{th} + \int_0^1 \frac{s^{r+1/2}}{\sqrt{1 - s}} \Big\{ &g_1(t(1 + h), ts(1 + h), \\
&u(ts(1 + h)), v(ts(1 + h))) \\
&+ t \frac{\partial g_1}{\partial t}(t + \theta(th), ts(1 + h), u(ts(1 + h)), v(ts(1 + h))) \\
&+ ts \frac{\partial g_1}{\partial s}(t, ts + \eta(tsh), u(ts(1 + h)), v(ts(1 + h))) \Big\}\, ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
w(t, h) = F_2(t, h) + \frac{1}{t^{r+3/2}} \int_0^t \frac{s^{r+1}}{\sqrt{t - s}} \Big\{ &\frac{\partial g_2}{\partial u}(t, s, \tilde{u}(s), v(s + h)) z(s, h) \\
&+ \frac{\partial g_2}{\partial v}(t, s, u(s), \tilde{v}(s)) w(s, h) \Big\}\, ds,
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
F_2(t, h) = \frac{f_2(t(1 + h)) - f_2(t)}{th} + \int_0^1 \frac{s^{r+1}}{\sqrt{1 - s}} \Big\{ &\frac{\partial g_2}{\partial t}(t + \alpha(th), \\
&ts(1 + h), u(ts(1 + h)), v(ts(1 + h))) \\
&+ \frac{\partial g_2}{\partial s}(t, ts + \beta(tsh), u(ts(1 + h), v(ts(1 + h))) \Big\}\, ds,
\end{aligned}
$$

$0 < \alpha(th) < th$, $0 < \beta(tsh) < tsh$, $\tilde{u}(s)$ lies between $u(s)$ and $u(s + sh)$, and $\tilde{v}(s)$ lies between $v(s)$ and $v(s + sh)$. On defining

$$
\begin{aligned}
z(0, h) &= \lim_{t \to 0} F_1(t, h) \\
&= F_1(0, h)
\end{aligned}
$$

and similarly

$$
w(0, h) = F_2(0, h) + \frac{(-\frac{1}{2})!(r + 1)!}{(r + \frac{3}{2})!} \frac{\partial g_2}{\partial u}(0, 0, u(0), v(0)) z(0, h),
$$

it follows from the application of Lemma 2.4 to the system (3.4), (3.5) that $z(t, h)$ and $w(t, h)$ are continuous on $0 \leqq t \leqq T - \bar{\delta}$. Define

$$Q_1(t, h) = F_1(t, h) - F_1(t) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+3/2}}{\sqrt{t - s}}$$

$$\cdot \left\{ \left[\frac{\partial g_1}{\partial u}(t, s, \hat{u}(s), v(s + sh)) - \frac{\partial g_1}{\partial u}(t, s, u(s), v(s)) \right] U(s) \right.$$

$$\left. + \left[\frac{\partial g_1}{\partial v}(t, s, u(s), \hat{v}(s)) - \frac{\partial g_1}{\partial v}(t, s, u(s), v(s)) \right] V(s) \right\} ds,$$

$$Q_2(t, h) = F_2(t, h) - F_2(t) + \frac{1}{t^{r+3/2}} \int_0^t \frac{s^{r+1}}{\sqrt{t - s}}$$

$$\cdot \left\{ \left[\frac{\partial g_2}{\partial u}(t, s, \tilde{u}(s), v(s + sh)) - \frac{\partial g_2}{\partial u}(t, s, u(s), v(s)) \right] U(s) \right.$$

$$\left. + \left[\frac{\partial g_2}{\partial v}(t, s, u(s), \tilde{v}(s)) - \frac{\partial g_2}{\partial v}(t, s, u(s), v(s)) \right] V(s) \right\} ds.$$

Let ε be a positive real number. Since $sg_1(t, s, u, v)$ and $g_2(t, s, u, v)$ have compact support, there exists an h_0 such that for $0 < h \leqq h_0$,

$$|Q_1(t, h)| \leqq \varepsilon, \qquad |Q_2(t, h)| \leqq \varepsilon \tag{3.6}$$

for $0 \leqq t \leqq T - \bar{\delta}$. Hence, the application of Lemma 2.5 to the systems (3.3) and (3.4), (3.5) and the subsequent use of (3.6) yields

$$|z(t, h) - U(t)| \leqq Q_1(t, h) + \frac{1}{2t^{r+1}} \int_0^t R(t - s)s^{r+1}\{|Q_1(s, h)| + \sqrt{s}|Q_2(s, h)|\} ds$$

$$\leqq \varepsilon\left(1 + \frac{1}{2t^{r+1}} \int_0^t R(t - s)s^{r+1}(1 + \sqrt{s})\, ds\right),$$

$$|w(t, h) - V(t)| \leqq |Q_2(t, h)| + \frac{1}{2t^{r+3/2}} \int_0^t R(t - s)s^{r+1}\{|Q_1(s, h)| + \sqrt{s}|Q_2(s, h)|\} ds$$

$$\leqq \varepsilon\left(1 + \frac{1}{2t^{r+3/2}} \int_0^t R(t - s)s^{r+1}(1 + \sqrt{s})\, ds\right)$$

for $0 \leqq t \leqq T - \bar{\delta}$, where $R(t)$ is the resolvent associated with the kernel $2Lt^{-1/2}$. From Miller and Feldstein [4, Lemmas 2 and 4], it follows that there exists a positive constant C such that

$$R(t) \leqq Ct^{-1/2} \quad \text{a.e. on } 0 \leqq t \leqq T.$$

Hence

$$\left.\begin{aligned} |z(t, h) - u(t)| &\leqq D\varepsilon \\ |w(t, h) - v(t)| &\leqq D\varepsilon \end{aligned}\right\} \qquad 0 \leqq t \leqq T - \bar{\delta}, \quad D = \text{const}.$$

Thus $z(t, h) \to U(t)$ and $w(t, h) \to V(t)$ uniformly on $0 \leqq t \leqq T - \bar{\delta}$. Since δ is arbitrary, $U(t)$ and $V(t)$ are the continuous right derivatives of $u(t)$ and $v(t)$ respectively on $0 \leqq t < T$. In addition, from the uniform convergence to $U(t)$ and $V(t)$, respectively, it follows that for any interval $I = \{t : \bar{\delta} \leqq t \leqq T - \bar{\delta}\}$ the sets $\{z(\cdot, h) : 0 < h < \delta\}$ and $\{w(\cdot, h) : 0 < h < \delta\}$ are equicontinuous and hence

$$\lim_{h\to 0} z(t, h) = \lim_{h\to 0} z(t - h, h) = U(t),$$

$$\lim_{h\to 0} w(t, h) = \lim_{h\to 0} w(t - h, h) = V(t)$$

uniformly on I. This implies that $U(t)$ and $V(t)$ are the left-hand derivatives of $u(t)$ and $v(t)$ respectively on I. A simple argument shows that $U(T)$ and $V(T)$ are the left derivatives of $u(t)$ and $v(t)$ respectively at $t = T$. This completes the proof.

THEOREM 3.2. *If in* (1.1),

(i) $f_1(t), f_2(t) \in C^n[0, T]$,

(ii) *$g(t, s, y)$ is n times continuously differentiable with respect to t and s on $0 \leqq s \leqq t \leqq T$ and $2n$ times continuously differentiable with respect to y for all y, and*

(iii) *$g(t, s, y)$ is Lipschitz continuous with respect to y for all y and $0 \leqq s \leqq t \leqq T$, then $u(t), v(t) \in C^n[0, T]$, where $u(t), v(t)$ is the solution of* (1.2) *and furthermore $u^{(m)}(t), v^{(m)}(t)$ is the solution of the system of equations obtained by formally differentiating* (1.2) *m times.*

Proof. From Corollary 2.1 it follows that $sg_1(t, s, u, v)$ and $g_2(t, s, u, v)$ are n times continuously differentiable. The result follows from induction and Lemma 3.1.

A stronger result can be obtained if f_1, f_2 and g are analytic.

THEOREM 3.3. *Let*

(i) *$f_1(t), f_2(t)$ be real analytic in a neighborhood of $0 \leqq t \leqq T$,*

(ii) *$g(t, s, x)$ be real analytic in an open set containing all real ordered triples (t, s, x), $0 \leqq s \leqq t \leqq T$ and $|x| < \infty$, and*

(iii) *equation* (1.2) *have a unique continuous solution $u(t), v(t)$ in an open set containing the interval $0 \leqq t \leqq T$.*

Then $u(t)$ and $v(t)$ are analytic in an open set containing $0 \leqq t \leqq T$.

Proof. It follows from (ii) and (1.3) that $g_1(t, s, u, v)$ and $g_2(t, s, u, v)$ are real analytic in an open set containing the real ordered quadruples (t, s, u, v), $0 \leqq s \leqq t \leqq T$, $|u|, |v| \leqq Q$, $Q < \infty$.

For $\varepsilon > 0$ define

$$D(\varepsilon) = \{z : -\varepsilon \leqq \operatorname{Re} z \leqq T + \varepsilon, |\operatorname{Im} z| \leqq \varepsilon\},$$

$$P = \max\{|f_1(z)|, |f_2(z)| ; z \in D(\varepsilon)\},$$

$$E(\varepsilon) = \{(z, zs, y) : z \in D(\varepsilon), 0 \leqq s \leqq 1, |y| \leqq P + 1\},$$

$$F(\varepsilon) = \{(z, zs, p, q) : z \in D(\varepsilon), 0 \leqq s \leqq 1, |p|, |q| \leqq P + 1\},$$

$$M = \max\left\{|g(z, zs, y)|, \left|\frac{\partial g}{\partial x}(z, zs, y)\right| : z \in D(\varepsilon), 0 \leqq s \leqq 1, |y| \leqq 2P + 2\right\}$$

$$G(\varepsilon) = \{z : -\varepsilon \leqq \operatorname{Re} z \leqq \varepsilon, |\operatorname{Im} z| \leqq \varepsilon/2\}.$$

Choose ε such that $f_1(z)$ and $f_2(z)$ are analytic on $D(\varepsilon)$ and $g_1(z, w, u, v)$ and $g_2(z, w, u, v)$ are analytic on $F(\varepsilon)$. Let $H(\varepsilon)$ denote the set of all functions ϕ, real analytic in the interior of $G(\varepsilon)$, continuous on $G(\varepsilon)$ and satisfying $|\phi(z)| \leqq P + 1$, $z \in G(\varepsilon)$. Given ϕ, ψ in $H(\varepsilon)$, define

$$(3.7)\qquad \begin{aligned} \mathscr{R}_1(\phi, \psi)(z) &= f_1(z) + \int_0^1 z \frac{\sqrt{s}}{\sqrt{1-s}} g_1\left(z, sz, \phi(sz), \frac{\psi(sz)}{K}\right) ds, \\ \mathscr{R}_2(\phi, \psi)(z) &= K f_2(z) + K \int_0^1 \frac{1}{\sqrt{1-s}} g_2\left(z, sz, \phi(sz), \frac{\psi(sz)}{K}\right) ds, \end{aligned}$$

where $K = \sqrt{\varepsilon}$. As in Lemma 2.2, K is introduced to obtain a contraction mapping. From (1.3) it follows that

$$|\mathscr{R}_1(\phi, \psi)(z)| \leqq P + (\sqrt{\varepsilon}M)/2,$$
$$|\mathscr{R}_2(\phi, \psi)(z)| \leqq \sqrt{\varepsilon}(P + M/2).$$

Hence, if $\varepsilon < \varepsilon_0$, $\varepsilon_0 = \min\{1, 1/M^2\}$, then (3.7) is a mapping from $H \times H$ into itself. It can easily be verified that (3.7) is a contraction mapping and consequently $u(z)$ and $v(z)$ are real analytic in the interior of $G(\varepsilon)$.

This result can be extended in the following way. If $u(t)$ and $v(t)$ are real analytic in a neighborhood of $0 \leqq t \leqq \tau + \delta$, $\tau, \delta > 0$, then (1.2) is rewritten as

$$\begin{aligned} u(\tau + t) &= \tilde{f}_1(t) + \int_0^t \frac{\sqrt{s+\tau}}{\sqrt{t-s}} g_1(\tau + t, \tau + s, u(\tau + s), v(\tau + s))\, ds, \\ v(\tau + t) &= \tilde{f}_2(t) + \frac{1}{\sqrt{\tau + t}} \int_0^t \frac{1}{\sqrt{t-s}} g_2(\tau + t, \tau + s, u(\tau + s), v(\tau + s))\, ds, \\ &\qquad t \in [0, T - \tau], \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_1(t) &= f_1(\tau + t) + \int_0^\tau \frac{\sqrt{s}}{\sqrt{\tau + t - s}} g_1(\tau + t, s, u(s), v(s))\, ds, \\ \tilde{f}_2(t) &= f_2(\tau + t) + \frac{1}{\sqrt{\tau + t}} \int_0^\tau \frac{1}{\sqrt{\tau + t - s}} g_2(\tau + t, s, u(s), v(s))\, ds. \end{aligned}$$

Clearly $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ are real analytic in the interior of $R(\tilde{\varepsilon}) = \{z : 0 \leqq \operatorname{Re} z \leqq \tilde{\varepsilon}, |\operatorname{Im} z| \leqq \tilde{\varepsilon}/2\}$ for some $\tilde{\varepsilon} > 0$. From (iii), (3.7) can be replaced by

$$(3.8)\qquad \begin{aligned} \tilde{\mathscr{R}}_1(\phi, \psi)(z) &= u(\tau + z) + z^{1/2} \int_0^1 \frac{\sqrt{sz + \tau}}{\sqrt{1-s}} \{g_1(\tau + z, \tau + sz, \phi(\tau + sz), \psi(\tau + sz)) \\ &\qquad - g_1(\tau + z, \tau + sz, u(\tau + sz), v(\tau + sz))\}\, ds, \\ \tilde{\mathscr{R}}_2(\phi, \psi)(z) &= v(\tau + z) + \frac{z^{1/2}}{\sqrt{\tau + z}} \int_0^1 \frac{1}{\sqrt{1-s}} \{g_2(\tau + z, \tau + sz, \phi(\tau + sz), \\ &\qquad \psi(\tau + sz)) - g_2(\tau + z, \tau + sz, u(\tau + sz), v(\tau + sz))\}\, ds. \end{aligned}$$

As previously, $u(\tau + t)$ and $v(\tau + t)$ can be shown to be analytic in the interior of $R(\tilde{\varepsilon})$ for $\tilde{\varepsilon}$ sufficiently small. If δ is chosen such that $\tilde{\varepsilon} - 2\delta > 0$, this process can be continued with $\tilde{\tau} = \tau + \tilde{\varepsilon} - \delta$.

Since $u(z)$ and $v(z)$ are bounded for $z \in D(\varepsilon)$, it follows from the form of (3.8) that the interval $[\varepsilon, T]$ can be covered by a finite number of applications of the above process.

4. Remark. Equation (1.1) can be rewritten in the form

$$y(t) = f_1(t^2) + tf_2(t^2) + \int_0^t \frac{2s}{\sqrt{t^2 - s^2}} g(t^2, s^2, y(s))\, ds, \qquad 0 \leqq t \leqq \sqrt{T}, \tag{4.1}$$

where $y(t) = x(t^2)$. Using the results of § 3, it follows that if f_1, f_2 and g are suitably smooth, then $y(t)$ is smooth. This provides a justification for the finite difference schemes for (4.1) which have been suggested by Noble [5, § 4, pp. 234–243].

Acknowledgment. The authors wish to thank Dr. M. R. Osborne for valuable comments on the draft of this paper.

REFERENCES

[1] P. L. CHAMBRE, *Nonlinear heat transfer problem*, J. Appl. Phys., 30 (1959), pp. 1683–1688.

[2] N. LEVINSON, *A nonlinear Volterra equation arising in the theory of superfluidity*, J. Math. Anal. Appl., 1 (1960), pp. 1–11.

[3] P. LINZ, *Numerical methods for Volterra integral equations with singular kernels*, SIAM J. Numer. Anal., 6 (1969), pp. 365–374.

[4] R. K. MILLER AND A. FELDSTEIN, *Smoothness of solutions of Volterra integral equations with weakly singular kernels*, this Journal, 2 (1971), pp. 242–258.

[5] B. NOBLE, *The numerical solution of nonlinear integral equations and related topics*, Nonlinear Integral Equations, P. M. Anselone, ed., University of Wisconsin Press, Madison, 1964, pp. 215–318.

[6] F. G. TRICOMI, *Integral Equations*, Interscience, New York, 1957.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

ANALYSIS OF WALSH TRANSFORMS USING INTEGRATION BY PARTS*

C. K. YUEN†

Abstract. Each Walsh transform of a smooth function can be expressed as a weighted average of a derivative of the function. The weighting function is an integral of the Walsh function. Its mean and maximum values can be found easily so that we can estimate the size of Walsh transforms without having to actually compute them.

1. Introduction. Although Walsh functions have been with us for some fifty years [1] and were recently subject to extensive studies by mathematicians [2], [3], [4], communication engineers, statisticians, physicists and even bio-medical workers [5], mathematical analysis of Walsh transforms remains difficult. While many common functions have Fourier, Laplace and Z transforms that can be expressed in closed form and be evaluated easily, their Walsh transforms rarely are. Walsh series expansions of even simple functions like x^2 and x^3 are already very complex [6]. The reason is that Walsh functions are discontinuous, so that functions that have simple Walsh transforms are usually discontinuous themselves.

No amount of clever manipulation can eliminate this basic mismatch between conventional analytical techniques and Walsh functions, though for particular purposes ways can be found to get around this difficulty. The "integration by parts" technique we present is an example. It will be proved that each Walsh transform of a well-behaved function can be expressed as a weighted average of one of its derivatives. The weighting function is none other than the Walsh function integrated by the same number of times. The mean and the maximum of each weighting function can be easily determined giving us a simple method for estimating the size of a Walsh transform.

The developments now taking place in various fields of science and engineering, with a few exceptions, make use of Walsh transforms of random variables rather than analytical functions. Also, "exact" Walsh transforms, as defined in this paper, are seldom used. In their place are approximations computed using sampled values. Thus our technique must be modified to be applicable to these cases. Such extensions will be discussed in a separate paper to appear in a computer journal [7] while the present work will concentrate on laying the mathematical groundwork.

Our discussion begins with the definition of Walsh functions and some related quantities, followed by several basic theorems. We then derive the integration by parts technique and prove several properties of the weighting function. In the final section these results are shown to be useful in estimating Walsh transforms and analyzing the convergence of the Walsh spectrum.

* Received by the editors February 24, 1972, and in revised form September 1, 1972.

† Basser Department of Computer Science, University of Sydney, Sydney, Australia. Now at The Computer Centre, Australian National University, Canberra, Australia.

2. Walsh functions and Walsh spectrum. Walsh functions were first defined by J. L. Walsh in 1923 [1] using a set of recursive relations. Subsequently Paley [8] redefined them as products of another orthogonal set called Rademacher functions. Paley's definition is different from Walsh's only in ordering. However, his functions obey a simpler kind of recursive relation, though this difference was never explicitly pointed out.

Researchers in most engineering applications prefer Walsh's ordering while Paley's orderings were found more convenient in mathematical discussions. Our frequent use of the simpler recursive relation will illustrate this convenience. Another point on which engineers and mathematicians differ is notation for Walsh functions. The former, as a rule, denote the ith Walsh function by wal (i, x). This is never used by any paper in mathematical journals, which have no universally agreed notation for either Walsh's or Paley's functions. We shall align ourselves with the engineers as far as notations are concerned. Thus we shall denote Paley's functions by pal (i, t).

DEFINITION 1. For $k = 1, 2, \cdots$ and $t \in [0, 1)$ we define the kth *Rademacher function* as [9]

$$R_k(t) = (-1)^{t_k},$$

where t_k is the kth binary digit of t:

$$t = t_1 2^{-1} + t_2 2^{-2} + \cdots, \qquad t_k = 0, 1.$$

As t goes from 0 to $1-$, t_k takes values 0 or 1 alternatingly so $R_k(t)$ alternates between 1 and -1. It can be seen that Rademacher functions are square waves. The larger k is, the faster t_k alternates, and the shorter is the period of the wave.

Walsh–Paley functions are defined as products of Rademacher functions. Given n Rademacher functions, there are 2^n possible products containing from 0 to all n factors. These are conveniently labeled from 0 to $2^n - 1$, as follows.

DEFINITION 2 [8]. For $i = 0, 1, 2, \cdots$ and $t \in [0, 1)$, the ith *Walsh–Paley function* is

$$\mathrm{pal}\,(i, t) = \prod_{k=1}^{\infty} [R_k(t)]^{i_k} = \prod_{k=1}^{\infty} (-1)^{i_k t_k},$$

with

$$i = i_1 2^0 + i_2 2^1 + \cdots + i_k 2^{k-1} + \cdots, \qquad i_k = 0, 1.$$

In other words, $R_k(t)$ is a factor of pal (i, t) if and only if $i_k = 1$. Clearly, if i contains r 1's in its binary representation, then pal(i, t) is the product of r Rademacher functions. This leads to the following definition.

DEFINITION 3. The *rank*[1] of pal (i, t) is the number of 1's in the binary digits of i:

$$r(i) = \sum_{k=1}^{\infty} i_k.$$

DEFINITION 4. The *degree*[1] of pal (i, t) is the highest order of Rademacher functions that are its factors:

$$d(i) = \max\{k : i_k = 1\}.$$

DEFINITION 5. The *negligibility* of pal (i, t) is defined as:

$$p(i) = r(i) + \sum_{k=1}^{\infty} i_k k.$$

For example, $6 = 2^2 + 2$, so $i_3 = i_2 = 1$, and

$$r(6) = 2, \quad d(6) = 3 \quad \text{and } p(6) = 7.$$

DEFINITION 6. The *dyadic inner product* of integer i with t is defined as [9]:

$$i * t = \sum_{k=1}^{\infty} i_k t_k \bmod 2.$$

The above definition allows us to express Walsh–Paley functions in a concise form.

THEOREM 1.

$$\text{pal}(i, t) = (-1)^{i*t}.$$

The following two theorems have been proved elsewhere [8].

THEOREM 2. *Walsh–Paley functions are orthonormal and complete over* $[0, 1)$:

$$\int_0^1 \text{pal}(i, t)\,\text{pal}(j, t)\,dt = \delta_{ij},$$

$$\sum_{i=0}^{\infty} \text{pal}(i, t)\,\text{pal}(i, t') = \delta(t - t').$$

THEOREM 3. *The Walsh series, defined by*

$$f_n(t) = \sum_{i=0}^{n} F_i\,\text{pal}(i, t),$$

with

$$F_i = \int_0^1 \text{pal}(i, T) f(T)\,dT$$

and $f(T)$ *being an integrable function over* $[0, 1)$, *converges uniformly to* $f(t)$ *as* $n \to \infty$ *wherever* $f(t)$ *is continuous.*

F_i will be called the ith Walsh–Paley transform, or Walsh transform for short, of $f(t)$. Its rank, degree and negligibility will be taken to be the same as those of pal (i, t).

[1] Rank has been variously called "vielfalt" or "multiplicity". We follow Polyak and Schreider [9] in naming it "rank"; "degree" is called "order" in Polyak and Schreider, a name we consider to be confusing.

We now prove a theorem similar to the recursive relation of Walsh's original functions.

THEOREM 4.

$$\operatorname{pal}(2i, t) = \operatorname{pal}(2i+1, t) = \operatorname{pal}(i, 2t), \qquad 0 \leqq t < \tfrac{1}{2},$$
$$\operatorname{pal}(2i, t) = \operatorname{pal}(i, 2t-1), \qquad \tfrac{1}{2} \leqq t < 1,$$
$$\operatorname{pal}(2i+1, t) = -\operatorname{pal}(i, 2t-1), \qquad \tfrac{1}{2} \leqq t < 1.$$

Proof. The integers i, $2i$ and $2i + 1$ may be expanded in binary representation as

$$i = i_1 2^0 + i_2 2^1 + \cdots + i_k 2^{k-1} + \cdots,$$
$$2i = 0 \cdot 2^0 + i_1 2^1 + \cdots + i_{k-1} 2^{k-1} + \cdots,$$
$$2i + 1 = 1 \cdot 2^0 + i_1 2^1 + \cdots + i_{k-1} 2^{k-1} + \cdots.$$

If $0 < t \leqq \frac{1}{2}$,

$$t = 0 \cdot 2^{-1} + t_2 2^{-2} + \cdots + t_k 2^{-k} + \cdots,$$
$$2t = t_2 2^{-1} + t_3 2^{-2} + \cdots + t_k 2^{-k+1} + \cdots,$$

so

$$(2i) * t = (2i+1) * t = t_2 i_1 + t_3 i_2 + \cdots + t_{k+1} i_k + \cdots = i * (2t).$$

And if $\frac{1}{2} \leqq t < 1$,

$$t = 1 \cdot 2^{-1} + t_2 2^{-2} + \cdots,$$
$$2t - 1 = t_2 2^{-1} + t_3 2^{-2},$$

which gives

$$(2i) * t = i * (2t - 1)$$

and

$$(2i + 1) * t = 1 + i * (2t - 1).$$

Using Theorem 1 we have the desired result.

We shall also need a theorem proved by Fine [11].

THEOREM 5. *The fuction $f(t) = t$ has the Walsh series expansion*

$$t = \frac{1}{2} - \sum_{m=1}^{\infty} 2^{-m-1} R_m(t).$$

We note that this contains only Walsh–Paley functions of rank 1. We also have the following.

COROLLARY.

$$t^k = [\tfrac{1}{2} - \sum 2^{-m-1} R_m(t)]^k.$$

This will be used to prove the following theorem.

THEOREM 6.

$$\int_0^1 t^k \operatorname{pal}(i,t)\,dt = \begin{cases} 0 & \text{if } k < r(i), \\ (-1)^{r(i)} r(i)!\,2^{-p(i)} & \text{if } k = r(i). \end{cases}$$

Proof. The first half has been proved by Polyak and Schreider [10], but we reproduce the argument below. We have

$$\begin{aligned} t^k = {} & \left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^{k-1} \sum_{m=1}^{\infty} 2^{-m-1} R_m(t) \\ & + \frac{k(k-1)}{2}\left(\frac{1}{2}\right)^{k-2} \sum_{m=1}^{\infty} \sum_{m'=1}^{\infty} 2^{-m-m'-2} R_m(t) R_{m'}(t) + \cdots \\ & + (-1)^k \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} 2^{-m_1-m_2-\cdots-m_k-k} R_{m_1}(t) \cdots R_{m_k}(t). \end{aligned}$$

Integrate this with $\operatorname{pal}(i, t)$, which has rank $r(i)$. If $k < r(i)$, then the above expression contains only Walsh functions of rank less than $r(i)$, so they must be orthogonal to $\operatorname{pal}(i, t)$. This proves the first equality. Now if $k = r(i)$, then the last term is the only term containing products of $r(i)$ Rademacher functions. Thus

$$\begin{aligned} \int_0^1 t^k \operatorname{pal}(i,t)\,dt = {} & (-1)^r \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \\ & \cdot 2^{-m_1-m_2-\cdots-m_r-r} \int_0^1 R_{m_1}(t) \cdots R_{m_r}(t) \operatorname{pal}(i,t)\,dt. \end{aligned}$$

To have a nonzero integral the product of the Rademacher functions must be exactly $\operatorname{pal}(i, t)$. Thus the m's must be such that

$$i_{m_1} = i_{m_2} = \cdots = i_{m_r} = 1.$$

Since i contains r 1's and there are r m's, all m's must be different. There are $r!$ ways of choosing them, each giving the same contribution. Thus

$$\begin{aligned} \int t^{r(i)} \operatorname{pal}(i,t)\,dt &= (-1)^r r!\,2^{-r-m_1-m_2-\cdots-m_r} \int [\operatorname{pal}(i,t)]^2\,dt \\ &= (-1)^r r!\,2^{-r-\Sigma i_n n}, \end{aligned}$$

where we have used the fact that $i_n = 1$ for $n = m_1, \cdots, m_r$. It can be readily seen that the above is simply

$$(-1)^r r!\,2^{-p}.$$

To illustrate the above theorem, we note that the function $f(t) = t$ contains no Walsh–Paley function of rank above 1, and, since $\operatorname{pal}(2^k, t) = R_{k+1}(t)$, we have

$$\int t \operatorname{pal}(2^k, t)\,dt = -2^{-k-2}.$$

On the other hand, $r(2^k) = 1$ and $p(2^k) = k + 2$. We have thus verified the theorem for the special case of t^1. It will be shown later that the theorem itself is a special case of Theorem 7, which applies to Walsh transforms of functions that include integer powers of t.

3. Integration by parts. We begin developing our integration by parts technique by defining a set of functions which will later be seen to be weighting functions which, when integrated with derivatives of a function, give its Walsh transforms.

DEFINITION 7.

$$y(i, t) = \frac{1}{[r(i) - 1]!} \int_0^t \mathrm{pal}\,(i, T)(t - T)^{r-1}\,dT, \quad i = 1, \quad r = r(i).$$

Several of its properties will be proved.

LEMMA 1.

$$y(i, t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{r-1}} \mathrm{pal}\,(i, t_r)\,dt_r\,dt_{r-1} \cdots dt_1.$$

Proof. Differentiate $y(i, t)$ k times with respect to t giving

$$y^{(k)}(i, t) = \frac{1}{(r - k - 1)!} \int_0^t \mathrm{pal}\,(i, T)(t - T)^{r-k-1}\,dT, \qquad k < r - 1,$$

$$y^{(r-1)}(i, t) = \int_0^t \mathrm{pal}\,(i, T)\,dT,$$

$$y^{(r)}(i, t) = \mathrm{pal}\,(i, t).$$

We also have from Theorem 6

$$y(i, 1) = y^{(k)}(i, 1) = 0 \quad \text{for all } k < r,$$

and of course,

$$y(i, 0) = y^{(k)}(i, 0) = 0.$$

Integrating $y^{(r)}(i, t)$ r times, taking into account the above boundary conditions, we have

$$y(i, t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{r-1}} \mathrm{pal}\,(i, t_r)\,dt_r\,dt_{r-1} \cdots dt_1.$$

The lemma immediately leads to our main result, that the integrand in the definition of Walsh transforms can be integrated by parts.

THEOREM 7. *For the function $f(t)$ which is smooth (i.e., continuous and having continuous derivatives) to order r over* $[0, 1]$,

$$F_i = (-1)^r \int_0^1 f^{(r)}(t) y(i, t)\,dt, \qquad i \geqq 1, \quad r = r(i).$$

Proof. We expand $f(t)$ in a Taylor series about $t = 1$ up to the power $r - 1$, giving

$$f(t) = f(1) + f'(1)(t - 1) + \cdots + f^{(r-1)}(1)\frac{(t - 1)^{r-1}}{r!} + s_r(t),$$

where $s_r(t)$, the remainder, is given by

$$s_r(t) = \frac{1}{(r-1)!}\int_1^t (t-T)^{r-1} f^{(r)}(T)\,dT.$$

When the series is integrated term by term with pal (i, t), only the remainder gives a nonzero result according to Theorem 6. So

$$F_i = \frac{1}{(r-1)!}\int_0^1 \operatorname{pal}(i,t)\int_1^t (t-T)^{r-1} f^{(r)}(T)\,dT\,dt.$$

Interchanging the order of integration gives

$$F_i = \frac{1}{(r-1)!}\int_0^1 f^{(r)}(T)\int_T^0 (t-T)^{r-1}\operatorname{pal}(i,t)\,dt\,dT$$

$$= (-1)^r \int_0^1 f^{(r)}(T) y(i,T)\,dT.$$

We see that in the expression for $F_i, f(t)$ has been differentiated $r(i)$ times while pal (i, T) has been integrated, as in integration by parts.

The above theorem would have been a mere mathematical curiosity but for the fact that $y(i, t)$ has some important properties: it is nonnegative and its maximum and mean can be found easily, as proved below.

LEMMA 2. $y(i, t) \geqq 0$ *for all* t *and* $i \geqq 1$.

Proof. The lemma is proved by induction using Theorem 4. First we have

$$y(1,t) = \int_0^t \operatorname{pal}(1,T)\,dT = \begin{cases} t, & 0 \leqq t < \frac{1}{2}, \\ 1-t, & \frac{1}{2} \leqq t < 1. \end{cases}$$

Now we show that if the theorem holds for $y(i, t)$, it also holds for $y(2i, t)$ and $y(2i + 1, t)$. We have $r(2i) = r(i)$ and $r(2i + 1) = r(i) + 1$. So

$$y(2i,t) = \int_0^t \operatorname{pal}(2i,T)(t-T)^{r-1}\,dT.$$

Applying Theorem 4, we have for $0 \leqq t < \frac{1}{2}$,

$$y(2i,t) = \int_0^t \operatorname{pal}(i,2T)(t-T)^{r-1}\,dT.$$

Changing the variable from T to $T' = 2T$ we have

$$y(2i,t) = \int_0^{2t} \operatorname{pal}(i,T')(t-\tfrac{1}{2}T')^{r-1}\tfrac{1}{2}\,dT'$$

$$= 2^{-r}\int_0^{2t} \operatorname{pal}(i,T')(2t-T')^{r-1}\,dT'$$

$$= 2^{-r} y(i,2t) \geqq 0.$$

For $\frac{1}{2} \leqq t < 1$,

$$y(2i, t) = \int_0^{1/2} \mathrm{pal}(i, 2T)(t - T)^{r-1}\, dT + \int_{1/2}^{t} \mathrm{pal}(i, 2T - 1)(t - T)^{r-1}\, dT.$$

The first term is zero according to Theorem 6, while the second is

$$\int_0^{2t-1} \mathrm{pal}(i, T')(t - \tfrac{1}{2}T' - \tfrac{1}{2})^{r-1}\tfrac{1}{2}\, dT' = 2^{-r}y(i, 2t - 1) \geqq 0.$$

Then we have, for $0 \leqq t < \frac{1}{2}$,

$$\begin{aligned} y(2i + 1, t) &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_r} \mathrm{pal}(i, 2t_{r+1})\, dt_{r+1}\, dt_r \cdots dt_1 \\ &= 2^{-r} \int_0^t y(i, 2t_1)\, dt_1 \geqq 0, \end{aligned}$$

while for $\frac{1}{2} \leqq t < 1$,

$$\begin{aligned} y(2i + 1, t) &= 2^{-r} \int_0^{1/2} y(i, 2t_1)\, dt_1 - 2^{-r} \int_{1/2}^{t} y(i, 2t_1 - 1)\, dt_1 \\ &= 2^{-r} \int_{t-1/2}^{1/2} y(i, 2t_1)\, dt_1 \geqq 0. \end{aligned}$$

Thus the lemma is true for all i.

LEMMA 3.

$$\int_0^1 y(i, t)\, dt = 2^{-p(i)}.$$

Proof.

$$\int_0^1 y(i, t)\, dt = \frac{1}{(r - 1)!} \int_0^1 \int_0^t \mathrm{pal}(i, T)(t - T)^{r-1}\, dT\, dt.$$

Exchanging the order of integration, and taking into account $t \geqq T$, we get

$$\begin{aligned} \int_0^1 y(i, t)\, dt &= \frac{1}{(r - 1)!} \int_0^1 \mathrm{pal}(i, T) \int_T^1 (t - T)^{r-1}\, dt\, dT \\ &= \frac{1}{(r - 1)!} \int_0^1 \mathrm{pal}(i, T) \frac{(1 - T)^r}{r}\, dT. \end{aligned}$$

By Theorem 6, only T^r gives nonzero contribution. Thus

$$\int_0^1 y(i, t)\, dt = \frac{1}{r!} \int_0^1 \mathrm{pal}(i, T)(-T)^r\, dT = 2^{-p(i)}.$$

LEMMA 4. $\max y(i,t) = 2^{1-p(i)}$ *for* $i \geqq 1$.

Proof. We need only prove that $\max y(i, t) = 2\int_0^1 y(i, t)\, dt$. Again induction is used. We first note that

$$\max (1, t) = y(1, \tfrac{1}{2}) = \tfrac{1}{2},$$

while

$$2^{-p(1)} = \tfrac{1}{4}.$$

Thus the lemma is true for $i = 1$. Now we already showed (proof of Lemma 2)

$$y(2i, t) = 2^{-r}y(i, 2t) \quad \text{or} \quad 2^{-r}y(i, 2t - 1),$$

so if the lemma holds for some i, it also holds for $2i$. Next we note

$$y(2i + 1, t) = \begin{cases} 2^{-r}\displaystyle\int_0^t y(i, 2T)\, dT, & 0 \leqq t < \tfrac{1}{2}, \\ 2^{-r}\displaystyle\int_{t-1/2}^{1/2} y(i, 2T)\, dT, & \tfrac{1}{2} \leqq t < 1. \end{cases}$$

Since $y(i, T)$ is never negative, $y(2i + 1, t)$ is biggest when the range of integration takes the whole domain of $y(i, T)$. This occurs for $t = \frac{1}{2}$, for which

$$y(2i + 1, \tfrac{1}{2}) = 2^{-r-1}\int_0^1 y(i, T)\, dT = 2^{-r-1-p(i)}.$$

But as $p(2i + 1) = r(i) + 2 + p(i)$, the lemma holds for $2i + 1$ as well.

We have thus succeeded in showing that F_i is a weighted average of the $r(i)$th derivative of $f(t)$, and that the mean and maximum of the weighting function $y(i, t)$ are readily computed. The usefulness of Theorem 7 applied in conjunction with Lemmas 2–4 is illustrated by examples in the next section.

4. Some applications. (a) In the closing paragraph of § 1 we mentioned that Theorem 6 is a special case of Theorem 7. This will now be shown.

Suppose $f(t) = t^k$ and we wish to find its Walsh transform F_i with $r(i)$ greater than k. Now if we differentiate t^k r times we have $f^{(r)} = 0$, thus $F_i = 0$. And if $r(i) = k$, then $f^{(r)} = r!$, so that $F_i = (-1)^r r! 2^{-p(i)}$. We have thus derived Theorem 6 from Theorem 7.

(b) It is possible to estimate the size of each Walsh transform of a function given its derivatives. Since $y(i, t)$ is nonnegative, we have

$$|F_i| \leqq \max |f^{(r)}(t)| \int_0^1 y(i, t)\, dt = \max |f^{(r)}|/2^{p(i)},$$

or

$$|F_i| \leqq \max y(i, t) \int_0^1 |f^{(r)}(t)|\, dt = \int_0^1 |f^{(r)}(t)|\, dt/2^{p(i)-1}.$$

We see that if $p(i)$ is large F_i has a small absolute value, and hence can be dropped from the Walsh series representing $f(t)$. This explains the name "negligibility".

To see when $p(i)$ is large, we note

$$\begin{aligned} p(2^k) &= k + 2, & p(2^k - 1) &= \tfrac{1}{2}k(k + 3), \\ r(2^k) &= 1, & r(2^k - 1) &= k, \\ d(2^k) &= k + 1, & d(2^k - 1) &= k. \end{aligned}$$

It can be seen that $p(i)$ is large when both rank and degree are large. But if rank is small then $p(i)$ can be small unless degree is exceptionally large.

If $f(t)$ and all its derivatives are slowly varying and free of isolated high and low values, then we may expect

$$F_i \sim (-1)^{r_2 - p(i)} \int_0^1 f^{(r)}(t)\, dt,$$

as a rough estimate. Using $f = \sin((\pi/2)t)$, we have computed 32 Walsh transforms, their upper bounds and estimated values. These are shown in Fig. 1.

(c) Convergence of the Walsh spectrum can also be studied. We note that both F_i and F_{2i} are weighted averages of $f^{(r)}$, and

$$\max y(i, t) = 2^r \cdot \max y(2i, t),$$

$$\int_0^1 y(i, t)\, dt = 2^r \int_0^1 y(2i, t)\, dt.$$

Thus how fast F_i converges depends on its rank. For $i = 2^k$, F_i is likely to be halved when i is doubled, while for $i = 2^k - 1$, F_i is likely to decrease by a factor of 2^{-k}.

It can be observed that the convergence of Walsh transforms is quite different from that of Fourier transforms. This is why the attempt to generalize discrete Walsh transforms to make i a continuous variable, so as to make them analytic functions of i, is fairly difficult, though formally possible. It is likely that many techniques in complex Fourier transform theory would have no counterparts in Walsh transform theory.

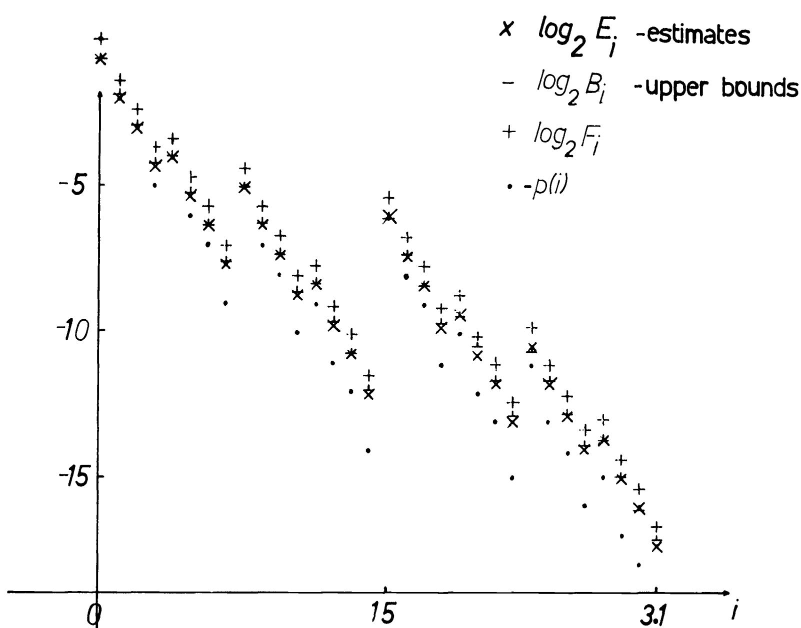

FIG. 1

(d) Finally we examine the effect of scaling. Suppose $f(t) = g(at)$ with $a < 1$. Then we have the upper bounds on $|F_i|$ and $|G_i|$,

$$|F_i| \leqq \max |g^{(r)}| 2^{-p(i)} a^r,$$

$$|G_i| \leqq \max |g^{(r)}| 2^{-p(i)}.$$

Thus the Walsh transforms of $f(t)$ are smaller than those of $g(t)$ for all the i's, with the reduction effect increasing with rank. We see that if a function has a Walsh series converging too slowly and containing too many significant high rank terms, we can divide the interval into sections, scale these to $[0, 1)$ and obtain a faster convergent series for each interval. However, if the original series contains only low rank terms, then dividing and scaling does not have much beneficial effect.

REFERENCES

[1] J. L. WALSH, *A closed set of normal orthogonal functions*, Amer. J. Math., 45 (1923), pp. 5–24.

[2] M. COHN, *Walsh functions, sequency, and Gray codes*, SIAM J. Appl. Math., 21 (1971), pp. 442–447.

[3] J. S. BYRNES AND D. A. SWICK, *Instant Walsh functions*, SIAM Rev., 12 (1970), p. 137.

[4] G. R. REDINBO, *A note on the construction of generalized Walsh functions*, this Journal, 2 (1971), pp. 166–167.

[5] Proc. of Symposium on Applications of Walsh Functions, Washington, D.C., 1970–1972. (National Technical Information Service Nos. AD-707 431, AD-727 000 and AD-744 650.)

[6] K. L. BRUNEAU AND B. R. CLARK, *Walsh–Fourier series for x^2, x^3, x^4 on $-\frac{1}{2} \times \frac{1}{2}$*, IEEE Trans. Electromag. Compat., 13 (1971), pp. 155–157.

[7] C. K. YUEN, *Upper bounds on Walsh transforms*, IEEE Trans. Computers, C-21 (1972), pp. 1273–1280.

[8] R. E. A. C. PALEY, *A remarkable series of orthogonal functions*, Proc. London Math. Soc., 2 (1932), no. 34, pp. 241–279.

[9] H. RADEMACHER, *Einige satze über Reihen von allegemeinen Orthogonalfunktionen*, Math. Ann., 87 (1922), pp. 112–138.

[10] B. T. POLYAK AND Y. A. SCHREIDER, *The application of Walsh functions in approximate calculations*, Problems in Theory of Mathematical Machines, collection 2, Bazilevskii, ed., Fizmatgiz, Moscow, 1966, pp. 176–190. (Abstract in Math. Rev., 31 (1966), pp. 1096–1097.)

[11] N. J. FINE, *On the Walsh functions*, Trans. Amer. Math. Soc., 65 (1949), pp. 372–414.

[12] ———, *The generalized Walsh functions*, Ibid., 69 (1950), pp. 66–77.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

INEQUALITIES INVOLVING A FUNCTION AND ITS INVERSE*

R. P. BOAS, JR. AND M. B. MARCUS†

Abstract. The paper presents a simple technique for establishing a class of inequalities, some of which arise in connection with ε-entropy and its applications in probability, and which include a generalization of Young's inequality.

We present a simple technique for establishing a class of inequalities to which we were led by some special cases arising in connection with ε-entropy and its applications in probability. We now give some examples.

Example 1. $t(1 - s) \leqq e^{-s} + t \log t \ (0 < s < 1; t > 0)$.

Example 2. If f is nonincreasing and positive, f^{-1} is a generalized inverse of f, and $\int^\infty f(u)\,du$ converges, then

$$\int_0^t f^{-1}(y)\,dy \leqq at + \int_a^\infty f(u)\,du.$$

Example 3. If $t_k \geqq 0$ and $\sum t_k$ converges, the sums

$$S_1 = \sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=n}^\infty t_k\right)^{1/2}, \qquad S_2 = \sum_{n=1}^\infty t_n\left(\frac{1}{n}\sum_{k=n}^\infty t_k\right)^{-1/2}$$

satisfy $\frac{1}{2}S_1 \leqq S_2 \leqq 2S_1$.

Example 4. Let $\sum_{k=1}^\infty t_k \leqq 1$. Define $M(\varepsilon) = \max\{n : \sum_{k=n}^\infty t_k \geqq \varepsilon\}$ if $\sum_{k=1}^\infty t_k \geqq \varepsilon$; $M(\varepsilon) = 0$ otherwise. Each of the sums S_1, S_2 of Example 3 and

$$\sum_{n=0}^\infty e^{-n/2}\{M(e^{-n})\}^{1/2}$$

is bounded by a multiple of each other one (independent of the t_k).

Uses of the function M can be found in Mitjagin [6] and Dudley [3]. Dudley's Theorem 7.1 suggested Example 4.

We shall be dealing with a nonincreasing positive function f, not necessarily continuous, whose domain contains the finite interval $[a, b]$. A generalized inverse f^{-1} of f is a function satisfying

$$\sup\{x : f(x) > y\} \leqq f^{-1}(y) \leqq \inf\{x : f(x) < y\}. \tag{1}$$

Two generalized inverses of the same function differ at most on a countable set.

We start with the following (presumably well-known) convergence theorem.

THEOREM 1. *If f is nonincreasing on $[a, \infty)$, f^{-1} is a generalized inverse of f, and $\int^\infty f(x)\,dx$ converges, then $\int_0 f^{-1}(y)\,dy$ converges, and conversely.*

Theorem 1 is geometrically obvious; see Fig. 1.

* Received by the editors January 18, 1972, and in revised form June 12, 1972.

† Department of Mathematics, Northwestern University, Evanston, Illinois 60201.

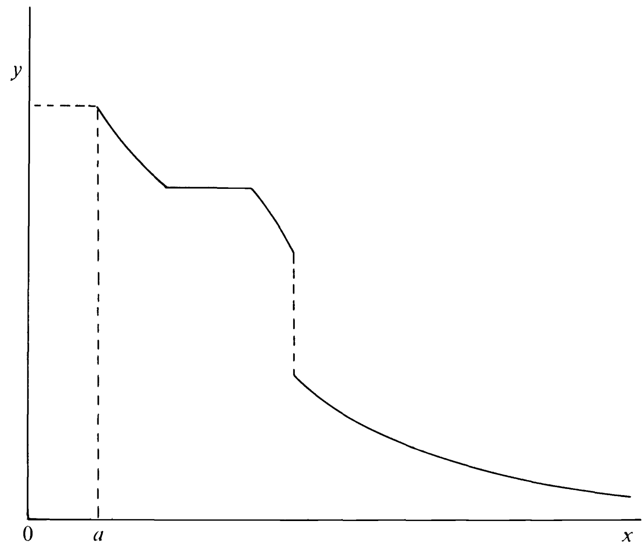

FIG. 1. *The area between the graph and the x-axis is finite if and only if the area between the graph and the y-axis is finite*

Proof. If f is continuous and strictly decreasing, the formula for integration by parts can be written

$$\int_a^b f(x)\,dx = bf(b) - af(a) + \int_{f(b)}^{f(a)} f^{-1}(y)\,dy. \tag{2}$$

In the general case, f is positive, nonincreasing, and not necessarily continuous. We can construct, in an obvious way, a sequence of bounded, strictly decreasing continuous functions g_k such that $g_k(x) \to g(x)$ and $g_k^{-1}(y) \to g^{-1}(y)$, where $g(x) = f(x)$ and $g^{-1}(y) = f^{-1}(y)$ except for a countable number of points. Since (2) holds for g_k and g_k^{-1}, it holds in the general case by bounded convergence.

The hypotheses imply that $xf(x) \to 0$ as $x \to \infty$. Consequently, we can let $b \to \infty$ and $f(b) \to 0$ in (2). The converse follows in the same way.

COROLLARY 1. *If p and r are positive numbers,*

$$\int^{\infty} \{f(u^r)\}^{1/p}\,du \quad \text{and} \quad \int_0 \{f^{-1}(y^p)\}^{1/r}\,dy \tag{3}$$

converge or diverge together.

Proof. This is an immediate consequence of Theorem 1 when f is strictly decreasing and continuous, since $g(u) = \{f(u^r)\}^{1/p}$ and $h(y) = \{f^{-1}(y^p)\}^{1/r}$ are inverses of each other. For the general case we have to verify that one of the generalized inverses of $g(u)$ is $\{f^{-1}(y^p)\}^{1/r}$, where f^{-1} is a generalized inverse of f. Now one of the definitions of $g^{-1}(y)$ is

$$\begin{aligned}\inf\{x : g(x) < y\} &= \inf\{x : [f(x^r)]^{1/p} < y\}\\ &= \inf\{x : f(x^r) < y^p\}\\ &= \inf\{x^{1/r} : f(x) < y^p\};\end{aligned}$$

the last expression is indeed a value of $\{f^{-1}(y^p)\}^{1/r}$.

COROLLARY 2. *If $p > 1$, $q = p/(p-1)$, then*

$$\int^{\infty} \{u^{-1}f(u)\}^{1/p}\,du \quad \textit{and} \quad \int_0 \{y^{-1}f^{-1}(y)\}^{1/q}\,dy$$

converge or diverge together.

Proof. This is obtained by taking $r = q$ in Corollary 1 and changing the variable of integration.

COROLLARY 3. *If $p > 1$, $q = p/(p-1)$, $c > 1$, then*

$$\int^{\infty} \{f(c^t)\}^{1/p} c^{t/q}\,dt \quad \textit{and} \quad \int^{\infty} \{f^{-1}(c^{-s})\}^{1/q} c^{-s/p}\,ds$$

converge or diverge together.

Proof. This is Corollary 2 with $u = c^t$ and $y = c^{-s}$.

Now let $\sum t_k$ be a convergent series of nonnegative terms; to simplify the discussion we assume that $\sum_{k=1}^{\infty} t_k \leqq 1$. Set

$$f(x) = \sum_{k=n}^{\infty} t_k, \qquad n - 1 < x \leqq n; \qquad f(0) = 1; \tag{4}$$

define

$$M(\varepsilon) = \max\left\{n : \sum_{k=n}^{\infty} t_k \geqq \varepsilon\right\} = \sup\{x : f(x) \geqq \varepsilon\}. \tag{4a}$$

The function M is a generalized inverse of f.

If we interpret Corollary 2 in terms of $\sum t_k$ and M, we see that

$$S_1 = \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{1/p} \quad \text{and} \quad S_3 = \sum_{n=0}^{\infty} c^{-n/p}\{M(c^{-n})\}^{1/q} \quad (c > 1) \tag{5}$$

converge or diverge together. However, what is wanted in applications is more, namely that the sums in (5) are mutually comparable, i.e., that each is bounded by a multiple of the other (the multiple being independent of $\{t_k\}$). That the stronger result is true illustrates how series inequalities are sometimes simpler than their integral analogues.

To obtain the stronger result, we return to (2), let $b \to \infty$, and take account of the term $af(a)$ at each step. From the proof of Theorem 1, we obtain

$$\int_a^{\infty} f(x)\,dx = -af(a) + \int_0^{f(a)} f^{-1}(y)\,dy.$$

Following the proof of Corollary 1, we have

$$\int_a^{\infty} \{f(u^r)\}^{1/p}\,du = -a\alpha + \int_0^{\alpha} \{f^{-1}(y^p)\}^{1/r}\,dy,$$

where $\alpha = \{f(a^r)\}^{1/p}$. Take $r = q$, $a = 1$, and then put $u^q = s$, $y^p = v$, so that $\alpha = \{f(1)\}^{1/p}$ and

$$\frac{1}{q}\int_1^{\infty} \{s^{-1}f(s)\}^{1/p}\,ds = -\alpha + \frac{1}{p}\int_0^{\alpha^p} \{v^{-1}f^{-1}(v)\}^{1/q}\,dv.$$

Interpreting this in terms of $\sum t_k$ and M as in (4) and (4a), we have

$$(6)\qquad \frac{1}{q}\int_1^\infty \{s^{-1}f(s)\}^{1/p}\,ds + \left(\sum_{k=1}^{n} t_k\right)^{1/p} = \frac{1}{p}\int_0^{\alpha^p} \{\varepsilon^{-1}M(\varepsilon)\}^{1/q}\,d\varepsilon.$$

We now proceed to compare each of the integrals in (6) with a corresponding sum.

On one hand, we have

$$(7)\qquad \sum_{k=2}^{\infty}\left(\frac{1}{k}\sum_{j=k}^{\infty} t_j\right)^{1/p} \leqq \int_1^\infty \{s^{-1}f(s)\}^{1/p}\,ds \leqq \sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{j=k}^{\infty} t_j\right)^{1/p}.$$

On the other hand, $M(\varepsilon) = 0$ when $\varepsilon > \alpha^p = f(1)$, and hence when $\varepsilon > 1$ we have

$$\int_0^{\alpha^p} \{\varepsilon^{-1}M(\varepsilon)\}^{1/q}\,d\varepsilon = \sum_{n=1}^{\infty}\int_{c^{-n}}^{c^{-n+1}} \{\varepsilon^{-1}M(\varepsilon)\}^{1/q}\,d\varepsilon.$$

Now

$$\int_{c^{-n}}^{c^{-n+1}} \{\varepsilon^{-1}M(\varepsilon)\}^{1/q}\,d\varepsilon = (\log c)\int_{n-1}^{n} \{M(c^{-t})\}^{1/q}c^{-t/p}\,dt,$$

and hence

$$(8)\qquad \begin{aligned} (\log c)\{M(c^{-n+1})\}^{1/q}c^{-n/p} &\leqq \int_{c^{-n}}^{c^{-n+1}} \{\varepsilon^{-1}M(\varepsilon)\}^{1/q}\,d\varepsilon \\ &\leqq (\log c)\{M(c^{-n})\}^{1/q}c^{-(n-1)/p}, \\ (\log c)c^{-1/p}\sum_{n=0}^{\infty}\{M(c^{-n})\}^{1/q}c^{-n/p} &\leqq \int_0^{\alpha^p} \{\varepsilon^{-1}M(\varepsilon)\}^{1/q}\,d\varepsilon \\ &\leqq (\log c)c^{1/p}\sum_{n=1}^{\infty}\{M(c^{-n})\}^{1/q}c^{-n/p}. \end{aligned}$$

Comparing (7) and (8), we get from (6):

$$\frac{1}{q}\sum_{k=2}^{\infty}\left(\frac{1}{k}\sum_{j=k}^{\infty} t_j\right)^{1/p} + \left(\sum_{k=1}^{\infty} t_k\right)^{1/p} \leqq p^{-1}(\log c)c^{1/p}\sum_{n=1}^{\infty}\{M(c^{-n})\}^{1/q}c^{-n/p}$$

and

$$\frac{1}{p}(\log c)c^{-1/p}\sum_{n=0}^{\infty}\{M(c^{-n})\}^{1/q}c^{-n/p} \leqq \left(\sum_{k=1}^{\infty} t_k\right)^{1/p} + \frac{1}{q}\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{k=1}^{\infty} t_j\right)^{1/p}.$$

Therefore we have the required result about (5), namely, the following corollary.

COROLLARY 4. *If $\sum t_k$ is a convergent series of positive terms with*

$$\sum_{k=1}^{\infty} t_k \leqq 1,$$

if $p > 1$ and $q = p/(p-1)$, $c > 1$, and S_1, S_3 are defined by (5), then

$$q^{-1}c^{-1/p}S_1 \leqq p^{-1}(\log c)S_3 \leqq c^{1/p}(1 + q^{-1})S_1.$$

Corollary 1 enables us to provide an elementary proof of a useful lemma ([5], Russian p. 20; English p. 10).

COROLLARY 5. *If* $\{a_n\}$ *is a nonincreasing sequence of positive numbers and* $M(s) = \sup\{n: a_n \geqq s\}$, *then*

$$\limsup_{s \to 0} \frac{\log M(s)}{\log(1/s)} = \inf\{\alpha: \sum a_n^\alpha < \infty\}.$$

Proof. Define $f(x) = a_n$ for $n - 1 < x \leqq n$; then $M(s)$ is a generalized inverse of f. Interpreting (3) in this notation with $p = 1/\alpha$ and $r = 1$, we have that

$$\int^\infty \{f(t)\}^\alpha \, dt \quad \text{and} \quad \int_0 M(\delta^{1/\alpha}) \, d\delta$$

converge or diverge together. Suppose that they converge. Then since M decreases, $\delta M(\delta^{1/\alpha}) \to 0$ as $\delta \to 0$; in other words, $M(s)s^\alpha \to 0$ as $s \to 0$. Therefore,

$$\alpha \geqq \limsup_{s \to 0} \frac{\log M(s)}{\log(1/s)} = \eta,$$

say. Then for s small enough, $M(s) \leqq s^{-\eta-\varepsilon}$, $\varepsilon > 0$. Therefore

$$\int M(s^{1/(\eta + 2\varepsilon)}) \, ds < \infty,$$

and using (3) again, we see that $\sum a_n^{\eta + 2\varepsilon} < \infty$ for all $\varepsilon > 0$. This completes the proof.

Theorem 1 and its consequences are in fact corollaries of a generalization of Young's inequality.

THEOREM 2. *Let* f *be nonincreasing and positive and let* t *be in the domain of* f^{-1}, *a generalized inverse of* f. *Then*

$$bf(b) + \int_{f(b)}^{t} f^{-1}(y) \, dy \leqq at + \int_a^b f(u) \, du \tag{9}$$

and

$$af(a) + \int_a^b f(u) \, du \leqq bt + \int_t^{f(a)} f^{-1}(y) \, dy, \tag{10}$$

with equality only when $f(a^+) \leqq t \leqq f(a^-)$ *in* (9) *or* $f(b^+) \leqq t \leqq f(b^-)$ *in* (10). (See Fig. 2.) *Furthermore, when* f *is nondecreasing and positive,* (9) *and* (10) *hold with the direction of the inequality reversed.*

Young's inequality [8] is (10) with a nondecreasing function and reversed inequality (Young considered only strictly increasing differentiable functions). The usual statement of Young's inequality [4, p. 111], [7, p. 48] is the special case when $a = f(a) = 0$. Generalized inverses were introduced in this context in [2]. We omit the proof of Theorem 2, which will be presented elsewhere.

In our final theorem, we introduce a third sum which is comparable to the sums in (5). This theorem, like Theorem 1, is a result of an integration by parts in which the monotonicity of the function allows some simplification. However, it is

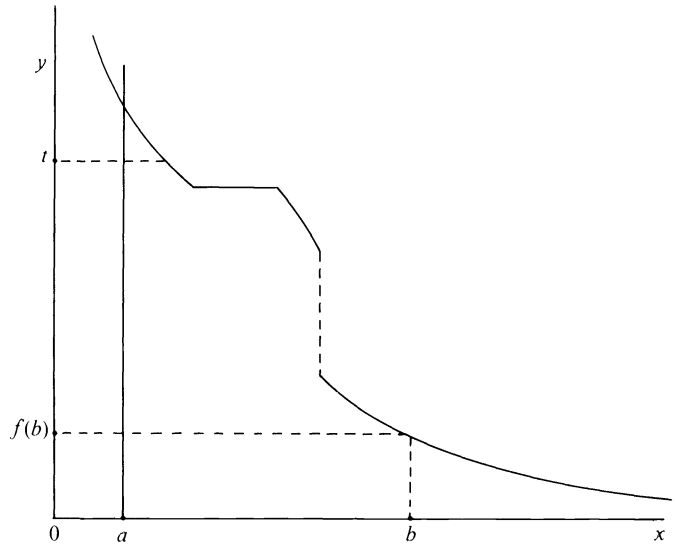

FIG. 2. *Pictorial representation of* (9)

still simpler to prove the theorem by means of a summation by parts; that is the method we shall use.

THEOREM 3. *Let* $\sum t_k$ *be a convergent series of nonnegative terms. If* $p > 1$ *and* $q = p/(p-1)$, *the sums*

$$S_1 = \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{1/p}, \qquad S_2 = \sum_{n=1}^{\infty} t_n\left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{-1/q}$$

satisfy

$$(1/q)S_1 \leqq S_2 \leqq pS_1.$$

Proof. If S_1 converges,

$$\begin{aligned} S_1 &= \sum_{n=1}^{\infty} n\left\{\left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{1/p} - \left(\frac{1}{n+1}\sum_{k=n+1}^{\infty} t_k\right)^{1/p}\right\} \\ &\geqq \sum_{n=1}^{\infty} n^{1/q}\left\{\left(\sum_{k=n}^{\infty} t_k\right)^{1/p} - \left(\sum_{k=n+1}^{\infty} t_k\right)^{1/p}\right\} \\ &\geqq \frac{1}{p}\sum_{n=1}^{\infty} t_n\left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{-1/q}. \end{aligned}$$

In the other direction, we have

$$\begin{aligned} S_1 &= \sum_{n=1}^{\infty} n\left\{\left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{1/p} - \left(\frac{1}{n+1}\sum_{k=n+1}^{\infty} t_k\right)^{1/p}\right\} \\ &\leqq \sum_{n=1}^{\infty} n\left\{\left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{1/p} - \left(\frac{1}{n}\sum_{k=n+1}^{\infty} t_k\right)^{1/p}\right\} + \frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{\infty} t_k\right)^{1/p}. \end{aligned}$$

Therefore,

$$S_1 \leqq q \sum_{n=1}^{\infty} n^{1/q} \left\{ \left(\sum_{k=n}^{\infty} t_k \right)^{1/p} - \left(\sum_{k=n+1}^{\infty} t_k \right)^{1/p} \right\}$$
$$\leqq q \sum_{n=1}^{\infty} t_n \left(\frac{1}{n} \sum_{k=n}^{\infty} t_k \right)^{-1/q}.$$

The last inequality follows because

$$b^{1/p} - (b - a)^{1/p} \leqq ab^{-1/q}.$$

It is interesting to observe that $\sum t_k^{1/p} < \infty$ is a necessary condition for S_1 and S_2 to be finite [4, p. 255, Theorem 345]. In fact, if we apply Hölder's inequality with index $1/p$ to S_2, we find

$$S_2 \geqq (\sum t_k^{1/p})^p S_1^{1-p}.$$

It is also known that S_1 is finite when $\{t_k\}$ decreases and $\sum t_k^{1/p} < \infty$ [1].

REFERENCES

[1] R. P. Boas, Jr., *Inequalities for monotonic series*, J. Math. Anal. Appl., 1 (1960), pp. 121–126.
[2] F. Cunningham, Jr. and N. Grossman, *On Young's inequality*, Amer. Math. Monthly, 78 (1971), pp. 781–783.
[3] R. M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Functional Anal., 1 (1967), pp. 290–330.
[4] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[5] B. Ja. Levin, *Distribution of zeros of entire functions*, GITTL, Moscow, 1956; English transl., Transl. Math. Monographs, vol. 5, Amer. Math. Soc., Providence, 1964.
[6] B. S. Mitjagin, *Approximate dimension and bases in nuclear spaces*, Uspehi Mat. Nauk., 16 (1961), no. 4 (100), pp. 63–312 = Russian Math. Surveys, 16 (1961), no. 4, pp. 59–127.
[7] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York, 1970.
[8] W. H. Young, *On classes of summable functions and their Fourier series*, Proc. Roy. Soc. Ser. A, 87 (1912), pp. 225–229.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

IMBEDDING A CLASS OF LINEAR INTEGRAL EQUATIONS THROUGH THE FIRST CRITICAL POINT*

ALAN SCHUMITZKY† AND TOM WENSKA‡

Abstract. In this paper, we investigate the continuation of an imbedded solution ($x(t) = x(t, \alpha)$) of

$$x(t) = f(t) + \int_0^\alpha k(t, s)x(s)\, ds = f(t) + (K_\alpha x)(t)$$

through its first "critical point" $\alpha = c$. Under the assumption that the Fredholm resolvent $\Gamma_\alpha = (I - K_\alpha)^{-1} - I$ has a simple pole in its meromorphic expansion about $\alpha = c$, we obtain a simple eigenspace corresponding to $\lambda = 1$ for the operator K_c; and in accordance with the Fredholm alternative, we have an imbedded solution for $\alpha > c$ for forcing functions orthogonal to the one-dimensional eigenspace of the adjoint operator K_c^*. The principal technique is the explicit solving of the Bartle–Schmidt bifurcation equation.

1. Introduction. We are concerned with linear Fredholm integral equations of the second kind:

$$x(t) = f(t) + \int_0^\alpha k(t, s)x(s)\, ds. \tag{1}$$

By an "imbedding" of (1) we mean the consideration of the solution $x(t) = x(t, \alpha)$ as a function of the interval-length parameter α. The imbedding formalism leads to an equivalent formulation of (1) as an initial value problem. The initial conditions are specified at $\alpha = 0$ by $x(t, 0) = f(t)$. New and effective numerical procedures have been obtained in this way (cf. [1], [3] and the references cited therein). In addition the imbedding concept has led to a general and unified theory for factorization of integral operators (cf. [13]). Nevertheless, there is one intrinsic limitation of the imbedding formalism from the practical point of view. This limitation can be described as follows.

Assume f and k are continuous and let c be the first "critical length" of (1), i.e., c is the largest number a such that the operator $I - K_\alpha$ is invertible for all $\alpha \in [0, a)$. (K_α is the integral operator defined by $K_\alpha x(t) = \int_0^\alpha k(t, s)x(s)\, ds$.) For general forcing function f, the imbedded solution $x(t, \alpha)$ tends to ∞ (in absolute value) as α tends to c. If originally (1) were to be solved for $\alpha = b > c$, then the initial value methods, which start at $\alpha = 0$, never get past $\alpha = c$.

The main question to which we address ourselves in this paper is: When can an imbedded solution get past the first critical length? Viewed in this light, the problem becomes one of bifurcation theory (cf. Wenska [5]). The Fredholm alternative shows that a necessary condition is that the forcing function f be orthogonal to the null-space of the adjoint of $(I - K_c)$. It turns out that this condition is not, in general, sufficient. A crucial role here is played by the nature of the singularities of the Fredholm resolvent $\Gamma_\alpha = (I - K_\alpha)^{-1} - I$ considered as a function of α.

* Received by the editors January 27, 1972, and in revised form August 7, 1972.

† Department of Mathematics, University of Southern California, Los Angeles, California 90007.
‡ Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822.

Our main result is as follows: *If $k(t, s)$ is symmetric, real analytic on $0 \leqq t \leqq s \leqq b$, $c < b$, if Γ_α is meromorphic in α and has a simple pole at $\alpha = c$, and if f satisfies the necessary orthogonality condition, then* (1) *has a unique imbedded solution which exists for $\alpha > c$.*

A word about the hypotheses is in order. The symmetry assumption on k is merely a convenience to make an already complicated proof more tractable. It could be omitted without violating the conclusion. The analyticity assumption on k and meromorphy assumption on Γ_α are necessary ingredients of the bifurcation machinery and are satisfied in a wide class of problems. The drastic assumption is, however, the one concerning the simplicity of the pole. Unfortunately the result is not true without it. (A counterexample is given in § 5.) However, the argument we give can be used for higher order poles at the expense of more labor and a more complicated conclusion.

A brief summary of the paper follows: In § 2 we state precisely our hypotheses. Section 3 contains a description of two classes of kernels which satisfy the meromorphy assumptions; namely, the class of kernels analytic on the square $[0, b] \times [0, b]$ and the class of kernels analytic on the triangle $0 \leqq t \leqq s \leqq b$ and fundamental in the sense of Schumitzky [3]. This latter class is important for applications in the area of prediction and control. In § 4 a characterization is given of those kernels for which Γ_α has a simple pole at $\alpha = c$; namely, a necessary and sufficient condition for this is that no eigenfunction $e(t)$ of K_c (corresponding to eigenvalue $\lambda = 1$) vanishes at $t = c$. This result may be of independent interest. A corollary of this result is that a simple pole in Γ_α implies a simple eigenvalue at $\lambda = 1$. An example is given to show that the converse is false. In § 5 we obtain an imbedded solution to (1) at $\alpha = c$.

Sections 6 through 9 concern themselves with continuation past $\alpha = c$. Section 6 outlines the general bifurcation technique. Sections 7 and 8 are devoted to the identification and solution of the "bifurcation equation." And § 9 concludes the paper with a statement of the main result and a formula for the aforementioned continuation.

2. Assumptions. In this section we make assumptions on the kernel k and on its resolvent. We shall adhere to these assumptions throughout the remainder of this paper. Our first assumption is the following:

(A_1) The kernel $k(t, s)$ is symmetric and real analytic in (t, s) on $0 < t < s < b$ and admits continuous extension onto $[0, b] \times [0, b]$.

By virtue of (A_1), the operator K_α defined on $C[0, \alpha]$ by

$$(K_\alpha x)(t) = \int_0^\alpha k(t, s)x(s)\, ds, \qquad 0 \leqq t \leqq \alpha,$$

is continuous from $C[0, \alpha]$ into itself.

Let $\Gamma_\alpha(t, s)$ be the Fredholm resolvent kernel of K_α, i.e., the kernel of the operator $[I - K_\alpha]^{-1} - I$.

Let c be the first critical length of $k(t, s)$, i.e.,

$$c = \sup \{s \mid \text{for every } \alpha \in [0, s], [I - K_\alpha]^{-1} \text{ exists in the algebra of bounded linear operators on } C[0, \alpha]\}.$$

The Fredholm alternative characterizes c as the first point a such that there exists $V \not\equiv 0$ and $V(t) = \int_0^a k(t, s)V(s)\, ds$. McNabb and Schumitzky [1] have given equivalently that c is the first point a such that the mapping $\alpha \to \|\Gamma_\alpha\|$ becomes unbounded for $\alpha \in [0, a]$.

Our second and third assumptions are as follows:

(A_2) $\Gamma_\alpha(t, s)$ is meromorphic in α near c; i.e., there exists a sequence of bounded kernels

$$\{A_n(t, s)\}_{n=-N}^{\infty} \quad \text{such that} \quad \Gamma_\alpha(t, s) = \sum_{n=-N}^{\infty} A_n(t, s)(c - \alpha)^n;$$

(A_3) $\Gamma_\alpha(t, s)$ has a simple pole at $\alpha = c$; i.e., $N = 1$.

Assumption (A_1) is completely straightforward. In the next two sections we elucidate the meaning of the latter two assumptions in terms of properties of k.

3. The meromorphy assumption. We give two classes of kernels which imply Assumption (A_2):

(a) *Kernels analytic on the square* $[0, b] \times [0, b]$. We consider first finite rank kernels of the form

$$k(t, s) = \sum_{i=1}^{n} g_i(t)h_i(s),$$

$\{g_i\}, \{h_i\}$ analytic on $[0, b]$ and respectively linearly independent. Then the singularities of Γ_α in fact are related to the zeros of the determinant

$$\left| \delta_{ij} - \int_0^\alpha g_i(s)h_j(s)\, ds \right| = D(\alpha),$$

and hence the singularities of Γ_α are the zeros of an entire function. Hence Γ_α does in fact have a meromorphic expansion.

Next we consider the general analytic k; the technique is basically in the style of Riesz and Nagy [2].

Suppose $k(t, s)$ is analytic on $[0, b] \times [0, b]$. Then $k(t, s)$ can be uniformly approximated by its truncated power series on $[0, b] \times [0, b]$. Let $k_n(t, s)$ be the nth sum, i.e.,

$$k_n(t, s) = \sum_{i=1}^{n} \sum_{j+k=i} A_{j,k} t^j s^k.$$

Then $k_n(t, s) \to k(t, s)$ uniformly, and it is known (see Wenska [5]) that $K_{n,\alpha} \to K_\alpha$ in the operator topology. Let N be such that $\sup_{\alpha \in [0,b]} \|K_{N,\alpha} - K_\alpha\| < 1/2$. Then $(I - (K_\alpha - K_{N,\alpha}))$ is invertible since $K_\alpha - K_{N,\alpha}$ has norm less than 1, and $(I - (K_\alpha - K_{N,\alpha}))^{-1} = I + (K_\alpha - K_{N,\alpha}) + (K_\alpha - K_{N,\alpha})^2 + \cdots = I + F_\alpha$, where the series is absolutely convergent in the operator norm. We can therefore write

(2) $$(I - K_\alpha) = (I - (K_\alpha - K_{N,\alpha}))(I - (I - (K_\alpha - K_{N,\alpha}))^{-1}K_{N,\alpha});$$

and

$$\begin{aligned}(I - (K_\alpha - K_{N,\alpha}))^{-1}K_{N,\alpha}(t,s) &= (I + F_\alpha)K_{N,\alpha}\\ &= \sum_{i=1}^{N}\sum_{j+k=i} A_{j,k}(I + F_\alpha)t^k s^j\\ &= \sum_{i=1}^{N}\sum_{j+k=i} A_{j,k}g_k(t,\alpha)(t)s^j,\end{aligned}$$

where $g_k(t, \alpha)$ is analytic in α being just the sum of convolutions of an analytic kernel and t^k. Hence $(I + F_\alpha)K_{N,\alpha}$ is of finite rank. It is also clear from (2) that $I - K_\alpha$ is invertible if and only if $(I - (I + F_\alpha)K_{N,\alpha})$ is invertible. Re-indexing the summands of $(I - (I + F_\alpha)K_{N,\alpha})$ we see that the invertibility of $(I - (I + F_\alpha)K_{N,\alpha})$ is related to the zeros of a determinant

$$\left|\delta_{ij} - \int_0^\alpha g_i(s)h_j(s)\,ds\right| = D_N(\alpha)$$

of order at most N^2, and each term of which is analytic in α. Letting $I + \Gamma_{N,\alpha}$ be the inverse of $(I - (I + F_\alpha)K_{N,\alpha})$ we have $I + \Gamma_\alpha = (I + \Gamma_{N,\alpha})(I + F_\alpha)$ and the singularities of Γ_α are in fact the zeros of $D_N(\alpha)$, and Γ_α has a meromorphic expansion.

(b) *Fundamental kernels.* Let $A(t)$, $B(t)$, $C(t)$ be $n \times m$, $n \times n$, $m \times m$ matrices respectively, whose entries are real analytic functions on the interval $[0, b]$; let F be an $n \times m$ constant matrix. Following Schumitzky [3], we say that the continuous $n \times m$ matrix kernel $Q(t, s)$ is fundamental relative to (A, B, C, F) if $Q(t, s)$ is continuously differentiable on each of the sets $0 \leqq t < s < b$, $0 \leqq s < t < b$ and $0 \leqq t = s < b$, and satisfies the system of equations

$$\frac{\partial}{\partial t}Q(t,s) = B(t)Q(t,s), \qquad t < s,$$

$$\frac{\partial}{\partial s}Q(t,s) = Q(t,s)C(s), \qquad s < t,$$

$$\frac{d}{dt}Q(t,t) = A(t) + B(t)Q(t,t) + Q(t,t)C(t), \qquad 0 < t,$$

$$Q(0,0) = F.$$

Now let $H(t)$ and $G(t)$ be $1 \times n$ and $m \times 1$ real analytic matrices respectively on $[0, b]$. Then the kernel $k(t, s) = H(t)Q(t, s)G(s)$ is scalar-valued and satisfies Assumption (A_1). Further, it is proved by McNabb and Schumitzky [4] that Γ_α satisfies the meromorphy assumption (A_2).

A typical example of kernels of this type is given by

$$k(t,s) = \sum_{i=1}^{n} h_i(t)g_i(s)\exp(-\gamma_i|t - s|),$$

where $H(t) = (h_1(t), h_2(t), \cdots, h_n(t))$, $G(t) =$ transpose of $(g_1(t), g_2(t), \cdots, g_n(t))$ and $Q(t, s) = \operatorname{diag}(\exp(-\gamma_1|t - s|), \exp(-\gamma_2|t - s|), \cdots, \exp(-\gamma_n|t - s|))$. Here $Q(t, s)$ is fundamental relative to (A, B, C, F), where $-\frac{1}{2}A = B = C = \operatorname{diag}(-\gamma_1, -\gamma_2, \cdots, -\gamma_n)$. (For further applications of kernels of this type see Schumitzky [3], and Kailath and Anderson [11].)

4. The simple pole assumption (A_3). The following result characterizes those kernels whose meromorphic resolvents have simple poles.

THEOREM. *Γ_α has a simple pole at $\alpha = c$ if and only if no eigenfunction of K_c (corresponding to $\lambda = 1$) vanishes at $t = c$.*

Proof. Let Γ_α have a pole of order $N \geqq 1$ at $\alpha = c$. Thus

$$\Gamma_\alpha(t, s) = \frac{A_{-N}(t, s)}{(c - \alpha)^N} + O\left(\left(\frac{1}{c - \alpha}\right)^{N-1}\right). \tag{3}$$

Then the Bellman–Krein formula (see [3]) gives

$$\begin{aligned} \frac{\partial}{\partial \alpha}\Gamma_\alpha(t, s) &= \frac{NA_{-N}(t, s)}{(c - \alpha)^{N+1}} + O\left(\left(\frac{1}{c - \alpha}\right)^{N}\right) \\ &= \Gamma_\alpha(t, \alpha)\Gamma_\alpha(s, \alpha) \\ &= \frac{A_{-N}(t, \alpha)A_{-N}(s, \alpha)}{(c - \alpha)^{2N}} + O\left(\left(\frac{1}{c - \alpha}\right)^{2N-1}\right). \end{aligned}$$

Multiplying through by $(c - \alpha)^{2N}$, we have

$$NA_{-N}(t, s)(c - \alpha)^{N-1} = A_{-N}(t, \alpha)A_{-N}(s, \alpha) + O(c - \alpha). \tag{4}$$

If $N = 1$, then

$$A_{-1}(t, s) = A_{-1}(t, c)A_{-1}(s, c). \tag{5}$$

Setting $s = c$, then either $A_{-1}(t, c) \equiv 0$ or $A_{-1}(c, c) = 1$. However, $A_{-1}(t, c) \equiv 0$ for every t implies $A_{-1}(t, s) \equiv 0$ from (5), which contradicts the order of the pole. Thus $A_{-1}(c, c) = 1$. If $N > 1$, then (4) implies $A_{-N}(t, c) \equiv 0$, $t \in [0, c]$.

At this point we claim that $A_{-N}(\cdot, s_0)$ is an eigenfunction of K_c for some s_0. The following is one of the resolvent equations for $\Gamma_\alpha(t, s)$:

$$\Gamma_\alpha(t, s) = k(t, s) + \int_0^\alpha k(t, \theta)\Gamma_\alpha(\theta, s)\, d\theta, \qquad 0 \leqq t, \quad s \leqq \alpha.$$

Let $A_{-N}(t, s) = A(t, s)$. Then

$$\frac{A(t, s)}{(c - \alpha)^N} + B(t, s, \alpha) = k(t, s) + \int_0^\alpha k(t, \theta)\frac{A(\theta, s)}{(c - \alpha)^N} + B(\theta, s, \alpha)\, d\theta,$$

where $B(t, s, \alpha) = O(1/(c - \alpha)^{N-1})$. Multiplying through by $(c - \alpha)^N$, we have

$$A(t, s) = \int_0^\alpha k(t, \theta)A(\theta, s)\, d\theta + O(c - \alpha)$$

or $A(t, s) = \int_0^c k(t, \theta)A(\theta, s)\, d\theta$; since $A(t, s) \not\equiv 0$ by hypothesis, our claim is verified. In particular for $N = 1$, $A(t, s_0) = A(t, c)A(s_0, c)$ and $A(t, c)$ is an eigenfunction.

Hence, if $N > 1$, then $A_{-N}(t, s_0)$ is an eigenfunction and $A_{-N}(c, s) = 0$, i.e., there exists an eigenfunction which vanishes. If $N = 1$, we claim that $A_{-1}(t, c)$ is the only eigenfunction (up to scalar multiples) and $A_{-1}(c, c) = 1$. Let $A_{-1}(t, c) = A(t)$. K_c is compact by the Arzela–Ascoli theorem, and $I - K_c$ has a finite-dimensional null space U. If $A(t)$ does not span U, then there exists a $g \in U, g \not\equiv 0$ and $\int_0^c g(t)A(t)\,dt = 0$, i.e., $g \perp A$.

Since K_c is self-adjoint, g is also in the null space of $I - K_c^*$. Hence the Fredholm alternative implies that there is no solution to the equation

$$x(t) = g(t) + \int_0^c k(t, s)x(s)\,ds. \tag{6}$$

However, consider

$$\begin{aligned} x(t, \alpha) &= g(t) + \int_0^\alpha \Gamma_\alpha(t, \theta)g(\theta)\,d\theta \\ &= g(t) + \frac{A(t)}{c - \alpha}\int_0^\alpha g(\theta)A(\theta)\,d\theta + \int_0^\alpha B(t, \theta, \alpha)g(\theta)\,d\theta. \end{aligned} \tag{7}$$

The integral $\int_0^\alpha B(t, \theta, \alpha)g(\theta)\,d\theta$ is analytic in α about c since $B(t, s, \alpha)$ is also, and hence is bounded. Likewise,

$$\lim_{\alpha \to c} \frac{1}{c - \alpha} A(t) \int_0^\alpha A(\theta)g(\theta)\,d\theta = -A(t)A(c)g(c).$$

Hence, $(A(t)/(c - \alpha))\int_0^\alpha A(\theta)g(\theta)\,d\theta$ is bounded. Thus the right-hand side of (7) is bounded, and Wenska [5] showed that a solution to (6) exists, and this contradiction verifies our claim.

So if $N > 1$, an eigenfunction vanishes at $t = c$, and if $N = 1$, then there is only one eigenfunction and its value at $t = c$ is not equal to zero.

COROLLARY. *If* Γ_α *has a simple pole at* $\alpha = c$, *then*

(a) 1 *is a simple eigenvalue,*

(b) $A_{-1}(t, s) = e(t)e(s)$, *where* $e(t)$ *is the eigenfunction corresponding to eigenvalue* 1, *and* $e(c) = 1$.

Remark 1. With regard to the corollary, we note that a simple eigenvalue at $\lambda = 1$ does not imply that Γ_α has a simple pole at $\alpha = c$. Consider the following example: $k(t, s) = 3(1 - t)(1 - s)$, where $c = 1$, $e(t) = (1 - t)$ and $\Gamma_\alpha(t, s) = 3(1 - t)(1 - s)/(1 - \alpha)^3$.

Remark 2. If $k(t, s) = H(t)Q(t, s)G(s)$, where Q is fundamental (§ 2), and G and H have constant coefficients, then it can be shown, using the Ricatti formalism of Schumitzky [3], that Γ_α has a simple pole at $\alpha = c$. Kernels in this class include those of the form $k(t, s) = \sum_{n=1}^k a_n e^{-\gamma_n |t-s|}$. Such examples are important in the theory of filtering and control [11].

5. The solution $x(t, \alpha)$ **at a critical length.** In [5], Wenska showed that if $x(t, \alpha)$ is a function jointly continuous on $[0, b] \times [0, c)$, uniformly bounded and

$$x(t, \alpha) = f(t) + \int_0^\alpha k(t, s)x(s, \alpha)\,ds, \qquad \alpha \in [0, c),$$

then a solution $x_0(t)$ to the equation

$$x(t) = f(t) + \int_0^c k(t, s)x(s)\, ds$$

exists and is the uniform limit of some subsequence $x(t, a_n)$, where $a_n \to c$.

THEOREM. *If k satisfies* (A_1), (A_2) *and* (A_3), *and if* $\int_0^c f(t)A(t)\, dt = 0$, *then there exists a unique function* $x(t, \alpha)$, *jointly continuous on* $[0, c] \times [0, c]$, *which satisfies*

$$x(t) = f(t) + \int_0^\alpha k(t, s)x(s)\, ds \quad \textit{for} \quad 0 \leqq t, \alpha \leqq c.$$

Note. Since $I - K_c$ is self-adjoint on $C[0, c]$ imbedded as a subspace of $L_2[0, c]$, $A(\cdot)$ generates the null space of $(I - K_c)^*$, and the condition $\int_0^c A(s)f(s)\, ds = 0$ is the Fredholm condition of being orthogonal to the adjoint null space.

Proof. Let U be the integral operator defined on $C[0, c]$ by

$$Ux(t) = \int_0^c \frac{A(t)A(s)x(s)\, ds}{\int_0^c A^2(s)\, ds}.$$

Since c is the first critical point of k, $x(t, \alpha)$ is given uniquely on $[0, c] \times [0, c)$ by $x(t, \alpha) = [(I + \Gamma_\alpha)f](t)$. By hypothesis, $Uf \equiv 0$; hence, $x(t, \alpha) = [(I + \Gamma_\alpha)(I - U)f]$ and by the remark prefacing the statement of the theorem, we need only show that $\Gamma_\alpha(I - U)f$ remains bounded in sup norm about c.

$$[\Gamma_\alpha(I - U)f](t) = \frac{A(t)\int_0^\alpha A(s)f(s)\, ds}{(c - \alpha)\int_0^c A^2(s)\, ds} + \int_0^\alpha B(t, s, \alpha)f(s)\, ds$$

and

$$\lim_{\alpha \to c} [\Gamma_\alpha(I - U)f](t) = \frac{-A(t)A(c)f(c)}{\int_0^c A^2(s)\, ds} + \int_0^c B(t, s, c)f(s)\, ds$$

and on $\{f | Uf \equiv 0\}$, $\|\Gamma_\alpha(I - U)f\| \leqq \|f\|\{\|A\|^2 + \|B\|c\}$, where $\|\Gamma_\alpha(I - U)f\|$, $\|f\|$ and $\|A\|$ are the sup norms on $[0, c]$, whereas the $\|B\|$ represents the sup norm on $[0, c] \times [0, c] \times [0, c]$. Hence, $x(t, \alpha)$ is uniformly bounded on $[0, c] \times [0, c]$ and

$$\begin{aligned} x(t, c) &= \lim_{\alpha \to c} f(t) + \int_0^\alpha \Gamma_\alpha(t, s)f(s)\, ds \\ &= f(t) - \frac{A(c)f(c)A(t)}{\int_0^c A^2(s)\, ds} + \int_0^c B(t, s, c)f(s)\, ds. \end{aligned}$$

The following should be noted: the simplicity of the pole is crucial at this point. In the example discussed in § 4, $\Gamma_\alpha(t, s) = 3(1 - t)(1 - s)/(1 - \alpha)^3$, and $e(t) = (1 - t)$. But given any f orthogonal to $e(t)$ such that $f(1) \neq 0$, then $\lim_{\alpha \to c} \|\Gamma_\alpha f\| = \infty$.

6. Continuation—bifurcations à la Bartle. Now that we have a solution to the "critical equation," we examine the problem of getting past the critical point by means of the bifurcation mechanism of Bartle [6].

The set-up for the general bifurcation problem is the following: Let B_1 and B_2 be Banach spaces and Φ a nonlinear operator from $B_1 \times B_2$ into B_1. Suppose that $\Phi(x, y) = Lx + F(x, y)$, where

(i) L is a bounded linear operator from B_1 into itself;
(ii) the dimensions of the null space of L and its adjoint are equal and finite;
(iii) $F(x, y)$ is defined and continuous for $\|x\|$ and $\|y\|$ small, $F(0, 0) = 0$;
(iv) $\|F(x_1, y) - F(x_2, y)\| \leqq M(x_1, x_2, y)\|x_1 - x_2\|$, where M is a nonnegative real-valued function which goes to zero with its arguments.

The object of bifurcation theory is to find solutions of

$$Lx + F(x, y) = 0 \tag{8}$$

in a neighborhood of (0, 0).

The following facts are easily verified algebraically. $L: B_1 \to B_1$ implies that B_1 is subject to two direct sum decompositions:

$$B_1 = D \oplus N(L), \qquad B_1 = R(L) \oplus C,$$

where $N(L)$ is the null space of L, $R(L)$ the range of L, D the "essential" domain and C the complementary space of $R(L)$. These decompositions induce a decomposition on the dual structure:

$$B_1^* = R^*(L) \oplus C^*, \qquad B_1^* = D^* \oplus N^*(L).$$

Moreover, $L^*: B_1^* \to B_1^*$, and $N(L^*) = C^*$. Hence C is the dual of C^* and is finite-dimensional and admits a continuous projection Z. Let N be a projection on $N(L)$.

Since $N(L)$ and $N(L^*)$ have the same dimensions, there exists an onto injection $U: N(L) \to C$ and $L + U: B_1 \to B_1$, 1–1 onto. Hence a continuous inverse exists $(L + U)^{-1} = L_U^{-1}$, and

$$L_U^{-1}L = I - N.$$

Equation (8) is then equivalent to

$$x + L_U^{-1}F(x, y) = u \quad \text{for some} \quad u \in N(L). \tag{9}$$

If we fix u in (9), the conditions on F allow us to choose y sufficiently small so that $L_U^{-1}F(x, y)$ is contractive, and for fixed u, (9) may be solved by iteration. Let $V_y(u)$ be the unique solution to (9).

THEOREM (Bartle [6]). *A solution $V_y(u)$ to* (9) *is a solution of* (8) *if and only if*

$$ZF(V_y(u), y) = 0. \tag{10}$$

Equation (10) is called the bifurcation equation. Also, in the self-adjoint case, it should be noted that $N(L) = N(L^*) = C$, and the maps U, N and Z are identical.

7. Specific application to integral equations. In this section we make the identifications with Bartle's theorem. We first extend the various mappings and functions to a larger domain. Define:

(i) $K_\alpha : C[0, b] \to C[0, b]$ by

$$(K_\alpha x)(t) = \int_0^\alpha k(t, s)x(s)\, ds, \qquad 0 \leqq t \leqq b.$$

(ii) $\Gamma_\alpha(t, s)$ by the solutions of the following:

$$\Gamma_\alpha(t, s) = k(t, s) + \int_0^\alpha k(t, \theta)\Gamma_\alpha(\theta, s)\, d\theta, \qquad 0 \leqq t \leqq b, \quad 0 \leqq s \leqq \alpha,$$

$$\Gamma_\alpha(t, s) = k(t, s) + \int_0^\alpha \Gamma_\alpha(t, \theta)k(\theta, s)\, d\theta, \qquad 0 \leqq t, \quad s \leqq b.$$

(iii)
$$A(t) = \begin{cases} A(t) & \text{for } 0 \leqq t \leqq c, \\ \int_0^c k(t, s)A(s)\, ds & \text{for } c \leqq t \leqq b. \end{cases}$$

(iv)
$$x(t, \alpha) = f(t) + \int_0^\alpha k(t, s)x(s, \alpha)\, ds \quad \text{for} \quad \alpha \leqq t \leqq b.$$

A closer look at the Bartle analysis indicates that the Banach space structure of B_1 is needed to guarantee the existence of the solution of (9) obtained by successive iteration. In our application we shall use the pre-Hilbert space $C[0, b]$ and a solution will exist since convergence will occur in the uniform sense as well as the L^2 sense.

To get to the bifurcation problem, we wish to get $x(t, c + \alpha)$ as $x(t, c) + \bar{x}(t, \alpha)$. Then

$$x(t, c) + \bar{x}(t, \alpha) = f(t) + \int_0^{c+\alpha} k(t, s)\{x(s, c) + \bar{x}(s, \alpha)\}\, ds$$

and

$$\bar{x}(t, \alpha) = \int_0^c k(t, s)\bar{x}(s, \alpha)\, ds + \int_c^{c+\alpha} k(t, s)\{x(s, c) + \bar{x}(s, \alpha)\}\, ds$$

or

$$\bar{x}(t, \alpha) - \int_0^c k(t, s)\bar{x}(s, \alpha)\, ds - \int_c^{c+\alpha} k(t, s)\{x(s, c) + \bar{x}(s, \alpha)\}\, ds = 0.$$

The identifications with Bartle's analysis are the following:

(i)
$$L = I - K_c,$$

(ii)
$$F(x, \alpha) = -\int_c^{c+\alpha} k(\cdot, s)\{x(s, c) + x(s)\}\, ds.$$

The Lipschitz condition of (iv) is effected via the mean continuity of k implied by the continuity of $k(t, s)$ (see Wenska [5]), and the $\mathscr{L}_2$ properties of k (see Zaanen [7]).

Since K_c is compact, the dimension of its eigenspace is finite and equal to the dimension of the adjoint null space. Moreover, every eigenfunction of K_c in

$C[0, c] \subset L^2[0, c]$ can be extended to an eigenfunction of K_c as an operator on $C[0, b] \subset L^2[0, b]$ by the extension

$$e(t) = \begin{cases} e(t) & \text{for} \quad 0 \leqq t \leqq c, \\ \int_0^c k(t, s)e(s)\, ds & \text{for} \quad c \leqq t \leqq b. \end{cases}$$

Likewise every eigenfunction of K_c in $C[0, b] \subset L^2[0, b]$ satisfies the extension equation. The following lemma shows that all the eigenfunctions of K_c are in $C[0, b]$.

LEMMA. *If* $x \in L_2[0, b]$ *and* $x = K_c x$, *where* $k(t, s)$ *is continuous on* $[0, b] \times [0, b]$, *then* $x \in C[0, b]$.

Proof. Since k is continuous on $[0, b] \times [0, b]$, it is uniformly continuous. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|h| < \delta$ implies that $|k(t + h, s) - k(t, s)| < \varepsilon/c\|x\|_2$; then for $|h| < \delta$ we have

$$\begin{aligned} |x(t + h) - x(t)| &= \left| \int_0^c \{k(t + h, s) - k(t, s)\}x(s)\, ds \right| \\ &\leqq \int_0^c |k(t + h, s) - k(t, s)|\, |x(s)|\, ds \\ &\leqq \left(\int_0^c |k(t + h, s) - k(t, s)|^2\, ds \right)^{1/2} \|x\|_2 \\ &< \left(\frac{\varepsilon}{c\|x\|_2} \right) c\|x\|_2 = \varepsilon. \end{aligned}$$

Hence, every eigenfunction of K_c is in $C[0, b]$ and as a result there is only one eigenfunction, the extension of $A(t)$, henceforth denoted as $\hat{A}(t)$.

To see the adjoint null space structure, we know that $A(s)$ generates the adjoint null space of K_c on $C[0, c] \subset L^2[0, c]$. Thus, for every $x \in L^2[0, c]$,

$$0 = ((I - K_c)x, A) = \int_0^c \left\{ x(t) - \int_0^c k(t, s)x(s)\, ds \right\} A(t)\, dt.$$

Hence, consider

$$A^*(s) = \begin{cases} A(s) & \text{for} \quad 0 \leqq s \leqq c, \\ 0 & \text{for} \quad c < s \leqq b. \end{cases}$$

Then for $x \in L^2[0, b]$,

$$\begin{aligned} \int_0^b A^*(t) \left\{ x(t) - \int_0^c k(t, s)x(s)\, ds \right\} &= \int_0^c A(t) \left\{ x(t) - \int_0^c k(t, s)x(s)\, ds \right\} dt \\ &= 0 \end{aligned}$$

and $A^* \in N(I - K_c^*) \subset L_2[0, b]$.

The simplicity of the eigenfunction as well as the equality of the dimensions of the null spaces imply that up to scalar multiples, A^* is the only element in

$N(I - K_c^*)$, and that the operator U in the Bartle machinery is an integral operator with kernel

$$U(t, s) = \frac{\hat{A}(t)A^*(s)}{\int_0^c A^2(t)\,dt}.$$

$I - K + U$ is now invertible. Let $I + \Gamma_U$ be its inverse. Let $\Gamma_U(t, s)$ be the resolvent kernel of $K - U$ on $C[0, b] \subset L^2[0, b]$. Consider the resolvent kernel equations of $\Gamma_U(t, s)$:

$$\Gamma_U(t, s) = k(t, s) - U(t, s) + \int_0^c k(t, \theta)\Gamma_U(\theta, s)\,d\theta - \frac{\hat{A}(t)}{\int_0^c A^2(t)\,dt}\int_0^c A^*(\theta)\Gamma_U(\theta, s)\,d\theta, \qquad 0 \leqq t, \quad s \leqq b$$

$$= k(t, s) - U(t, s) + \int_0^c \Gamma_U(t, \theta)k(\theta, s)\,d\theta - \frac{\int_0^c \Gamma_U(t, \theta)\hat{A}(\theta)\,d\theta A^*(s)}{\int_0^c A^2(t)\,dt}.$$

Hence, it is obvious that $\Gamma_U(t, s)$ has the property that on $[0, c]^2$, $\Gamma_U(t, s)$ satisfies the resolvent kernel relations for the resolvent of $K_c - U$ on $C[0, c] \subset L^2[0, c]$, and by uniqueness they are one and the same. Since $K_c - U$ is symmetric on $C[0, c]$, it is known that $\Gamma_U(t, s) = \Gamma_U(s, t)$ for $0 \leqq s, t \leqq c$.

The equivalent bifurcation problem appears as

$$v(t, \alpha) = \left((I + \Gamma_U)\int_c^{c+\alpha} k(\cdot, s)\{x(s, c) + v(s, \alpha)\}\,ds\right)(t) + \xi\hat{A}(t). \tag{11}$$

Writing out the operator convolutions, we have

$$v(t, \alpha) = \xi\hat{A}(t) + \int_c^{c+\alpha} k(t, s)x(s, c)\,ds + \int_0^c\int_c^{c+\alpha} \Gamma_U(t, \theta)k(\theta, s)x(s, c)\,ds\,d\theta$$
$$+ \int_c^{c+\alpha} k(t, s)v(s, \alpha)\,ds + \int_0^c\int_c^{c+\alpha} \Gamma_U(t, \theta)k(\theta, s)v(s, \alpha)\,ds\,d\theta.$$

Hence,

$$v(t, \alpha) = \xi\hat{A}(t) + \int_c^{c+\alpha}\left\{k(t, s) + \int_0^c \Gamma_U(t, \theta)k(\theta, s)\,d\theta\right\}\{x(s, c) + v(s, \alpha)\}\,ds.$$

The resolvent equations for Γ_U imply that

$$K_c + \Gamma_U K_c = \Gamma_U + U + \Gamma_U U.$$

The following lemma gives further simplification.

LEMMA. $\Gamma_U U = 0$.

Proof. $U = (I + \Gamma_U)(I - K_c + U)U = (I + \Gamma_U)U$, and we are done. Hence,

$$k(t, s) + \int_0^c \Gamma_U(t, \theta)k(\theta, s)\,d\theta = \Gamma_U(t, s) + \frac{\hat{A}(t)A^*(s)}{\int_0^c A^2(t)\,dt},$$

and

$$v(t,\alpha) = \xi\hat{A}(t) + \int_c^{c+\alpha} \left\{\Gamma_U(t,s) + \frac{\hat{A}(t)A^*(s)}{\int_0^c A^2(t)\,dt}\right\}\{x(s,c) + v(s,\alpha)\}\,ds.$$

But $A^* \equiv 0$ on $(c, c+\alpha]$, so

$$v(t,\alpha) = \xi\hat{A}(t) + \int_c^{c+\alpha} \Gamma_U(t,s)x(s,c)\,ds + \int_c^{c+\alpha} \Gamma_U(t,s)v(s,\alpha)\,ds.$$

Considering this last equation as a Fredholm equation with kernel $\Gamma_U(t,s)$ and forcing term

$$\xi\hat{A}(t) + \int_c^{c+\alpha} \Gamma_U(t,s)x(s,c)\,ds,$$

for α small we can use the Neumann expansion for $\Delta_\alpha(t,s)$, the resolvent kernel of $\Gamma_U(t,s)$ on $[c, c+\alpha]$. Therefore,

$$\begin{aligned} v(t,\alpha) = v_\xi(t,\alpha) = \xi\hat{A}(t) &+ \int_c^{c+\alpha} \Gamma_U(t,s)x(s,c)\,ds + \xi\int_c^{c+\alpha} \Delta_\alpha(t,s)\hat{A}(s)\,ds \\ &+ \int_c^{c+\alpha}\int_c^{c+\alpha} \Delta_\alpha(t,\theta)\Gamma_U(\theta,s)x(s,c)\,ds\,d\theta \end{aligned}$$

and

$$\begin{aligned} F(v_\xi(t,\alpha),\alpha) &= \int_c^{c+\alpha} k(t,s)x(s,c)\,ds + \xi\int_c^{c+\alpha} k(t,s)\hat{A}(s)\,ds \\ &\quad + \int_c^{c+\alpha}\int_c^{c+\alpha} k(t,\theta)\Gamma_U(\theta,s)x(s,c)\,ds\,d\theta \\ &\quad + \xi\int_c^{c+\alpha}\int_c^{c+\alpha} k(t,\theta)\Delta_\alpha(\theta,s)\hat{A}(s)\,ds\,d\theta \\ &\quad + \int_c^{c+\alpha}\int_c^{c+\alpha}\int_c^{c+\alpha} k(t,\theta_1)\Delta_\alpha(\theta_1,\theta_2)\Gamma_U(\theta_2,s)x(s,c)\,ds\,d\theta_2\,d\theta_1 \\ &= I_1 + \xi I_2 + I_3 + \xi I_4 + I_5. \end{aligned}$$

The Z projection in the Bartle machinery is generated by the dual basis of the adjoint null space. Hence,

$$Zx(t) = \frac{A^*(t)}{\int_0^c A^2(t)\,dt}\int_0^b A^*(s)\,x(s)\,ds$$

and

$$ZF(v_\xi(\cdot,\alpha),\alpha)(t) = \frac{A^*(t)}{\int_0^c A^2(t)\,dt}\left\{\int_0^c A(s)\left(\sum_1^5 c_i I_i(s)\right)ds\right\},$$

where

$$c_i = \begin{cases} 1 & \text{for } i = 1,3,5, \\ \xi & \text{for } i = 2,4. \end{cases}$$

Hence the bifurcation problem is reduced to finding a ξ such that

$$\int_0^c A(s)\left(\sum c_i I_i(s)\right) ds = 0.$$

8. Solution of the bifurcation equation—computation of ξ. In this section we solve specifically the bifurcation equations in a neighborhood of c. Some simplifications are in order.

$$\text{(i)} \int_0^c A(\theta) I_1(\theta)\, d\theta = \int_0^c \int_c^{c+\alpha} A(\theta) k(\theta, s) x(s, c)\, ds\, d\theta$$
$$= \int_c^{c+\alpha} \left\{ \int_0^c k(\theta, s) \hat{A}(\theta)\, d\theta \right\} x(s, c)\, ds$$
$$= \int_c^{c+\alpha} A(s) x(s, c)\, ds.$$

Similarly,

$$\text{(ii)} \int_0^c A(\theta) I_2(\theta)\, d\theta = \int_c^{c+\alpha} \hat{A}^2(s)\, ds,$$

$$\text{(iii)} \int_0^c A(\theta) I_3(\theta)\, d\theta = \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta) \Gamma_U(\theta, s) x(s, c)\, ds\, d\theta,$$

$$\text{(iv)} \int_0^c A(\theta) I_4(\theta)\, d\theta = \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta) \Delta_\alpha(\theta, s) \hat{A}(s)\, ds\, d\theta,$$

$$\text{(v)} \int_0^c A(\theta) I_5(\theta)\, d\theta = \int_c^{c+\alpha} \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta_1) \Delta_\alpha(\theta_1, \theta_2) \Gamma_U(\theta_2, s) x(s, c)\, ds\, d\theta_2\, d\theta_1.$$

Noting also the resolvent relations for Δ_α, we have

$$\int_c^{c+\alpha} \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta_1) \Delta_\alpha(\theta_1, \theta_2) \Gamma_U(\theta_2, s) x(s, c)\, ds\, d\theta_2\, d\theta_1$$
$$= \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta_1) \{\Delta_\alpha(\theta_1, s) - \Gamma_U(\theta_1, s)\} x(s, c)\, ds\, d\theta_1.$$

Hence the term involving I_3 cancels out, and the bifurcation equation is the following:

$$\int_c^{c+\alpha} \hat{A}(s) x(s, c)\, ds + \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta) \Delta_\alpha(\theta, s) x(s, c)\, ds\, d\theta$$
$$+ \xi \left(\int_c^{c+\alpha} \hat{A}^2(s)\, ds + \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta) \Delta_\alpha(\theta, s) \hat{A}(s)\, ds\, d\theta \right) = 0.$$

Consider now the coefficient of ξ. Call it $M_{c+\alpha}$.

$$\left| \int_c^{c+\alpha} \hat{A}^2(s)\,ds + \int_c^{c+\alpha} \int_c^{c+\alpha} \hat{A}(\theta)\Delta_\alpha(\theta, s)\hat{A}(s)\,ds\,d\theta \right|$$

$$\geqq \left(1 - \int_c^{c+\alpha} \int_c^{c+\alpha} |\Delta_\alpha(t, s)|^2\,ds\,dt\right) \int_c^{c+\alpha} \hat{A}^2(s)\,ds$$

by two applications of Schwarz' inequality. Consider $\Gamma_U(t, s) \in \mathscr{L}^2[c, b]$ (see Zaanen [7]). It is clear that $\Delta_\alpha(t, s) \in \mathscr{L}^2[c, c + \alpha]$. Imbed $\Delta_\alpha(t, s) \in \mathscr{L}^2[c, b]$ by defining

$$\Delta_\alpha(t, s) = \begin{cases} \Delta_\alpha(t, s) & \text{if} \quad c \leqq t, \quad s \leqq c + \alpha, \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to show that $\alpha \to \Delta_\alpha(t, s)$ is a continuous mapping into $\mathscr{L}^2[c, b]$, and $\Delta_0(t, s) = 0$ a.e. by its resolvent equation. Hence,

$$\lim_{\alpha \to 0} \int_c^b \int_c^b |\Delta_\alpha(t, s) - 0|^2\,ds\,dt = 0$$

and

$$\int_c^b \int_c^b |\Delta_\alpha(t, s)|^2\,dt\,ds \geqq \frac{1}{(b - c)^2}\left(\int_c^b \int_c^b |\Delta_\alpha(t, s)|\,ds\,dt\right)^2$$
$$= \frac{1}{(b - c)^2}\left(\int_c^{c+\alpha} \int_c^{c+\alpha} |\Delta_\alpha(t, s)|\,ds\,dt\right)^2.$$

Hence there is an entire neighborhood of 0 such that

$$\int_c^{c+\alpha} \int_c^{c+\alpha} |\Delta_\alpha(t, s)|^2\,ds\,dt < 1.$$

The analyticity of k on $[0, b]^2$ implies the analyticity of $\hat{A}$. Hence for all $\alpha > 0$,

$$\int_c^{c+\alpha} \hat{A}^2(s)\,ds > 0.$$

Thus for α sufficiently small, $M_{c+\alpha} > 0$ and solving for ξ, we have

$$\xi(\alpha) = \frac{-\{\int_c^{c+\alpha} \hat{A}(s)x(s, c)\,ds + \int_c^{c+\alpha}\int_c^{c+\alpha} \hat{A}(\theta)\Delta_\alpha(\theta, s)x(s, c)\,ds\,d\theta\}}{(\int_c^{c+\alpha} \hat{A}^2(s)\,ds + \int_c^{c+\alpha}\int_c^{c+\alpha} \hat{A}(\theta)\Delta_\alpha(\theta, s)\hat{A}(s)\,ds\,d\theta)}.$$

The following proposition discusses some properties of $\xi(\alpha)$.

PROPOSITION. *$\xi(\alpha)$ is differentiable in α and* $\lim_{\alpha \to 0} \xi(\alpha) = 0$.

We need the following lemma in the proof of this proposition.

LEMMA. *If $\int_0^c A(t)f(t)\,dt = 0$, then $x(c, c) = 0$.*

Proof of lemma. If $f'(c)$ exists, then

$$\frac{x(t,\alpha)-x(t,c)}{\alpha-c}=\frac{\hat{A}(t)}{\alpha-c}\left\{\frac{\int_0^\alpha A(s)f(s)\,ds}{(c-\alpha)\int_0^c A^2(s)\,ds}+\frac{f(c)A(c)}{\int_0^c A^2(s)\,ds}\right\}$$
$$+\frac{1}{\alpha-c}\int_0^\alpha \{B(t,s,\alpha)-B(t,s,c)\}f(s)\,ds$$
$$-\frac{1}{\alpha-c}\int_\alpha^c B(t,s,\alpha)f(s)\,ds.$$

Our initial assumption about the meromorphic structure of Γ_α implies the following:

$$\lim_{\alpha\to 0^-}\frac{x(t,\alpha)-x(t,c)}{\alpha-c}=\frac{\hat{A}(t)}{\int_0^c A^2(s)\,ds}(Af)'(c)+B(t,c,c)f(c)-\int_0^c A_1(t,s)f(s)\,ds$$
$$=\frac{d}{d\alpha^-}x(t,c)<\infty$$

for $t\in[0,b]$;

$$\frac{d}{d\alpha}x(t,\alpha)=\frac{d}{d\alpha}\left\{\frac{\hat{A}(t)}{\int_0^c A^2(s)\,ds}\,\frac{\int_0^\alpha A(s)f(s)\,ds}{(c-\alpha)}+\int_0^\alpha B(t,s,\alpha)f(s)\,ds\right\}$$
$$=\frac{\hat{A}(t)}{\int_0^c A^2(s)\,ds}\left\{\frac{A(\alpha)f(\alpha)}{c-\alpha}+\frac{\int_0^\alpha A(s)f(s)\,ds}{(c-\alpha)^2}\right\}$$
$$+B(t,\alpha,\alpha)f(\alpha)-\sum_1^\infty n\int_0^\alpha A_n(t,s)(c-\alpha)^{n-1}f(s)\,ds$$

and

$$\lim_{\alpha\to c^-}\frac{d}{d\alpha}x(t,\alpha)=\frac{d}{d\alpha^-}x(t,c).$$

But it is known (see, for instance, Kagiwada, Kalaba and Schumitzky [9]) that

$$\frac{d}{d\alpha}x(t,\alpha)=x(\alpha,\alpha)\Gamma_\alpha(t,\alpha),$$

and it is also known (see McNabb and Schumitzky [1]) that Γ_α is unbounded in norm as $\alpha\to c$. Since

$$\lim_{\alpha\to c^-}\frac{d}{d\alpha}x(t,\alpha)<\infty,$$

it must be that $\lim_{\alpha\to c}x(\alpha,\alpha)=0$. For general f, the result follows by choosing a sequence of $\{f_n\}\subset C'[0,b]$ converging to f (for instance, an ε-mollification of f) and applying collectively-compact arguments (see Wenska [5]).

Proof of proposition. For the differentiability of $\xi(\alpha)$, just compute formally the derivatives by quotient rule, and notice that the only, perhaps, unjustified differentiation is the one involving Δ_α. But the Bellman–Krein formula allows us to so justify it.

To show that $\lim_{\alpha \to 0^+} \xi(\alpha) = 0$, consider

$$\lim_{\alpha \to 0^+} -\frac{(1/\alpha)\{\int_c^{c+\alpha} \hat{A}(s)x(s, c)\, ds + \int_c^{c+\alpha}\int_c^{c+\alpha} A(\theta)\Delta_\alpha(\theta, s)\hat{A}(s)\, ds\, d\theta\}}{(1/\alpha)\{\int_c^{c+\alpha} \hat{A}^2(s)\, ds + \int_c^{c+\alpha} \hat{A}(\theta)\Delta_\alpha(\theta, s)\hat{A}(s)\, ds\, d\theta\}} = \frac{-x(c, c)}{\hat{A}(c)},$$

since $\Delta_\alpha(\theta, s) \to 0$ a.e. The corollary of § 3 and the last lemma give the conclusion.

9. The continuation theorem. Sections 7, 8 and 9 implement the requirements of Bartle's theorem, and prove the following.

THEOREM. *If k satisfies* (A_1), (A_2) *and* (A_3), *then given f orthogonal to the null space of $I - K_c$, an imbedding $x(t, \alpha)$ exists on a neighborhood Λ which properly includes* $[0, c]$, *where*

$$x(t, \alpha) = f(t) + \int_0^\alpha k(t, s)x(s, \alpha)\, ds \quad \textit{for all} \quad \alpha \in \Lambda.$$

It should be noted in the proof of this last theorem that the solution past the critical length is given by the following:

$$x(t, c + \alpha) = x(t, c) + \int_c^{c+\alpha} \Delta_\alpha(t, s)x(s, c)\, ds - \left\{\hat{A}(t) + \int_c^{c+\alpha} \Delta_\alpha(t, s)\hat{A}(s)\, ds\right\}$$
$$\cdot \left\{\frac{\int_c^{c+\alpha} \hat{A}(s)x(s, c)\, ds + \int_c^{c+\alpha}\int_c^{c+\alpha} \hat{A}(t)\Delta_\alpha(t, s)x(s, c)\, ds\, dt}{\int_c^{c+\alpha} \hat{A}^2(s)\, ds + \int_c^{c+\alpha}\int_c^{c+\alpha} \hat{A}(t)\Delta_\alpha(t, s)\hat{A}(s)\, ds\, dt}\right\},$$

where $\Delta_\alpha(t, s)$ is the resolvent of $\Gamma_U(t, s)$ computed by Neumann expansion, and $\Gamma_U(t, s)$ the resolvent of

$$K(t, s) - U(t, s) = k(t, s) - \frac{\hat{A}(t)A^*(s)}{\int_0^c A^2(s)\, ds}.$$

It should also be noted that $x(t, c) + \zeta\hat{A}(t)$ is also a solution to the critical equation, but the imbedded solution is the only solution which vanishes at $t = c$.

REFERENCES

[1] A. McNabb and A. Schumitzky, *Factorization of operators II: A nonlinear Volterra method for numerical solution of linear Fredholm equations*, J. Comp. Syst. Sci., 4 (1970), pp. 103–128.

[2] R. Riesz and B. Nagy, *Functional Analysis*, Frederick Ungar, New York, 1955.

[3] A. Schumitzky, *On the equivalence between matrix Riccati equations and Fredholm resolvents*, J. Comp. Syst. Sci., 2 (1968), pp. 76–87.

[4] A. McNabb and A. Schumitzky, *Fundamental matrix for multi-point boundary value problems in the complex plane*, to appear.

[5] T. Wenska, *Imbedding of non-linear integral equations*, J. Math. Anal. Appl., 2 (1972), pp. 1–29.

[6] R. G. Bartle, *Singular points of functional equations*, Trans. Amer. Math. Soc., 75 (1953), pp. 366–384.

[7] A. C. Zaanen, *Linear Analysis*, North-Holland, Amsterdam, 1964.

[8] H. Kagiwada, R. Kalaba and A. Schumitzky, *A representation for the solution of Fredholm integral equations*, Proc. Amer. Math. Soc., 23 (1969), pp. 37–40.

[9] R. BELLMAN, *Functional equations in the theory of dynamic programming VII: A partial differential equation for the Fredholm resolvent*, Ibid., 8 (1957), pp. 435–440.

[10] M. G. KREIN, *On a new method for solving linear integral equations of the first and second kinds*, Dokl. Akad. Nauk SSSR, 100 (1955), pp. 413–416.

[11] B. D. ANDERSON AND T. KAILATH, *Some integral equations with nonsymmetric kernels*, SIAM J. Appl. Math., 20 (1971), pp. 659–669.

[12] R. GEESEY AND T. KAILATH, *An innovations approach to least-squares estimation, Part IV: Recursive estimation given the covariance functions*, to appear.

[13] A. MCNABB AND A. SCHUMITZKY, *Factorization of operators—I: Algebraic theory and examples*, J. Functional Anal., 9 (1972), pp. 262–295.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

LINEAR INTEGRAL EQUATIONS OF THE THIRD KIND*

G. R. BART† AND R. L. WARNOCK‡

Abstract. Linear integral equations of the third kind are studied as equations in two different spaces of generalized functions. In the first space D_τ, which consists of linear combinations of delta functions and continuous functions, the equation of the third kind has properties similar to those of the Fredholm equation of the second kind. The second space P_τ is comprised of linear combinations of delta functions and functions continuous except for poles, integration over the poles being defined by Cauchy's principal value. In P_τ the behavior of the third-kind equation is essentially different from that of second-kind Fredholm equations. Solutions in both D_τ and P_τ may be constructed explicitly via Fredholm theory. Examples showing the suitability of these spaces in physical problems are cited, and earlier literature on third-kind equations is surveyed briefly.

1. Introduction. We are interested in linear integral equations of the form

$$g(t)\varphi(t) - \lambda \int_a^b K(t, t')\varphi(t')\,dt' = f(t), \tag{1.1}$$

where $g(t)$ vanishes at least once in the interval $[a, b]$. Such equations are often called *equations of the third kind.* Several papers on the topic have appeared during the six decades following the early work of Hilbert [10] and Picard [15], but the amount of attention it has received hardly compares with that directed to equations of the second and first kinds. Most of the standard treatises on integral equations have little to say about equations of the third kind. Noteworthy exceptions are the books of Schmeidler [18] and Hellinger and Toeplitz [9].

One must begin, of course, with the question of what is meant by a solution of (1.1). Various answers have been proposed in the literature. We suggest a new definition of solution, which appears to be natural and appropriate in certain applications of third-kind equations that we have encountered.

We look for solutions in a space D_τ of generalized functions of the type

$$\varphi(t) = \sum_{i=1}^{n} \omega_i \delta(t - t_i) + y(t). \tag{1.2}$$

Here, $y(t)$ is a complex continuous function on the closed interval $[a, b]$, and $\delta(t)$ is Dirac's generalized function. The ω_i are complex numbers, and the t_i are the points at which $g(t)$ vanishes; we suppose that n is finite, and $t_i \neq a, b$. The space of test functions consists of all complex, continuous functions on $[a, b]$. The subscript τ of D_τ denotes the set of all $t_i : \tau = \{t_i | g(t_i) = 0\}$. The space D_τ is a Banach space with the following norm:

$$\|\varphi\| = \sup_{1 \leqq i \leqq n} |\omega_i| + \sup_{a \leqq t \leqq b} |y(t)|. \tag{1.3}$$

* Received by the editors December 14, 1971, and in revised form July 14, 1972. This work was supported in part by the National Science Foundation and by the U.S. Atomic Energy Commission.

† High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439, and Department of Physical Sciences, Mayfair College, Chicago, Illinois 60630.

‡ Department of Physics and Center for Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois 60616, and High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439.

In this space, one is able to obtain close analogues of the usual Fredholm theorems, provided that g, K, and f satisfy certain continuity conditions, and that $d = \det [K(t_i, t_j)]$ is not zero. The continuous part of the solution, $y(t)$, is obtained as the solution of a regular Fredholm equation, while the ω_i are explicit functionals of y. In other approaches, both the Fredholm-type theorems and the reduction to a regular Fredholm equation are lacking.

In the case $d = 0$ we have not been able to show that the very close analogy with Fredholm theory persists. We do establish some results which resemble Fredholm theory, however. We study this case both in the space D_τ, and in the space P_τ of generalized functions of the form

$$\varphi(t) = \sum_{i=1}^{n} \omega_i \delta(t - t_i) + P\left(\frac{x(t)}{g(t)}\right), \tag{1.4}$$

where P denotes Cauchy's principal value integration, $g(t)$ is the coefficient appearing in (1.1), and $x(t)$ is a complex continuous function. Similar solutions were discussed by van Kampen in connection with a problem of transport theory; see § 5.

For purposes of comparison, we begin with a brief survey of that earlier work on third-kind equations which has come to our attention. The papers mentioned deal with fairly general classes of equations. In § 5, we cite literature on various special equations of the third kind.

Hilbert [10] investigated a class of equations of the form (1.1) which he called "polar equations". For these equations $g(t)$ is piecewise continuous and takes on only the values ± 1, while $K(t, t')$ is symmetric and nonnegative. A number of authors have extended Hilbert's work in various ways; for a summary and references, see [9, §§ 21, 38, 44]. Schmeidler [18] devotes a chapter to treating the third-kind equation by Hilbert's method of infinitely many variables.

In the spirit of Hilbert's approach, Picard [15] handles the important case of (1.1) when $g(t) = t$, by substituting $\psi(t) = t\varphi(t)$ and replacing the integral equation by

$$\psi(s) - \lambda \int_a^{-\varepsilon} K(s, t)t^{-1}\psi(t)\, dt - \lambda \int_\eta^b K(s, t)t^{-1}\psi(t)\, dt = f(s), \tag{1.5}$$

where $0 < \varepsilon < -a$ and $0 < \eta < b$. He then treats (1.5) as a regular Fredholm equation and lets ε and η go to zero, in the solution, in such a way that $\ln(\eta/\varepsilon)$ goes to a nonzero finite constant limit C. If K is analytic, the solution converges to a linear function of C which satisfies the original integral equation for ψ, as the limit of (1.5). For extensions of this work, see the references in [9, § 21]. (In addition, Chvoles [4] has proved the convergence of Picard's limiting process under the assumption that K and f have first order derivatives satisfying a Lipschitz condition. Also, Trjitzinsky [21], [22] has developed a general theory of singular integral equations of the second kind using an approach similar to that of Picard.)

One should note, however, that it is not always expeditious to consider the equation for $\psi = g\varphi$, i.e.,

$$\psi(t) = f(t) + \lambda \int_a^b K(t, t')g^{-1}(t')\psi(t')\, dt', \tag{1.6}$$

in place of that for φ because, in general, the singularity structure given to ψ by the singular kernel in (1.6) will be very complicated [12]. Sometimes one may avoid the singular integration in (1.6) by using a principal value rule. Thus, the equation

$$\psi(t) + \text{P.P.} \int_a^b F(t, t', \varphi(t'))[g(t')]^{-\alpha-k}\, dt' = f(t), \tag{1.7}$$

where $0 < a < b$, $k > 0$ is an integer and $0 < \alpha < 1$, has been discussed by Vekua [24] when F is linear in φ and $g(t) = b - t$, as well as by Weiner [25], [26] when F is nonlinear but vanishes for $t' > t$ with $g(t) = |b - t|$.

2. The quasi-Fredholm case. The equation (1.1) is abbreviated as follows:

$$(g\mathbf{I} - \mathbf{K}_\lambda)\varphi = f. \tag{2.1}$$

By the adjoint of this equation, we mean

$$\bar{g}(t)\psi(t) - \bar{\lambda} \int_a^b K^*(t, t')\psi(t')\, dt' = \bar{f}(t), \tag{2.2}$$

or

$$(\bar{g}\mathbf{I} - \mathbf{K}_{\bar{\lambda}}^*)\psi = (g\mathbf{I} - \mathbf{K}_\lambda)^*\psi = \bar{f}, \tag{2.3}$$

where $\bar{g}$ is the complex conjugate of g, etc., and $K^*(t, t') = \bar{K}(t', t)$. If f is identically zero, the equation for φ or ψ is called homogeneous. Whenever the limits of integration are suppressed hereafter, the interval $[a, b]$ is to be understood.

We suppose that $g(t)$ and $f(t)$ are continuous in $[a, b]$, and that $K(t, t')$ is continuous in the square $[a, b] \times [a, b]$. We assume that $g(t)$ has at least one zero in (a, b), and $g(a)$, $g(b) \neq 0$. The derivatives $g'(t)$, $f'(t)$, $\partial K(t, t')/\partial t$, $\partial K(t, t')/\partial t'$, $\partial^2 K(t, t')/\partial t\, \partial t'$ are all assumed to exist and be continuous when t or t' is sufficiently close to a zero of $g(t)$. Also, $g'(t_i)$ must be nonzero if $g(t_i) = 0$. It will be clear that some of these conditions can be weakened without difficulty. For the work of this section in particular, the continuity of $g'(t)$ near t_i may be replaced by the requirement that $g'(t)$ be continuous near t_i except for a jump discontinuity at t_i. This entails a corresponding generalization of the space D_τ to allow for jumps of $y(t)$. Furthermore, one could drop the requirement that $g'(t)$ be continuous from the right or left, and assume instead that $g(t)$ has asymptotes of the type $g(t) \sim \alpha_{i\pm}|t - t_i|^{\beta_{i\pm}}$, $t \to t_i \pm 0$, $0 < \beta_{i\pm} < 1$.

Suppose that $g(t)$ has n zeros at $t = t_i$ in (a, b), $i = 1, 2, \cdots, n$. Let C be the set of all continuous complex functions defined on $[a, b]$, and let D_τ be the space of generalized functions defined in the Introduction. For the operators $g\mathbf{I}$, $\mathbf{K}_\lambda : D_\tau \to C$ we have the definitions

$$g\mathbf{I}\varphi(t) = g(t)y(t), \tag{2.4}$$

$$\mathbf{K}_\lambda\varphi(t) = \lambda \sum_{i=1}^{n} \omega_i K(t, t_i) + \lambda \int K(t, t')y(t')\, dt'. \tag{2.5}$$

Obviously, $\mathbf{K}_\lambda$ is completely continuous.

Although the adjoint operator $(g\mathbf{I} - \mathbf{K}_\lambda)^*$ may be viewed as just a second mapping from D_τ to C, the natural interpretation is in terms of dual spaces. Let D_τ^* and C^* be the dual spaces of D_τ and C, respectively. An element ψ of C^* is a continuous linear functional which assigns the complex number $\langle \psi, y\rangle$ to an element y of C. An element Ψ of D_τ^* is similarly a continuous linear functional assigning $\langle \Psi, \varphi\rangle$ to an element φ of D_τ. The adjoint operator $(g\mathbf{I} - \mathbf{K}_\lambda)^* : C^* \to D_\tau^*$ is related to $(g\mathbf{I} - \mathbf{K}_\lambda) : D_\tau \to C$ as follows:

$$\langle (g\mathbf{I} - \mathbf{K}_\lambda)^*\psi, \varphi\rangle = \langle \psi, (g\mathbf{I} - \mathbf{K}_\lambda)\varphi\rangle. \tag{2.6}$$

Just as we do not consider all $\varphi \in C^*$ which are solutions of (1.1), we shall not consider all $\psi \in C^*$ which are solutions of (2.2). Rather, we consider only $\psi \in D_\tau \subset C^*$.

THEOREM 1. *If $g(t)$ has only the n zeros at $t = t_i$, $i = 1, 2, \cdots, n$, in (a, b) with $g'(t_i) \neq 0$ and*

$$d = \det [K(t_i, t_j)] \neq 0, \tag{2.7}$$

then either (a) *both* (1.1) *and* (2.2) *have unique D_τ solutions, or* (b) *the homogeneous forms of* (1.1) *and* (2.2) *each have a finite number $k \geqq 1$ of linearly independent nontrivial D_τ solutions and* (1.1) *has a D_τ solution if and only if for every $\psi \in D_\tau$ satisfying the homogeneous adjoint equation*

$$(\bar{g}\mathbf{I} - \mathbf{K}_\lambda^*)\psi = 0, \tag{2.8}$$

one has

$$\int \bar{\psi}(t) f(t)\, dt = 0. \tag{2.9}$$

Note that in case (b) the general D_τ solution φ of (1.1) involves k arbitrary constants a_k and is obtained from a particular solution φ_0 by the formula

$$\varphi = \varphi_0 + a_1\varphi_1 + \cdots + a_k\varphi_k, \tag{2.10}$$

where the φ_k, $k > 0$, are linearly independent solutions of the homogeneous equation

$$(g\mathbf{I} - \mathbf{K}_\lambda)\varphi = 0. \tag{2.11}$$

The theorem is trivial if $\lambda = 0$ since then (b) always holds. Suppose, therefore, that $\lambda \neq 0$ and consider first $n = 1$ with, say, $t_1 = 0$. With $\varphi = \omega\delta + y$, equation (1.1) becomes

$$g(t)y(t) = f(t) + \omega\lambda K(t, 0) + \lambda \int K(t, t')y(t')\, dt'. \tag{2.12}$$

We obtain an expression for ω by evaluating this equation at $t = 0$. Since by assumption $K(0, 0) \neq 0$,

$$\omega = -\frac{f(0) + \lambda \int K(0, t')y(t')\, dt'}{\lambda K(0, 0)}, \tag{2.13}$$

and (2.12) becomes

$$g(t)y(t) = f(t) - f(0)\frac{K(t, 0)}{K(0,0)} + \lambda \int \left[K(t, t') - \frac{K(t, 0)K(0, t')}{K(0, 0)} \right] y(t')\, dt'. \tag{2.14}$$

Now if $\varphi = \omega\delta + y$ is a solution of (1.1), then ω is fixed in terms of y by (2.13) and y satisfies (2.14). Conversely, if y satisfies (2.14) and ω is defined by (2.13), then one can check that $\varphi = \omega\delta + y$ is a solution of (1.1). Thus, in this case, (1.1) and (2.14) are equivalent. Similarly, (2.2) is equivalent to an equation like (2.14). But by our assumptions on g, f, and K, clearly (2.14) is a regular Fredholm equation of the second kind in the space C (after dividing by g), since we may use l'Hospital's rule to evaluate the 0/0 limits when they occur. Hence, (2.14) may fail to have a unique continuous solution only if there exist a finite number $k \geqq 1$ of nontrivial continuous solutions to the equations

$$g(t)y(t) = \lambda \int \left[K(t,t') - \frac{K(t,0)K(0,t')}{K(0,0)} \right] y(t')\, dt', \tag{2.15}$$

$$z(t) = \bar{\lambda} \int \left[K^*(t,t') - \frac{K^*(t,0)K^*(0,t')}{K^*(0,0)} \right] \frac{z(t')}{\bar{g}(t')}\, dt'. \tag{2.16}$$

Note that (2.16) is obtained from (2.15) by first dividing by $g(t)$ and then taking the adjoint in the usual Fredholm sense. Equation (2.16) is different from (2.8), the adjoint homogeneous equation in D_τ. Suppose these nontrivial solutions exist and z is any solution of (2.16). Notice that $z(0) = 0$ and $z'(t)$ is continuous at $t = 0$. Hence, $\hat{y}(t) = z(t)/\bar{g}(t)$ satisfies the adjoint (with respect to D_τ) of (2.15) and is continuous. Consequently, if one defines

$$\varphi_i(t) = \omega_i\delta(t) + y_i(t), \tag{2.17}$$

$$\psi_i(t) = \hat{\omega}_i\delta(t) + \hat{y}_i(t) \tag{2.18}$$

with

$$\omega_i = -\frac{\int K(0,t)y_i(t)\, dt}{K(0,0)}, \tag{2.19}$$

$$\hat{\omega}_i = -\frac{\int K^*(0,t)\hat{y}_i(t)\, dt}{K^*(0,0)}, \tag{2.20}$$

in terms of each of the linearly independent solutions $y_i, \hat{y}_i, i = 1, \cdots, k$, then one can verify that the φ_i satisfy (2.11) and are linearly independent, while the ψ_i satisfy (2.8) and are linearly independent.

Thus, we find that in the theorem either (a) or the first part of (b) is true when $n = 1$. The second part of (b) is obtained by noting that (2.9) is equivalent to the well-known condition for (2.14) to have a solution when (2.15) has nontrivial solutions, i.e., since $z_i = \bar{g}\hat{y}_i$,

$$\begin{aligned} \int \bar{z}_i(t)\frac{1}{g(t)}\left[f(t) - f(0)\frac{K(t,0)}{K(0,0)} \right] dt &= \int \bar{\hat{y}}_i(t)f(t)\, dt + \bar{\hat{\omega}}_i f(0) \\ &= \int \bar{\psi}_i(t)f(t)\, dt = 0. \end{aligned} \tag{2.21}$$

Consider next the case when g has $n > 1$ zeros in (a, b) at points t_i near which $g'(t)$ is continuous and for which $g'(t_i) \neq 0$. We use (1.1) and (1.2) to write

$$g(t)y(t) = f(t) + \lambda \sum_{i=1}^{n} \omega_i K(t,t_i) + \lambda \int K(t,t')y(t')\, dt'. \tag{2.22}$$

The ω_i are fixed by the requirement that the right-hand side of (2.22) be zero when evaluated at each t_i. Thus,

$$\lambda \sum_{i=1}^{n} \omega_i K(t_j, t_i) = -f(t_j) - \lambda \int K(t_j, t) y(t)\, dt. \tag{2.23}$$

Since, by assumption,

$$d = \det [K(t_j, t_k)] \neq 0, \tag{2.24}$$

we have

$$\omega_i = -\Omega_i(\lambda d)^{-1}, \tag{2.25}$$

where Ω_i is the determinant of the matrix formed from the matrix $[K(t_j, t_k)]$ by the replacement of the ith column, $K(t_j, t_i)$, by $f(t_j) + \lambda \int K(t_j, t) y(t)\, dt$. If the expression for ω_i is substituted into (2.22), one obtains

$$g(t)y(t) = F(t) + \lambda \int M(t, t')y(t')\, dt', \tag{2.26}$$

where

$$F(t) = \frac{1}{d}\begin{vmatrix} f(t) & K(t, t_1) & \cdots K(t, t_n) \\ f(t_1) & K(t_1, t_1) & \cdots K(t_1, t_n) \\ \cdots & \cdots & \cdots \\ f(t_n) & K(t_n, t_1) & \cdots K(t_n, t_n) \end{vmatrix}, \tag{2.27}$$

$$M(t, t') = \frac{1}{d}\begin{vmatrix} K(t, t') & K(t, t_1) & \cdots K(t, t_n) \\ K(t_1, t') & K(t_1, t_1) & \cdots K(t_1, t_n) \\ \cdots & \cdots & \cdots \\ K(t_n, t') & K(t_n, t_1) & \cdots K(t_n, t_n) \end{vmatrix}. \tag{2.28}$$

Note that for $i = 1, 2, \cdots, n$,

$$F(t_i) = M(t_i, t') = M(t, t_i) = 0. \tag{2.29}$$

From this point on, one may proceed with (2.26) to (2.29) essentially as in the $n = 1$ case to complete the proof.

From Fredholm theory, we know that $y(t)$ in (2.26) is a meromorphic function of λ with no singularity at $\lambda = 0$. Thus, each ω_i and the solution φ of (1.1) is also a meromorphic function of λ with, perhaps, a pole at $\lambda = 0$.

3. A principal value interpretation of the integral equation. In this section we drop the requirement $d \neq 0$, but work in a different space, the space P_τ mentioned in the Introduction. Under the restrictions we have placed on g following (2.3), the Cauchy principal value integrals $P \int [x(t)/g(t)]\, dt$ will exist, provided that x is continuous and has a continuous derivative in a neighborhood of each zero of g. Let us define the space P_τ, $\tau = \{t_1, t_2, \cdots, t_n\}$, as the space of generalized functions of the form

$$\varphi(t) = \sum_{i=1}^{n} \omega_i \delta(t - t_i) + P[x(t)/g(t)], \tag{3.1}$$

where x is continuous and has a continuous derivative in a neighborhood of each zero t_i of g. The P in (3.1) indicates that Cauchy principal values are to be used when integrating expressions containing $1/g$. The space P_τ is closely related to D_τ. In fact, members of P_τ for which $x(t_i) = 0$, for all i, are also members of D_τ. We shall show how one can obtain particular solutions of (1.1) in P_τ for a wide class of possible kernels. But, because the homogeneous form of (1.1) *always* has a nontrivial solution in P_τ, a particular solution of (1.1) can never be a unique solution in P_τ. (One can sometimes find a unique solution of (1.1) by the following approach in that subspace of P_τ defined by requiring all the ω_i to vanish in (3.1).)

By substituting (3.1) in equation (1.1), one obtains

$$x(t) - \lambda P \int K(t, t') \frac{x(t')}{g(t')} dt' = f(t) + \lambda \sum_{i=1}^{n} \omega_i K(t, t_i). \tag{3.2}$$

This equation may be written in a more convenient form by making use of a set of continuous functions $p_i(t)$ which are required to have continuous derivatives in a neighborhood of each zero of g and to satisfy

$$p_i(t_j) = \delta_{ij}. \tag{3.3}$$

Thus,

$$x(t) - \lambda \int D(t, t') x(t') \, dt' = f(t) + \lambda \sum_{i=1}^{n} \hat{\omega}_i K(t, t_i), \tag{3.4}$$

where

$$D(t, t') = \left[K(t, t') - \sum_{i=1}^{n} K(t, t_i) p_i(t') \right] / g(t'), \tag{3.5}$$

$$\hat{\omega}_i = \omega_i + P \int \frac{p_i(t) x(t)}{g(t)} dt. \tag{3.6}$$

Note that (3.4) is a necessary condition on any solution x of (3.2). Since the $\hat{\omega}_i$ are constants, equation (3.4) is a regular Fredholm equation of the second kind for x, if the $\hat{\omega}_i$ are considered as given. Furthermore, any continuous solution of (3.4) automatically has a derivative, which is continuous in a neighborhood of each t_i.

Suppose that (3.4), with the right side set equal to zero, has no nontrivial continuous solutions. That is, suppose $(\mathbf{I} - \lambda \mathbf{D})^{-1}$ exists in the space of continuous functions. Then

$$x = x_0 + \sum_{i=1}^{n} \hat{\omega}_i x_i \tag{3.7}$$

is a particular solution of (3.4) with

$$x_0 = (\mathbf{I} - \lambda \mathbf{D})^{-1} f, \tag{3.8}$$

$$x_i = (\mathbf{I} - \lambda \mathbf{D})^{-1} \lambda K(\cdot, t_i). \tag{3.9}$$

To have (3.7) yield a solution of (1.1), it is required that the $\hat{\omega}_i$ satisfy the consistency condition obtained by substituting (3.7) into the right-hand side of (3.6).

Thus,

$$\hat{\omega}_i = \omega_i + P\int \frac{p_i(t)x_0(t)}{g(t)}\,dt + \sum_{j=1}^{n} \hat{\omega}_j P\int \frac{p_i(t)x_j(t)}{g(t)}\,dt. \tag{3.10}$$

Put

$$\Lambda_{ij} = P\int \frac{p_i(t)x_j(t)}{g(t)}\,dt. \tag{3.11}$$

Then (3.10) becomes

$$(\hat{\omega} - \Lambda\hat{\omega})_i = \omega_i + P\int \frac{p_i x_0}{g}\,dt. \tag{3.12}$$

If $\det[I - \Lambda] \neq 0$, this equation has a solution for arbitrary ω_i. Therefore, either the ω_i or the $\hat{\omega}_i$ may be regarded as free parameters. If the x_i, $i = 1, \cdots, n$, are linearly independent, then we have in (3.7) an n-dimensional manifold of solutions of (1.1) in the space P_τ. If we put $f = 0$, then $x_0 = 0$, and (3.7) yields a solution of the homogeneous form of (1.1). The ω_i may always be chosen so that this solution is nontrivial. If $\det[I - \Lambda] = 0$, then it is possible that there is a solution of (3.12) with some of the ω_i still arbitrary. But there is always a solution with $\hat{\omega}_i \neq 0$ for some i and

$$\omega_i = -P\int \frac{p_i x_0}{g}\,dt. \tag{3.13}$$

Again, if $\hat{\omega}_i \neq 0$, then $P\hat{\omega}_i x_i/g$ is a solution of the homogeneous form of (1.1), so our solution in P_τ is not unique.

Finally, if (3.4) with the right side set equal to zero has a nontrivial solution x, this directly yields a solution φ of the homogeneous form of (1.1) by taking

$$\omega_i = -P\int \frac{p_i(t)x(t)}{g(t)}\,dt. \tag{3.14}$$

Thus, by (3.6) we have $\hat{\omega}_i = 0$ and

$$\varphi(t) = -\sum_{i=1}^{n} \delta(t - t_i)P\int p_i(t')\frac{x(t')}{g(t')}\,dt' + P\left[\frac{x(t)}{g(t)}\right]. \tag{3.15}$$

There are many ways to choose the $p_i(t)$. For a particular kernel, this freedom might be exploited to ensure that $\mathbf{I} - \lambda\mathbf{D}$ has an inverse in the space of continuous functions. One simple choice for the p_i is

$$\begin{aligned} p_1(t) &= 1 \quad \text{if } n = 1, \\ p_i(t) &= \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(t - t_j)}{(t_i - t_j)} \quad \text{if } n > 1. \end{aligned} \tag{3.16}$$

If there exist n points τ_i such that

$$\Delta = \det[K(\tau_i, t_j)] \tag{3.17}$$

does not vanish, then another choice of the p_i is minus the coefficient of $K(t, t_i)$ in the following expression:

$$D(t, t')g(t') = \frac{1}{\Delta}\begin{vmatrix} K(t, t') & K(t, t_1) & \cdots K(t, t_n) \\ K(\tau_1, t') & K(\tau_1, t_1) & \cdots K(\tau_1, t_n) \\ \cdots & \cdots & \cdots \\ K(\tau_n, t') & K(\tau_n, t_1) & \cdots K(\tau_n, t_n) \end{vmatrix}. \quad (3.18)$$

This choice is motivated by analogy with § 2. If one evaluates (3.2) at the points $t = \tau_j$, solves for the ω_i, and substitutes for the ω_i in (3.2), there results

$$x(t) - \lambda \int D(t, t')x(t')\,dt' = H(t), \quad (3.19)$$

where

$$H(t) = \frac{1}{\Delta}\begin{vmatrix} f(t) & K(t, t_1) & \cdots K(t, t_n) \\ f(\tau_1) - x(\tau_1) & K(\tau_1, t_1) & \cdots K(\tau_1, t_n) \\ \cdots & \cdots & \cdots \\ f(\tau_n) - x(\tau_n) & K(\tau_n, t_1) & \cdots K(\tau_n, t_n) \end{vmatrix}. \quad (3.20)$$

Note that the n values $x(\tau_i)$ are unconstrained by (3.19) since $D(\tau_i, t') = 0$ and $H(\tau_i) = x(\tau_i)$. The linearity of H in $K(t, t_i)$ and in $x(\tau_i)$ implies that the quantities $x(\tau_i)$ are just a new parametrization of the manifold of solutions previously parametrized by the ω_i.

There is some overlap between the present approach and that based on the space D_τ. Suppose that we have the situation where the ω_i are arbitrary, and that

$$\det[x_i(t_j)] \neq 0. \quad (3.21)$$

This requires that the $x_i(t)$, $i = 1, \cdots, n$, be linearly independent. In that case, one can determine the $\hat{\omega}_i$ in (3.7) so that $x(t_i) = 0$. Since $x'(t)$ is continuous at $t = t_i$, $x(t)/g(t)$ is continuous near t_i and the principal value prescription is unnecessary. The resulting solution of (1.1) lies in D_τ. A similar conclusion may be stated in terms of the $x(\tau_i)$.

The main results of this section are summarized in the following theorems.

THEOREM 2. *Let $\varphi(t) \in P_\tau$ be a solution of the integral equation of the third kind* (1.1). *Then the corresponding $x(t)$ satisfies* (3.4), *where the $p_i(t)$ are arbitrary except for the restrictions stated.*

THEOREM 3. *There is always a nontrivial solution in P_τ of the homogeneous third-kind equation*

$$(g\mathbf{I} - \mathbf{K}_\lambda)\varphi = 0. \quad (3.22)$$

THEOREM 4. *Suppose that $(\mathbf{I} - \lambda\mathbf{D})^{-1}$ exists on the space of continuous functions $C[a, b]$, for a particular choice of the $p_i(t)$. Let $x_i(t)$, $i = 0, 1, 2, \cdots, n$, be defined by* (3.8) *and* (3.9). *Then, if* $\det(I - \Lambda) \neq 0$, *the general solution of* (1.1) *in P_τ is given by*

$$\varphi(t) = \sum_{i=1}^{n} \omega_i \delta(t - t_i) + P\left[\frac{x_0(t) + \sum_{i=1}^{n} \hat{\omega}_i x_i(t)}{g(t)}\right], \quad (3.23)$$

where the ω_i are arbitrary, and the $\hat{\omega}_i$ are obtained as the unique solution of (3.12).

THEOREM 5. *Under the hypotheses of Theorem* 4, *plus condition* (3.21), *there exists a unique choice of the* ω_i *such that the solution* (3.23) *belongs to* D_τ.

4. Analysis of the case det $[K(t_i, t_j)] = 0$**.** We now return to the analysis of (1.1) in the space D_τ. Although we know of no counterexample to the consequences of Theorem 1 in the case $d = \det[K(t_i, t_j)] = 0$, we have not been able to extend the theorem, as it stands, to this case. In this section, we shall examine an alternate approach to the third-kind equation yielding theorems which hold even if $d = 0$. The obstacle to extending Theorem 1 is the possibility of unequal dimensions for the null space of $g\mathbf{I} - \mathbf{K}_\lambda$ and its adjoint (acting on D_τ). We can show, however, that the dimensions are both finite numbers.

THEOREM 6. *The null space of* $g\mathbf{I} - \mathbf{K}_\lambda$, $N(g\mathbf{I} - \mathbf{K}_\lambda) \subset D_\tau$, *is finite dimensional.*

Proof. It suffices to show that every closed and bounded set in $N(g\mathbf{I} - \mathbf{K}_\lambda)$ is compact. Let $F \subset N(g\mathbf{I} - \mathbf{K}_\lambda)$ be closed and bounded. All we need show is that from every sequence $\{z_\alpha\}$ in F we can extract a convergent subsequence (the limit is, of course, in F since F is closed). Since the z_α belong to $N(g\mathbf{I} - \mathbf{K}_\lambda)$, they have the representation

$$z_\alpha(t) = \sum_{j=1}^{n} \omega_{\alpha j}\delta(t - t_j) + r_\alpha(t),$$

where

$$g(t)r_\alpha(t) = \lambda \sum_{j=1}^{n} \omega_{\alpha j}K(t, t_j) + \lambda \int K(t, t')r_\alpha(t')\, dt' \tag{4.1}$$

and

$$0 = \lambda \sum_{j=1}^{n} \omega_{\alpha j}K(t_i, t_j) + \lambda \int K(t_i, t')r_\alpha(t')\, dt'. \tag{4.2}$$

Let

$$p_i(t) = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{(t - t_j)}{(t_i - t_j)} \tag{4.3}$$

so that $p_i(t_j) = \delta_{ij}$. Multiply (4.2) by $p_i(t)$, sum over i, and subtract from (4.1) to get

$$\begin{aligned} r_\alpha(t) = {} & \frac{\lambda}{g(t)} \sum_{j=1}^{n} \omega_{\alpha j}\left[K(t, t_j) - \sum_{i=1}^{n} p_i(t)K(t_i, t_j)\right] \\ & + \frac{\lambda}{g(t)} \int \left[K(t, t') - \sum_{i=1}^{n} p_i(t)K(t_i, t')\right] r_\alpha(t')\, dt'. \end{aligned} \tag{4.4}$$

Now the sequence of n-tuples $\{\omega_\alpha = (\omega_{\alpha 1}, \omega_{\alpha 2}, \cdots, \omega_{\alpha n})\}$ certainly contains a convergent subsequence $\{\omega_{\alpha\beta}\}$. Since the integral operator in (4.4) is completely continuous on the space of continuous functions, it follows that the associated subsequence $\{r_{\alpha\beta}\}$ contains a convergent subsequence and, hence, so does $\{z_\alpha\}$. Similarly, the dimension of the null space of the adjoint operator is finite.

A useful property of the integral equation of the third kind is that if $g(t)\varphi(t)$ vanishes at one of the zeros of $g(t)$, say $t = t_i$, and $f(t_i) \neq 0$, then the inhomogeneous integral equation, (1.1), is equivalent to a homogeneous equation plus a subsidiary condition. That is, we have the following theorem.

THEOREM 7. *If* $t = t_i$ *is a zero of* g *in* (a, b) *and* $f(t_i) \neq 0$, *then* $\varphi(t)$ *satisfies both* (1.1) *and* $g(t_i)\varphi(t_i) = 0$ *if and only if* $\varphi(t)$ *satisfies both*

$$g(t)\varphi(t) = \lambda \int \left[K(t, t') - \frac{f(t)K(t_i, t')}{f(t_i)} \right] \varphi(t')\, dt' \tag{4.5}$$

and

$$\int K(t_i, t')\varphi(t')\, dt' \neq 0. \tag{4.6}$$

Proof. It is trivial to see that (4.6) is necessary. One can see that (4.5) is necessary by evaluating (1.1) at $t = t_i$, multiplying the result by $f(t)/f(t_i)$ and subtracting from (1.1). From any solution $\hat{\varphi}(t)$ of (4.5) which satisfies (4.6), one can check that one gets a solution of (1.1) by taking

$$\varphi(t) = N\hat{\varphi}(t), \tag{4.7}$$

where

$$N = -f(t_i)/\int \lambda K(t_i, t)\hat{\varphi}(t)\, dt. \tag{4.8}$$

Let us suppose that $t_i = 0$ is the only zero of g in (a, b) and impose the same assumptions on f, g, and K as in Theorem 1, except that $K(0, 0) = 0$ is allowed, but $f(0) = 0$ is not. Then, (4.5) may be interpreted as a regular homogeneous Fredholm equation which might have nontrivial continuous solutions. If there are continuous nontrivial solutions of (4.5), either there is one which satisfies (4.6) and thus gives rise to a solution of (1.1), or obviously there exists a nontrivial solution of (2.11). If there are no continuous nontrivial solutions of (4.5), put $\varphi(t) = \delta(t) + y(t)$ in (4.5) to get

$$g(t)y(t) = \lambda K(t, 0) - \lambda \frac{f(t)}{f(0)} K(0, 0) + \lambda \int \left[K(t, t') - \frac{f(t)}{f(0)} K(0, t') \right] y(t')\, dt'. \tag{4.9}$$

This equation must have a continuous solution $y(t)$ if (4.5) has none by the Fredholm alternative. One can verify that $\varphi = \delta + y$ must, therefore, be a solution of (4.5). This φ leads to a solution of (1.1) via Theorem 7 if (4.6) holds; that is, if

$$\int K(0, t)[\delta(t) + y(t)]\, dt \neq 0. \tag{4.10}$$

On the other hand, if the left-hand side in (4.10) is zero, then by inspection $\varphi = \delta + y$ satisfies the homogeneous equation, (2.11), which is identical with (4.9) for this φ. Hence, we have proved the $f(0) \neq 0$ part of the following theorem.

THEOREM 8. *If* $g(t)$ *has only one zero, say at* $t = 0$, *in* (a, b) *with* $g'(0) \neq 0$, *and possibly* $K(0, 0) = 0$, *then either* (a) *equation* (1.1) *has a unique* D_τ *solution, or* (b) *the homogeneous form of* (1.1) *has a finite number* $k \geqq 1$ *of linearly independent nontrivial* D_τ *solutions.*

The finiteness of k in Theorem 8 comes from Theorem 6. The $f(0) = 0$ case is reducible to the $f(0) \neq 0$ case by the substitution

$$\varphi(t) = \chi(t) + \bar{K}(0, t), \tag{4.11}$$

if $K(0, t)$ does not identically vanish. Thus,

$$g(t)\chi(t) = f_1(t) + \lambda \int K(t, t')\chi(t')\,dt' \tag{4.12}$$

with

$$f_1(t) = f(t) - g(t)\bar{K}(0, t) + \lambda \int K(t, t')\bar{K}(0, t')\,dt' \tag{4.13}$$

and

$$f_1(0) = \lambda \int |K(0, t)|^2\,dt \neq 0. \tag{4.14}$$

If $K(0, t) \equiv 0$, one only has to put $\varphi = \omega\delta + y$ in (2.11), which becomes a regular Fredholm equation for y,

$$y(t) = \frac{\omega\lambda K(t, 0)}{g(t)} + \frac{\lambda}{g(t)} \int K(t, t')y(t')\,dt', \tag{4.15}$$

to see that case (b) always holds. Here, one takes $\omega = 0$ or $\omega \neq 0$ as the situation requires.

For the case when g has $n > 1$ zeros in (a, b) at $t = t_i$, we have a second theorem like Theorem 8. First, however, we note the following lemma, which may be proved by induction.

LEMMA. *If* $n > 1$,

$$\prod_{\substack{r,s=1\\ r<s}}^{n} [f(t_r) - f(t_s)] \neq 0 \tag{4.16}$$

and

$$\sigma_i^n(t) = \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\frac{f(t) - f(t_j)}{f(t_i) - f(t_j)}\right], \tag{4.17}$$

then

$$\sum_{i=1}^{n} \sigma_i^n(t) f(t_i) = f(t). \tag{4.18}$$

We then have the following theorem.

THEOREM 9. *If* $g(t_i) = 0$ *for* $i = 1, 2, \cdots, n$ *with* $n \geqq 2$ *and*

$$\prod_{\substack{r,s=1\\ r<s}}^{n} [f(t_r) - f(t_s)] \neq 0, \tag{4.19}$$

then a necessary condition that $\varphi(t)$ *satisfy* (1.1) *and* $g(t_i)\varphi(t_i) = 0$, $i = 1, 2, \cdots, n$, *is that* φ *satisfy the homogeneous equation*

$$g(t)\varphi(t) = \lambda \int \left[K(t, t') - \sum_{i=1}^{n} \sigma_i^n(t) K(t_i, t')\right]\varphi(t')\,dt'. \tag{4.20}$$

Also a sufficient condition for (1.1) *to have a solution* $\varphi(t)$ *is that there exist at least n linearly independent solutions* $\varphi_j(t)$ *to* (4.20) *such that*

$$\det\left[\int K(t_i, t')\varphi_j(t')\,dt'\right] \neq 0. \tag{4.21}$$

The solution of (1.1) is given by

$$\varphi(t) = \sum_{i=1}^{n} \alpha_j \varphi_j(t), \tag{4.22}$$

where the α_j are determined by

$$\sum_{i=1}^{n} \alpha_j \int K(t_i, t')\varphi_j(t')\,dt' = -\frac{f(t_i)}{\lambda}. \tag{4.23}$$

This theorem is a direct consequence of the lemma since (4.20) follows from (1.1) by evaluating (1.1) at t_i, multiplying the result by $\sigma_i^n(t)$, summing over i, and subtracting the result from (1.1). The rest of the theorem is easy to verify. Note that the property $\sigma_i^n(t_j) = \delta_{ij}$ implies that the adjoint (with respect to D_τ) to the homogeneous equation (4.20) necessarily has at least n linearly independent solutions, namely, the $\delta(t - t_j)$.

5. Applications of equations of the third kind. We now mention examples of third-kind equations which are found in the literature. (i) In the scattering matrix theory of elementary particle interactions, one encounters the integral equation of Frye and Warnock [7]. This equation has the form (1.1), with $g(t)$ being the elasticity function. Under certain physical conditions, g acquires a zero, and the solution of physical interest lies in the space D_τ. (ii) Under simplifying assumptions, the Boltzmann transport equation reduces to an integral equation of the third kind. The physically meaningful solution of the latter, identified by van Kampen [23] and studied by several authors [2], [3], [23], [17], [8], lies in the space P_τ. (iii) Third-kind operators occur in certain physical problems as Fréchet derivatives of nonlinear operators; for instance, the nonlinear integral operator which appears in the unitarity equation of scattering theory [1]. An understanding of the Fréchet derivative is important in the solution of the nonlinear equation by the Newton–Kantorovich method. It is not yet clear, however, that our spaces of generalized functions are relevant to such problems. (iv) Various other special integral equations of the third kind appear in the literature; see [20], [16], [14], [5], [6], [13], [19], [11].

Acknowledgment. One of us (GRB) wishes to thank the Argonne National Laboratory Center for Educational Affairs and the Argonne High Energy Physics Division for their hospitality during an appointment in their Summer Faculty Research Program.

REFERENCES

[1] D. Atkinson, P. W. Johnson and R. L. Warnock, *Determination of the scattering amplitude from the differential cross section and unitarity*, Comm. Math. Phys., 28 (1972), pp. 133–158.
[2] K. M. Case, *Plasma oscillations*, Ann. Physics, 7 (1959), pp. 349–364.

[3] ———, *Elementary solutions of the transport equation and their applications*, Ibid., 9 (1960), pp. 1–23.

[4] A. R. CHVOLES, *On Fredholm's integral equations of the third kind*, Soobšč. Akad. Nauk Gruzin. SSR, 2 (1941), pp. 389–395.

[5] T. FÉNYES, *A note on the solution of integral equations of convolution type of the third kind by application of the operational calculus of Mikusiński*, Studia Sci. Math. Hungar., 2 (1967), pp. 81–89.

[6] ———, *On the operational solution of certain non-linear singular integral equations*, Ibid., 4 (1969), pp. 69–91.

[7] G. FRYE AND R. L. WARNOCK, *Analysis of partial wave dispersion relations*, Phys. Rev., 130 (1963), pp. 478–494.

[8] S. GALLONE AND G. GHILARDOTTI, *On the use of Case's general solution of the transport equation in neutron transport problems*, Nuovo Cimento, 31 (1964), pp. 203–218.

[9] E. HELLINGER AND O. TOEPLITZ, *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten*, Chelsea, New York, 1953.

[10] D. HILBERT, *Grundzüge einer allgemeinen Theorie der linear Integralgleichungen*, Chelsea, New York, 1953.

[11] K. JAECKEL, *Über die Eigenlösungen gewisser Integralgleichungen der Potentialtheorie*, J. Reine Angew. Math., 189 (1951), pp. 141–149.

[12] G. S. LITVINČUK, *Integral equations with analytic kernels*, Izv. Vyssh. Uchebn. Zaved. Matematika, 1958, no. 2 (3), pp. 197–209.

[13] L. I. PANOV, *Integral equations with kernels possessing a non-integrable singularity of arbitrary order*, Dokl. Akad. Nauk Tadžik. SSR, 10 (1967), no. 6, pp. 3–7.

[14] T. L. PEREL'MAN, *On asymptotic expansions of solutions of a certain class of integral equations*, Prikl. Mat. Meh., 25 (1961), pp. 1145–1147 = J. Appl. Math. Mech., 25 (1962), pp. 1709–1714.

[15] É. PICARD, *Un théorème général sur certaines équations intégrales de troisième espèce*, Comptes Rendus, 150 (1910), pp. 489–491.

[16] T. SATO, *Sur l'équation intégrale* $xu(x) = f(x) + \int_0^x K(x, t, u(t))\,dt$, J. Math. Soc. Japan, 5 (1953), pp. 145–153.

[17] D. H. SATTINGER, *A singular eigenfunction expansion in anisotropic transport theory*, J. Math. Anal. Appl., 15 (1966), pp. 497–511.

[18] W. SCHMEIDLER, *Integralgleichungen mit Anwendungen*, Physik und Technik, Akademische Verlagsgesellschaft, Leipzig, 1955.

[19] B. STANKOVIČ, *Solution d'une équation intégrale homogène*, Srpska Akad. Nauka. Zbornik Radova 35. Mat. Inst., 3 (1953), pp. 95–106.

[20] T. TAKESADA, *On the singular point of integral equations of Volterra type*, J. Math. Soc. Japan, 7 (1955), pp. 123–136.

[21] W. J. TRJITZINSKY, *General theory of singular integral equations with real kernels*, Trans. Amer. Math. Soc., 46 (1942), pp. 202–279.

[22] ———, *Some problems in the theory of singular integral equations*, Ann. of Math. (2), 41 (1940), pp. 584–619.

[23] N. G. VAN KAMPEN, *On the theory of stationary waves in plasmas*, Physica, 21 (1955), pp. 949–963.

[24] N. P. VEKUA, *Integral equations of Fredholm type with integrals in Hadamard's sense*, Trav. Inst. Math. Tbilissi, 7 (1940), pp. 113–146.

[25] K. WEINER, *Über nichtlineare Integralgleichungen mit Hadamard-Integralen mit variablen oberen Grenzen und festen Singularitäten*, Wiss. Beträge Martin-Luther-Univ. Halle-Wittenberg, M1 (1968), pp. 101–110.

[26] ———, *Über nichtlineare Volterrasche Integralgleichungen und Gleichungssysteme mit Hadamard-Integralen I*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe, 17 (1968), pp. 805–827.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

SINGULAR DIFFERENTIAL EQUATIONS IN HILBERT SPACE*

JOHN LAGNESE†

Abstract. A singular differential equation in a Hilbert space H is one of the form

$$A\frac{du(t)}{dt} - Bu(t) = f(t),$$

where A and B are linear operators in H which may be unbounded and A may have zero as a spectral point. In the current paper we obtain necessary and sufficient conditions for solvability and uniqueness of solution of such equations. Representation formulas and eigenfunction expansions for the solutions are also obtained. We show that our results apply to a large class of boundary value problems for certain nonclassical partial differential equations. This class contains in particular equations which occur in various physical problems such as fluid flow through a fissured rock, shear in second order fluids, soil mechanics and thermodynamics.

1. Introduction. Various problems in mathematical physics lead to boundary value problems for the equation

$$(1.1)\qquad (1 - \gamma\mathscr{A})\frac{\partial u}{\partial t} - \mathscr{B}u = f,$$

where γ is some nonzero real physical constant and $\mathscr{A}$ and $\mathscr{B}$ are elliptic partial differential operators in the space variables with the order of $\mathscr{A}$ at least as great as the order of $\mathscr{B}$. For example, (1.1) with $\mathscr{A} = \mathscr{B} =$ Laplacian occurs in the theory of flows of second order fluids [5], [14], consolidation of clay [13], and seepage of fluid through a fissured rock [2] and thermodynamics [4]; cf. [1]. In the next to last application, for example, the constant γ represents a characteristic of the fissured rock and decreases as the degree of fissuring increases. In the limit (i.e., when $\gamma = 0$), (1.1) coincides with the parabolic equation arising in the classical theory of seepage of a fluid under elastic conditions.

Let us consider $\mathscr{A}$ and $\mathscr{B}$ as certain unbounded operators A and B in, say, some Hilbert space H and write (1.1) as

$$(1.2)\qquad (1 - \gamma A)\frac{du}{dt} - Bu = f.$$

If γ^{-1} is not in the spectrum of A, (1.2) is equivalent to

$$\frac{du}{dt} - (1 - \gamma A)^{-1}Bu = (1 - \gamma A)^{-1}f,$$

and in applications the operator $(1 - \gamma A)^{-1}B$ is a bounded operator in some other function space. In this case the integration of (1.1) poses no serious problems and very strong and complete results are known concerning existence, uniqueness, regularity and various other properties of solutions, not only for (1.1) but for much more general classes of equations (see e.g. [8]–[12], [15]). However the situation is

* Received by the editors April 7, 1972.
† Department of Mathematics, Georgetown University, Washington, D.C. 20007.

more complex if γ^{-1} is a spectral point of A. As we shall show, in this case (1.1) may not have solutions (at least in the sense we shall use) even for very smooth functions f. And even when (1.1) has a solution satisfying some prescribed initial condition

$$u(0) = u_0 \in H, \tag{1.3}$$

the solution to this initial value problem may not be unique.

That anomalies of this sort can occur seems to have first been observed by Coleman, Duffin and Mizel [5] in the case where $\gamma = -1$ and $\mathscr{A} = \mathscr{B} = \partial^2/\partial x^2$. Among other things, they gave necessary conditions in order that (1.1) have a solution satisfying

$$u(x,0) = u_0(x), \qquad 0 < x < \pi, \quad u(0,t) = u(\pi,t) = 0, \quad t \geqq 0,$$

and also proved that the solution is unique whenever it exists. Later, Showalter [11, § 2] considered the singular problem (1.2) assuming $B = A = A^*$ and that A^{-1} exists and is compact (so that γ is a characteristic number of A of finite multiplicity). Necessary and sufficient conditions were given for the unique solvability of (1.2) subject to the initial condition (1.3). It was shown that existence of solutions to this problem implies uniqueness and it follows easily from the results of [11] that (1.2) always has solutions in this case whenever f is continuously differentiable in H. Similar results were obtained in [9, § 3] in the case where $B = A$ is not necessarily self-adjoint but has compact inverse, usually under the additional assumption that the root vectors of A form a basis of one sort or another for H.

The purpose of the present note is to give general existence-uniqueness theory for (1.2) when A is self-adjoint and γ^{-1} is an eigenvalue of A. The multiplicity of γ^{-1} may be infinite and, in the general theory developed in § 3, we place no further restrictions on A. As for B, we assume that $D(B)$ (the domain of B) and $D(B^*)$ contain $D(A)$ and that E, the eigenspace of A corresponding to γ^{-1}, is an invariant subspace of B. In applications to the differential equation (1.1), the first assumption on B means roughly that the order of $\mathscr{B}$ does not exceed that of $\mathscr{A}$ while the condition on the invariance of E under B is motivated by applications to physical problems. While this assumption could be replaced by some other or even eliminated altogether. it seems to be the condition which leads to the most natural and satisfactory results. We note that our results contain in particular those in [5], [11] mentioned above.

Almost all of our results give both necessary and sufficient conditions in order that (1.2) have some property or another. For example, we prove that (1.2) has a solution for every C^∞-function f if and only if E is contained in the range of B (Corollary 3.1); that solutions of (1.2) are uniquely determined by (1.3) if and only if the restriction of B to E is injective (Corollary 3.2), and so forth. In addition we obtain in Theorem 3.2 a representation formula for solutions of (1.2). We close § 3 by applying our results to the problem considered in [5] previously mentioned.

In § 4, under additional hypotheses on A and B, we obtain eigenfunction expansions for solutions of (1.2). Here we suppose that the spectrum of A consists of a discrete sequence of eigenvalues of finite multiplicity having no finite accumulation point and that B leaves invariant each eigenspace of A, a condition which we prove to be equivalent to $BA \subset AB$. We show that these hypotheses are satisfied by a large class of boundary value problems for (1.1). In particular they hold when B is the realization in $L^2(\Omega)$ of the elliptic operator $\mathscr{B}$ under Dirichlet boundary conditions, $\mathscr{A}$ is a certain polynomial in $\mathscr{B}$ and A the realization of $\mathscr{A}$ in $L^2(\Omega)$ under a certain system of normal boundary conditions.

In § 2 we state our hypotheses and develop some preliminary material. The final result of that section, Theorem 2.2, is the only one needed in § 3 and § 4.

Finally we remark that all of our results are valid when $\dim E = 0$, i.e., when γ^{-1} is in the resolvent set of A. But in this case more general results than those presented here are already known. However, it is the singular problem which is of primary interest here.

2. Preliminaries. H shall denote a complex Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. Let A be a closed, densely defined linear operator in H and B a linear operator in H such that

(2.1)*B is a closed operator and*

$$D(B) \cap D(B^*) \supset D(A).$$

LEMMA 2.1. *There are positive constants C and C^* such that for all $u \in D(A)$,*

$$|Bu| \leqq C(|u| + |Au|),$$

$$|B^*u| \leqq C^*(|u| + |Au|).$$

The proof can be found in [7, p. 121], for example, and is a simple consequence of the closed graph theorem.

Let $E = \{u : Au = 0\}$. Since A is a closed operator, E is a closed subspace of H and we may write

$$H = E \oplus H_1,$$

where $H_1 = \overline{Rg(A^*)}$, the closure of the range of A^*.

Let V_A (respectively, V_B) denote the linear space $D(A) \cap H_1$ (respectively, $D(B) \cap H_1$) endowed with the norm

$$|u|_{V_A} = |u| + |Au|$$

(respectively,

$$|u|_{V_B} = |u| + |Bu|).$$

V_A and V_B are Banach spaces each of which is dense in H_1, and the injections of V_A and V_B into H_1 are continuous. Moreover, $V_A \subset V_B$ and Lemma 2.1 implies that the injection of V_A into V_B is continuous.

Next we introduce the Banach space $V_{B^*} = D(B^*) \cap H_1$ with the graph norm

$$|u|_{V_{B^*}} = |u| + |B^*u|.$$

Then $V_A \subset V_{B^*}$ with continuous injection. Thus if V is the closure of V_A in V_{B^*} we have

$$V_A \subset V \subset H_1,$$

each space is dense in the one that follows and the injections of V_A into V and of V into H_1 are continuous. Thus if we identify the Hilbert space H_1 with its dual and denote by V' and V'_A the duals of V and V_A, respectively, we may write

$$V_A \subset V \subset H_1 \subset V' \subset V'_A,$$

each space being dense in the one that follows with continuous injection. We remark that the $V - V'$ duality $\langle \cdot , \cdot \rangle_V$ coincides with the inner product $(\cdot , \cdot)$ on $V \times H_1$ and similarly for the $V_A - V'_A$ duality $\langle \cdot , \cdot \rangle_{V_A}$ on $V_A \times H_1$.

In what follows we shall assume that A satisfies the following:

(2.2)*A is a self-adjoint operator having closed range.*

Condition (2.2) implies that zero lies either in the resolvent set or point spectrum of A. One can prove that for any self-adjoint operator A, the injection of V_A into H_1 is compact if and only if the following holds: $Rg(A)$ is closed and the spectrum of A consists of a discrete sequence of eigenvalues having no finite accumulation point, each nonzero eigenvalue being of finite multiplicity. This fact is implicit in the proof of the proposition in the Appendix.

Let A_1 denote the restriction of A to V_A, B_1 the restriction of B to V_B and let P be the orthogonal projection of H onto E. Then A_1 is a continuous bijection of V_A onto H_1 and $(I - P)B_1$ a continuous mapping of V_B into H_1.

LEMMA 2.2. *A_1 has a unique linear extension to a continuous bijection $\hat{A}_1$ of H_1 onto V'_A. $(I - P)B_1$ has a unique linear extension to a continuous mapping $\hat{B}_1$ of H_1 into V'.*

Proof. We first prove the statement concerning $(I - P)B_1$. Since this operator maps $V_B \subset H_1$ into $H_1 \subset V'$, we may consider $(I - P)B_1$ as a mapping from the dense set V_B in H_1 into V'. Considered in this way, we show that $(I - P)B_1$ is bounded.

Let $u \in V_B$ and $\langle \cdot , \cdot \rangle_V$ denote the $V - V'$ duality; this duality coincides with $(\cdot , \cdot)$ on $V \times H_1$. Thus for all $v \in V \subset D(B^*) \cap H_1$,

$$\langle v, (I - P)B_1 u\rangle_V = (v, (I - P)B_1 u) = (v, Bu) = (B^* v, u).$$

Taking the supremum over the set $\{v \in V : |v|_V = 1\}$ we obtain

$$|(I - P)B_1 u|_{V'} \leqq |u| \sup |B^* v| \leqq (\text{const.})|u|,$$

since the restriction of B^* to V is a bounded linear operator from V into H. Therefore $(I - P)B_1$ may be extended by continuity to a bounded linear operator $\hat{B}_1$ from H_1 into V'.

In a similar way one proves that A_1 is a bounded linear operator from the dense set V_A in H_1 into V'_A and may therefore be extended by continuity to a bounded linear operator $\hat{A}_1$ from H_1 into V'_A. Moreover, if $u \in V_A$ and $v = (A_1^{-1} u)/|A_1^{-1} u|_{V_A}$ we have

$$|A_1 u|_{V'_A} \geqq |\langle v, A_1 u \rangle_{V_A}| = |(A_1 v, u)| = \frac{|u|^2}{|A_1^{-1} u|_{V_A}} \geqq (\text{const.})|u|,$$

since A_1 is a continuous bijection of V_A onto H_1. Thus $\hat{A}_1$ has a bounded inverse from $Rg(\hat{A}_1) \subset V'_A$ onto H_1. But $Rg(A_1) = H_1$ is dense in V'_A and it follows that $Rg(\hat{A}_1) = V'_A$.

THEOREM 2.1. *Let* $f_1 \in C([0, \infty), V'_A)$ *and* $u_1 \in H_1$. *There is one and only one function* $u \in C'([0, \infty), H_1)$ *such that*

$$\hat{A}_1 u'(t) - \hat{B}_1 u(t) = f_1(t), \qquad t \geqq 0, \tag{2.3}$$

$$u(0) = u_1. \tag{2.4}$$

Proof. By Lemma 2.2, (2.3) is equivalent to

$$u'(t) - \hat{C}u(t) = (\hat{A}_1)^{-1} f_1(t), \qquad t \geqq 0,$$

where $\hat{C} = (\hat{A}_1)^{-1}\hat{B}_1$. Since $\hat{B}_1$ is a bounded operator from H_1 into V' and the injection of V' into V'_A is continuous, $\hat{C}$ is a bounded operator on H_1. Therefore $\hat{C}$ is the generator of a uniformly continuous group $\{e^{t\hat{C}}: -\infty < t < +\infty\}$ of bounded operators on H_1. It follows that (2.3), (2.4) has a unique solution given by

$$u(t) = e^{t\hat{C}}u_1 + \int_0^t e^{(t-s)\hat{C}}(\hat{A}_1)^{-1} f_1(s)\, ds.$$

THEOREM 2.2. *Let* $f \in C([0, \infty), H_1)$ *and* $u_1 \in V_B$. *There is one and only one function* $u \in C'([0, \infty), H_1)$ *such that* $u(t) \in D(B), u'(t) \in D(A), u(0) = u_1$ *and*

$$Au'(t) - (I - P)Bu(t) = f_1(t), \qquad t \geqq 0. \tag{2.5}$$

Moreover, $u \in C'([0, \infty), V_B)$ *and* $u' \in C([0, \infty), V_A)$.

Proof. Uniqueness follows from Theorem 2.1. For if $u \in C'([0, \infty), H_1)$ satisfies (2.5) and $u(0) = u_1$ then u is the solution of (2.3), (2.4).

The existence proof is similar to the proof of Theorem 2.1. In fact $(I - P)B_1$ is a bounded operator from V_B into H_1 and A_1 is a continuous bijection of V_A onto H_1. Since the injection of V_A into V_B is continuous, $A_1^{-1}(I - P)B_1$ is a bounded operator on V_B. It therefore follows that

$$A_1 u'(t) - (I - P)B_1 u(t) = f_1(t), \qquad t \geqq 0,$$

$$u(0) = u_1 \in V_B$$

has a unique solution in the class $C'([0, \infty), V_B)$ provided $f_1 \in C([0, \infty), H_1)$. Moreover,

$$u'(t) = A_1^{-1}[(I - P)B_1 u(t) + f_1(t)]$$

is easily seen to be continuous as a function in V_A. The proof is completed by recalling that $A_1 = A$ on V_A and $B_1 = B$ on V_B.

3. Singular differential equations. We consider the problem

$$Au'(t) - Bu(t) = f(t), \qquad t \geqq 0, \tag{3.1}$$

$$u(0) = u_0. \tag{3.2}$$

DEFINITION. A *solution* of (3.1) is a function $u \in C'([0, \infty), H)$ such that $u(t) \in D(B)$, $u'(t) \in D(A)$ and (3.1) is satisfied for all $t \geqq 0$.

Note that the definition requires $u(0) = u_0 \in D(B)$ but does not require Au' or Bu to be continuous. However, it turns out that each of these functions is continuous whenever $Au' - Bu \in C([0, \infty), H)$ (Corollary 3.4). We also remark that we could consider (3.1) on *any* interval $[t_0, t_1)$. All of our results remain valid after obvious changes in their statements and in the above definition.

Let B_E denote the restriction of B to the subspace E and let $E_0 = \{u \in E: Bu = 0\}$. This subspace is closed in E since B_E is a bounded operator. If we write

$$E = E_0 \oplus E_1$$

and let B_e be the restriction of B to E_1, then B_e maps E_1 one-to-one and boundedly onto $Rg(B_E)$. Moreover, B_e has a bounded inverse if and only if the following holds:

(3.3) *The range of B_E is closed in H.*

This condition is automatically satisfied if $\dim E < \infty$. It also holds whenever E is an invariant subspace of B and the injection into E of E_1, endowed with the norm of the graph of B, is compact (cf. the remark following (2.2)).

We now state the main results of this section.

THEOREM 3.1. *Assume* (2.1), (2.2) *and that E is an invariant subspace of B. Let $f \in C([0, \infty), H)$, $u_0 \in D(B)$ and set*

$$f_0(t) \equiv Pf(t), \quad f_1(t) \equiv (I - P)f(t).$$

Let u_1 be the unique solution in $C'([0, \infty), H_1)$ of (2.5) *satisfying*

$$u_1(0) = (I - P)u_0.$$

In order that (3.1), (3.2) *have a solution the following conditions are necessary*:

$$f_0 \in C'([0, \infty), E), \tag{3.4}$$

$$f(0) + Bu_0 \in H_1, \tag{3.5}$$

$$f_0(t) + PBu_1(t) \in Rg(B_E), \qquad t \geqq 0. \tag{3.6}$$

If in addition we suppose (3.3) *holds, then* (3.4), (3.5) *and* (3.6) *are also sufficient conditions in order that* (3.1), (3.2) *have a solution.*

From this result we obtain the following existence-nonexistence theorem for (3.1).

COROLLARY 3.1. *Assume* (2.1), (2.2), *that $B = B^*$ and that E is an invariant subspace of B. If $E \not\subset Rg(B)$, there is an analytic function $f:[0, \infty) \to H$ such that* (3.1) *has no solution on any interval $[0, t_1)$, $t_1 > 0$. If $E \subset Rg(B)$, then for any $f \in C'([0, \infty), H)$ equation* (3.1) *has a solution on $[0, \infty)$.*

The following gives a representation formula for solutions of (3.1).

THEOREM 3.2. *Assume* (2.1), (2.2) *and that E is an invariant subspace of B. Let $f \in C([0, \infty), H)$, $u_0 \in D(B)$ and suppose u is a solution of* (3.1), (3.2). *Then*

$$u(t) = -B_e^{-1}(f_0(t) + PBu_1(t)) + v(t) + u_1(t), \tag{3.7}$$

where $v \in C'([0, \infty), E_0)$ such that $v(0) = 0$. Conversely, if (3.3)–(3.6) *are satisfied, then* (3.7) *is a solution of* (3.1), (3.2) *for each such $v(t)$.*

Concerning uniqueness of solution we have the following.

COROLLARY 3.2. *Assume* (2.1), (2.2) *and that E is an invariant subspace of B. Then* (3.1), (3.2) *has at most one solution if and only if B_E is injective.*

When B_E is injective and $\dim E < \infty$, Theorems 3.1 and 3.2 yield the following concise result.

COROLLARY 3.3. *Assume* (2.1), (2.2) *and that* E *is an invariant subspace of* B. *In addition suppose* B_E *is injective and* $\dim E < \infty$. *Let* $f \in C([0, \infty), H)$ *and* $u_0 \in D(B)$. *Then* (3.1), (3.2) *has a solution if and only if* $f_0 \in C'([0, \infty), E)$ *and* $f(0) + Bu_0 \in H_1$. *When these conditions are satisfied, the solution is unique and given by*

$$u(t) = -B_E^{-1}(f_0(t) + PBu_1(t)) + u_1(t).$$

From Theorem 3.2 we also obtain the following regularity result (cf. Theorem 2.2).

COROLLARY 3.4. *Assume* (2.1), (2.2), (3.3) *and that* E *is an invariant subspace of* B. *Let* $f \in C([0\ \infty), H)$ *and suppose* u *is a solution of* (3.1). *Then* $u \in C'([0, \infty), D(B))$ *and* $u' \in C([0, \infty), D(A))$ *where* $D(A)$ *and* $D(B)$ *are equipped with the norms of the graph of* A *and* B, *respectively*.

Proof of Theorem 3.1. Let u be a solution of (3.1), (3.2) and write

$$u(t) = v_0(t) + v_1(t), \qquad t \geqq 0, \tag{3.8}$$

where

$$v_0(t) \equiv Pu(t), \quad v_1(t) \equiv (I - P)u(t).$$

Then $v_0 \in C'([0, \infty), E)$ and $v_1 \in C'([0, \infty), H_1)$. Substituting (3.8) into (3.1) gives

$$Av_1'(t) - B(v_0(t) + v_1(t)) = f(t), \qquad t \geqq 0. \tag{3.9}$$

Since E is an invariant subspace of B and H_1 an invariant subspace of A, the last equation is equivalent to the system

$$Av_1'(t) - (I - P)Bv_1(t) = f_1(t), \qquad t \geqq 0, \tag{3.10}$$

$$Bv_0(t) + PBv_1(t) = -f_0(t), \qquad t \geqq 0. \tag{3.11}$$

As $v_1(0) = (I - P)u_0$ it follows from (3.10) and Theorem 2.2 that $v_1(t) \equiv u_1(t)$ and in addition $v_1 \in C'([0, \infty), V_B)$. This last fact implies that $PBv_1 \in C'([0, \infty), E)$. Since B_E is a bounded operator we also have $Bv_0 \in C'([0, \infty), E)$ so that (3.11) implies $f_0 \in C'([0, \infty), E)$. Moreover, (3.11) is solvable for v_0 only if $f_0(t) + PBu_1(t) \in Rg(B_E)$ for all $t \geqq 0$.

Returning to (3.9) we see that $f(t) + Bu(t) = Av_1'(t) \in H_1$ for all $t \geqq 0$ and in particular $f(0) + Bu_0 \in H_1$.

We have therefore proved that (3.4), (3.5) and (3.6) are necessary conditions in order that (3.1), (3.2) have a solution. That these conditions, together with (3.3), are also sufficient is a consequence of Theorem 3.2 which we now prove.

Proof of Theorem 3.2. Suppose that $u(t)$ given by (3.8) is a solution of (3.1), (3.2). Then (3.4), (3.5) and (3.6) hold. As already noted, necessarily $v_1(t) \equiv u_1(t)$ and $v_0(t)$ satisfies (3.11). Using the decomposition $E = E_0 \oplus E_1$ we may write

$$v_0(t) = v(t) + w(t), \qquad v(t) \in E_0, \quad w(t) \in E_1.$$

Then $v \in C'([0, \infty), E_0)$ and

$$w(t) = -B_e^{-1}(f_0(t) + PBu_1(t)), \qquad t \geqq 0.$$

Condition (3.5) is equivalent to

$$f_0(0) + PBu_0 = 0$$

so that

$$w(0) = -B_e^{-1}(f_0(0) + PB(I - P)u_0) = Pu_0 = v_0(0). \tag{3.12}$$

Thus $v(0) = 0$ and

$$u(t) = -B_e^{-1}(f_0(t) + PBu_1(t)) + v(t) + u_1(t), \tag{3.13}$$

where $v \in C'([0, \infty), E_0)$ such that $v(0) = 0$.

Conversely, given such a function v we must show that (3.13) is a solution of (3.1), (3.2). Using (3.12) we easily see that $u(0) = u_0$. Since $f_0 \in C'([0, \infty), E)$, $u_1 \in C'([0, \infty), V_B)$ and B_e^{-1} is a bounded operator on $Rg(B_E)$ because of (3.3), the first term in (3.13), is of class $C'([0, \infty), E_1)$. Thus $u \in C'([0, \infty), H)$ and $Au'(t) = Au_1'(t)$ since the first two terms in (3.13) take values in E. Therefore

$$\begin{aligned} Bu(t) &= -(f_0(t) + PBu_1(t)) + Bu_1(t) \\ &= -(f_0(t) + PBu_1(t)) + (Au_1'(t) + PBu_1(t) - f_1(t)) \\ &= Au'(t) - f(t). \end{aligned}$$

Proof of Corollary 3.1. If there exists $f \in E - Rg(B)$, set $f(t) = tf$. Since $B = B^*$, $PBu_1(t) \equiv 0$ and so (3.6) does not hold for this $f(t)$ for any $t > 0$.

If $E \subset Rg(B)$ then $E = Rg(B_E)$ so that (3.3), (3.6) are satisfied. If $f \in C'([0, \infty), H)$ then (3.4) holds. We therefore have only to choose $u_0 \in E$ such that $Pf(0) + Bu_0 = 0$ in order to satisfy (3.5).

Proof of Corollary 3.2. If u is a solution of (3.1), (3.2) with homogeneous data, then $u_1(t) \equiv 0$ and by Theorem 3.2, $u \in C'([0, \infty), E_0)$. If B_E is injective we have $E_0 = \{0\}$ so that $u(t) \equiv 0$. Conversely, if the homogeneous problem has only the trivial solution then $E_0 = \{0\}$ since, otherwise, the function $u(t) = tv$ $(v \in E_0)$ gives a nontrivial solution to the homogeneous problem.

Proof of Corollary 3.3. Uniqueness follows from the previous corollary. Since $\dim E < \infty$, B_E is a bijection of E onto itself and so (3.3) and (3.6) are automatically satisfied. Now apply Theorems 3.1 and 3.2.

Proof of Corollary 3.4. $u(t)$ is given by (3.7) where $u_1 \in C'([0, \infty), V_B)$. The first two terms are differentiable with respect to the $|\cdot|$ norm and take values in E. Since $|Bw| \leqq C|w|$ on E it follows that these terms are of class $C'([0, \infty), D(B))$. Finally, $Au' = Bu + f$ is continuous as a function in H, that is, $u' \in C([0, \infty), D(A))$.

By way of example (other examples are discussed in the next section), we consider the problem

$$\left(1 + \frac{\partial^2}{\partial x^2}\right)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F(x, t), \qquad 0 < x < m\pi, \quad t \geqq 0, \tag{3.14}$$

$$u(x, 0) = u_0(x), \qquad 0 < x < m\pi, \tag{3.15}$$

$$u(0, t) = u(m\pi, t) = 0, \qquad t \geqq 0, \tag{3.16}$$

where m is a fixed positive integer. We take $H = L^2(0, m\pi)$, $A = 1 + d^2/dx^2$ and $B = d^2/dx^2$, where $D(A) = D(B)$ is the closure in the Sobolev space $H^2(0, m\pi)$ of the class of functions u in $C^2([0, m\pi])$ which satisfy (3.16). Then A and B are both self-adjoint and the eigenspace E is 1-dimensional: $E = \text{span}\,[\sqrt{2/m\pi}\sin x]$. If

$u \in E$ then $Bu = d^2u/dx^2 = -u \in E$ and $Bu = 0$ if and only if $u = 0$. Thus B leaves E invariant and $B_E = -I$ is obviously injective. Moreover, $Rg(A)$ is closed since, as is well known, the injection of $H^2(0, m\pi)$ into $L^2(0, m\pi)$ is compact (see Appendix). Let the mapping $t \to F(\cdot, t)$ of $[0, \infty)$ into $L^2(0, m\pi)$ be continuous and $u_0 \in D(B)$. Then from Corollary 3.3 we find that (3.14)–(3.16) has a solution if and only if the mapping $t \to f_0(\cdot, t)$ of $[0, \infty)$ into E defined by

$$f_0(x, t) = (PF)(x, t) = \frac{2}{m\pi} \sin x \int_0^{m\pi} F(\xi, t) \sin \xi \, d\xi$$

is continuously differentiable and

$$\int_0^{m\pi} \left[F(\xi, 0) + \frac{d^2 u_0(\xi)}{d\xi^2} \right] \sin \xi \, d\xi = \int_0^{m\pi} [F(\xi, 0) - u_0(\xi)] \sin \xi \, d\xi = 0.$$

Moreover, the solution is necessarily unique. These conclusions imply the results of Theorems 3.1, 5.1, and 5.3 of [5].

We next use the representation formula (3.7) to obtain an explicit expression for the solution. Since B_E is injective we have $v(t) \equiv 0$ and $B_e = B_E = -I$. Also, $PBu_1(t) \equiv 0$ since B is self-adjoint and leaves E, and thus H_1, invariant. From (3.7) we therefore obtain

$$u(x, t) = \frac{2}{m\pi} \sin x \int_0^{m\pi} F(\xi, t) \sin \xi \, d\xi + u_1(x, t).$$

u_1 may be obtained as an expression in the eigenfunctions of A as in the next section or as in [11, § 2] and has the form

$$u_1(x, t) = \frac{2}{m\pi} \sum_{\substack{i=1 \\ i \neq m}}^{\infty} v_i(t) \sin \frac{ix}{m},$$

where

$$v_i(t) = \exp\left(\frac{i^2 t}{i^2 - m^2}\right) \int_0^{m\pi} u_0(\xi) \sin \frac{i\xi}{m} \, d\xi$$
$$+ \frac{m^2}{m^2 - i^2} \int_0^t \exp\left[(t - s)\frac{i^2}{i^2 - m^2}\right] \int_0^{m\pi} F(\xi, s) \sin \frac{i\xi}{m} \, d\xi \, ds.$$

The above series, together with the series obtained by termwise differentiation with respect to t, converges in $H^2(0, m\pi)$ uniformly on bounded subsets of $[0, \infty)$.

4. Eigenfunction expansions of solutions. In this section we obtain, for each solution of (3.1), (3.2), an expansion in the eigenfunctions of A. The coefficients in the expansion will be determined through systems of first order linear differential equations with constant coefficients. The conditions which we require on A and B are somewhat more restrictive than those of the last section. First of all we assume:

(4.1) *A is a self-adjoint operator such that the injection into H of $D(A)$, with the norm of the graph of A, is compact.*

The compactness of this injection implies that the range of A is closed and that the spectrum of A consists of a discrete sequence of eigenvalues, each of finite multiplicity, having no finite accumulation point. (See Appendix.) In particular, $\dim E < \infty$. If A is the realization in $H = L^2(\Omega)$ of a formally self-adjoint elliptic differential operator under Dirichlet boundary conditions, then (4.1) is satisfied (see example below).

We shall further assume:

(4.2) *B is a closed operator such that $D(B) \cap D(B^*) \supset D(A)$. Each eigenspace of A is invariant under B and $B|_E$ is a normal operator.*

Concerning this condition we have the following proposition.

PROPOSITION. *Let* (4.1) *hold and B be a densely defined operator such that $D(B) \cap D(B^*) \supset D(A)$. Then each eigenspace of A is invariant under B if and only if $BA \subset AB$.*

Proof. Let $\{\lambda_i\}_{i=1}^{\infty}$ be the sequence of *distinct* nonzero eigenvalues of A, arranged in order of nondecreasing magnitude, and $\{\phi_k\}_{k=n_i+1}^{n_i+1}$ be an orthonormal basis of eigenvectors of A for the eigenspace $E(\lambda_i)$. Here $n_1 < n_2 < \cdots$, where we set $n_1 = \dim E$. Because of (4.1), A_1^{-1} is a compact self-adjoint operator on H_1 whose eigenvalues are exactly $\{\lambda_i^{-1}\}_{i=1}^{\infty}$ and therefore

$$H_1 = \sum_{i=1}^{\infty} \oplus E(\lambda_i). \tag{4.3}$$

The condition $BA \subset AB$ clearly implies that each eigenspace of A is an invariant subspace of B. Conversely, suppose this condition is satisfied and $u \in D(BA)$. Writing $u = u_0 + u_1$ with $u_0 \in E$ and $u_1 \in H_1$, it suffices to show that $u_1 \in D(AB)$ and $BAu_1 = ABu_1$.

Since each $E(\lambda_i)$ is invariant under B it follows from (4.3) that each $E(\lambda_i)$ is also invariant under B^*. Thus from (4.3) we have

$$\begin{aligned} BAu_1 &= \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} (BAu_1, \phi_k)\phi_k = \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} (u_1, AB^*\phi_k)\phi_k \\ &= \sum_{i=1}^{\infty} \lambda_i \sum_{k=n_i+1}^{n_{i+1}} (u_1, B^*\phi_k) = \sum_{i=1}^{\infty} A\psi_i, \end{aligned}$$

where

$$\psi_i = \sum_{k=n_i+1}^{n_{i+1}} (Bu_1, \phi_k)\phi_k.$$

We have

$$Bu_1 = \lim_{n\to\infty} \sum_{i=1}^{n} \psi_i, \qquad BAu_1 = \lim_{n\to\infty} A\left(\sum_{i=1}^{n} \psi_i\right).$$

Since A is closed it follows that $Bu_1 \in D(A)$ and $ABu_1 = BAu_1$.

From (4.3) we see that the spectral expansion of A_1 is given by

$$A_1 u = \sum_{i=1}^{\infty} \lambda_i \sum_{k=n_i+1}^{n_{i+1}} (u, \phi_k)\phi_k, \qquad u \in D(A_1). \tag{4.4}$$

Let $\{\mu_i\}_{i=1}^{m_1} (m_1 \leqq n_1)$ be the nonzero eigenvalues of $B_E = B|_E$, taking into account algebraic multiplicities. Choose an orthonormal basis $\{\phi_i\}_{i=1}^{n_1}$ for E consisting of eigenvectors of B_E such that $\{\phi_i\}_{i=1}^{m_1}$ correspond to the nonzero eigenvalues. Then

$$B_e^{-1} = \sum_{i=1}^{m_1} \mu_i^{-1}(\cdot, \phi_i)\phi_i, \qquad B_E\phi_i = 0, \qquad i = m_1 + 1, \cdots, n_1.$$

In what follows we shall use, for $i = 1, 2, \cdots$, the notation

$$\mathscr{B}_i = [(\phi_k, B\phi_j)]_{j,k=n_i+1}^{n_{i+1}},$$
$$\mathring{U}_i = [(u_0, \phi_k)]_{k=n_i+1}^{n_{i+1}},$$
$$F_i(t) = [(f(t), \phi_k)]_{k=n_i+1}^{n_{i+1}}.$$

THEOREM 4.1. *Assume* (4.1) *and* (4.2). *Let* $u_0 \in D(B)$ *and* $f \in C([0, \infty), H)$. *Then* (3.1), (3.2) *has a solution if and only if*

$$(4.5) \qquad (f(\cdot), \phi_k) \in C'([0, \infty)), \qquad k = 1, 2, \cdots, m_1,$$

$$(4.6) \qquad (f(t), \phi_k) \equiv 0, \qquad k = m_1 + 1, \cdots, n_1,$$

$$(4.7) \qquad (f(0) + Bu_0, \phi_k) = 0, \qquad k = 1, 2, \cdots, n_1.$$

When these three conditions hold, a function $u:[0, \infty) \to H$ *is a solution of* (3.1), (3.2) *if and only if*

$$(4.8) \qquad u(t) = \sum_{k=1}^{m_1} \mu_k^{-1}(f(t), \phi_k)\phi_k + \sum_{k=m_1+1}^{n_1} g_k(t)\phi_k + \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} v_k(t)\phi_k,$$

where

$$g_k \in C'([0, \infty)), \quad g_k(0) = 0, \quad k = m_1 + 1, \cdots, n_1,$$

and $U_i(t) \equiv \{v_k(t)\}_{k=n_i+1}^{n_{i+1}}$ *is given by*

$$(4.9) \qquad U_i(t) = e^{t\lambda_i^{-1}\mathscr{B}_i}\mathring{U}_i + \lambda_i^{-1} \int_0^t e^{(t-s)\lambda_i^{-1}\mathscr{B}_i}F_i(s)\,ds, \qquad i = 1, 2, \cdots.$$

Proof. It is easily seen that (4.5), (4.6) and (4.7) are equivalent to the three conditions (3.4), (3.5) and (3.6) and, since $\dim E < \infty$, (3.3) also holds. Thus (3.1), (3.2) has a solution if and only if (4.5)–(4.7) hold. Formula (4.8) is obtained from (3.7). In fact $PBu_1(t) \equiv 0$, so the first term in (3.7) is exactly

$$\sum_{k=1}^{m_1} \mu_k^{-1}(f(t), \phi_k)\phi_k.$$

Moreover, the function v in (3.7) necessarily has the form of the second sum in (4.8).

Let u_1 be the unique solution in $C'([0, \infty), H_1)$ of

$$Au_1'(t) - Bu_1(t) = (I - P)f(t), \qquad t \geqq 0,$$

$$(4.10) \qquad u_1(0) = (I - P)u_0.$$

If we set $v_k(t) = (u_1(t), \phi_k)$, then

$$(4.11) \qquad u_1(t) = \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} v_k(t)\phi_k.$$

Using (4.3) and (4.4) we obtain from (4.10)

$$(4.12) \qquad \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} [\lambda_i v_k'(t) - (Bu_1(t), \phi_k) - (f(t), \phi_k)]\phi_k = 0.$$

Since B^* leaves $E(\lambda_i)$ invariant we have for $k = n_i + 1, \cdots, n_{i+1}$

$$(Bu_1(t), \phi_k) = (u_1(t), B^*\phi_k) = \sum_{j=n_i+1}^{n_{i+1}} v_j(t)(B^*\phi_k, \phi_j).$$

Therefore (4.12) implies that for $i = 1, 2, \cdots,$

$$\lambda_i v_k'(t) - \sum_{j=n_i+1}^{n_{i+1}} (\phi_k, B\phi_j)v_j(t) = (f(t), \phi_k), \qquad k = n_i + 1, \cdots, n_{i+1}. \tag{4.13}$$

The system (4.13), together with the initial conditions

$$v_k(0) = (u_0, \phi_k), \qquad k = n_i + 1, \cdots, n_{i+1}, \tag{4.14}$$

has a unique solution given by (4.9). It follows from (3.7) that a function u is a solution of (3.1), (3.2) if and only if u has the form (4.8) with the functions $v_k(t)$ determined by (4.13) and (4.14). The proof is complete.

Example. In a bounded open set $\Omega \subset \mathbb{R}^n$ we consider the problem

$$(1 - \gamma\mathscr{A}(x, D))\frac{\partial u}{\partial t} - \mathscr{B}(x, D)u = F(x, t), \qquad (x, t) \in \Omega \times [0, \infty), \tag{4.15}$$

$$\left(\frac{\partial}{\partial n}\right)^{j-1} u = 0, \qquad \left(\frac{\partial}{\partial n}\right)^{l+k-1} \frac{\partial u}{\partial t} = 0, \tag{4.16}$$

$$j = 1, \cdots, l; k = 1, \cdots, m - l; (x, t) \in \partial\Omega \times [0, \infty).$$

In (4.15), γ is a nonzero real number,

$$\mathscr{A}(x, D) = \sum_{|\alpha| \leqq 2m} a_\alpha(x)D^\alpha, \qquad \mathscr{B}(x, D) = \sum_{|\alpha| \leqq 2l} b_\alpha(x)D^\alpha,$$

where $l \leqq m$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ for any multi-integer $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i \geqq 0$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We suppose $\mathscr{A}$ and $\mathscr{B}$ are each elliptic in $\bar{\Omega}$, with $\mathscr{A}$ formally self-adjoint, that Ω is of class C^{2m} and that the coefficients in $\mathscr{A}$ and $\mathscr{B}$ are of class $C(\bar{\Omega})$. If $n = 2$ we assume additionally that $\mathscr{A}$ and $\mathscr{B}$ satisfy the roots condition (see e.g., [3]).

Let $H = L^2(\Omega)$ and $H^k(\Omega)$ be the Hilbert space consisting of functions in $L^2(\Omega)$ whose derivatives, in the distribution sense, to order k belong to $L^2(\Omega)$. The norm in $H^k(\Omega)$ is

$$|u|_k = \left(\sum_{|\alpha| \leqq k} |D^\alpha u|^2\right)^{1/2}.$$

We define unbounded operators A and B in H as follows: $D(A)$ is the closure in $H^{2m}(\Omega)$ of those functions u in $C^{2m}(\bar{\Omega})$ which satisfy on $\partial\Omega$

$$\left(\frac{\partial}{\partial n}\right)^{j-1} u = 0 \tag{4.17}$$

for $j = 1, 2, \cdots, m$. For $u \in D(A)$ we set

$$Au = \mathscr{A}(\cdot, D)u(\cdot).$$

Similarly, $D(B)$ is the closure in $H^{2l}(\Omega)$ of functions in $C^{2l}(\bar{\Omega})$ which satisfy (4.17) on $\partial\Omega$ for $j = 1, 2, \cdots, l$, and $Bu = \mathscr{B}(\cdot, D)u(\cdot)$. Clearly $D(B) \supset D(A)$ and it is well known [3] that B is a closed operator such that $D(B^*) = D(B)$, that A is self-adjoint and, in particular, that $1 - \gamma A$ satisfies (4.1) and a fortiori (2.2). Thus if we write $f(t) = F(\cdot, t)$ and define a solution of (4.15), (4.16) to mean a solution of

$$(I - \gamma A)u'(t) - Bu(t) = f(t),$$

the results of § 3 may be applied to (4.15), (4.16). In particular, the conclusions of Theorem 3.2 are valid for this problem provided only that the subspace $\{u \in D(A): \gamma Au = u\}$ is invariant under B.

Suppose $\mathscr{B}$ is formally self-adjoint, so that B is a self-adjoint operator in H. Then the conclusions of Theorem 4.1 are valid for (4.15), (4.17) provided only that $BA \subset AB$. This last condition holds in all applications in which (4.15), (4.16) occurs since in the applications we have $A = c_1 B + c_2 I$ $(c_1 \neq 0)$, that is, $l = m$ and $\mathscr{A}(x, D) = c_1\mathscr{B}(x, D) + c_2$.

More generally, suppose that

$$\mathscr{A}(x, D) = P(\mathscr{B}(x, D)),$$

where $P(z)$ is a polynomial of degree p in the complex variable z with real coefficients. We may suppose the leading coefficient in P equals unity and for simplicity we assume the coefficients in B are of class $C^\infty(\bar{\Omega})$. We wish to show that $A = P(B)$ so that, in particular, $BA = AB$ and the conclusions of Theorem 4.1 can be applied. However this will not in general be true when $p > 1$ unless we redefine the domain of A. Thus let B be defined as above; B is then a self-adjoint operator which satisfies (4.1) with respect to $D(B)$. We set $D(A) = D(B^p)$ and $Au = \mathscr{A}(\cdot, D)u(\cdot)$ for $u \in D(A)$. As is well known, $D(B^p) = D(P(B)) \subset H^{2pl}(\Omega)$ and it is clear that $A = P(B)$. However we must show that the operator A so defined satisfies (4.1), for the boundary conditions associated with A are no longer the Dirichlet conditions but rather

$$R_j(x, D)u = 0, \qquad x \in \partial\Omega, \quad j = 1, 2, \cdots, m,$$

where

$$R_{k+q}(x, D) = \left(\frac{\partial}{\partial n}\right)^{k-1} [B(x, D)]^q, \qquad k = 1, \cdots, l; \quad q = 0, 1, \cdots, p - 1.$$

To show that A satisfies (4.1) we first observe that $A^* = P(B^*) = P(B) = A$ ([6, Thm. XII.2.6 and Cor. XII.2.8]). Let $\{\mu_i\}_{i=1}^{\infty}$ be the nonzero eigenvalues of B enumerated according to nondecreasing magnitude, taking into account algebraic multiplicities, and $\{\psi_i\}_{i=1}^{\infty}$ be the corresponding orthonormal basis of eigenvectors of B for $Rg(B)$. Then B has the spectral resolution

$$Bu = \sum_{i=1}^{\infty} \mu_i(u, \psi_i)\psi_i, \qquad u \in D(B),$$

so that

$$Au = \sum_{i=1}^{\infty} P(\mu_i)(u, \psi_i)\psi_i, \qquad u \in D(A).$$

We therefore see that the nonzero spectrum of A is $\{P(\mu_i): i = 1, 2, \cdots; P(\mu_i) \neq 0\}$ and each point of this set is an eigenvalue of A of finite multiplicity. Writing

$$Au = \prod_{j=1}^{p} (B - \beta_j)u, \qquad u \in D(A),$$

we see that zero is in the spectrum of A if and only if some β_j is an eigenvalue of B and moreover

$$E = \{u \in D(A): Au = 0\} = \sum_{j=1}^{p} \oplus \hat{E}(\beta_j),$$

where $\hat{E}(\beta_j) = \{u \in D(B): Bu = \beta_j u\}$. Thus zero is either in the resolvent set of A or is an eigenvalue of A of finite multiplicity. If A_1 is the restriction of A to $D(A) \cap [H \ominus E]$ and B_j the restriction of B to $D(B) \cap [H \ominus \hat{E}(\beta_j)]$ we have

$$A_1 u = \prod_{j=1}^{p} (B_j - \beta_j)u, \qquad u \in D(A_1).$$

Since β_j is in the resolvent set of B_j it follows that A_1 has a bounded inverse from $H \ominus E$ onto $D(A_1)$. Thus A has closed range and so satisfies (4.1).

Appendix. The purpose of this section is to prove the following result.

PROPOSITION. *Let A be a self-adjoint operator in a Hilbert space H. Then the injection into H of $D(A)$, endowed with the graph norm of A, is compact if and only if $Rg(A)$ is closed and the spectrum of A consists of a discrete sequence of eigenvalues, each of finite multiplicity, having no finite accumulation point.*

Proof. Write $H = E \oplus H_1$, $E = \{u \in D(A): Au = 0\}$ and let A_1 be the restriction of A to $D(A) \cap H_1$. Let us suppose first of all that $Rg(A)$ is closed and the spectrum of A is as described in the proposition. Denote by $\{\lambda_i\}$ the nonzero eigenvalues of A enumerated in order of nondecreasing magnitude. Then A_1 maps $D(A_1)$ one-to-one onto $Rg(A) = H_1$ and the spectrum of A_1 consists of the points $\{\lambda_i\}$, each of which is an eigenvalue of A_1 of finite multiplicity. Therefore A_1^{-1} is a bounded, self-adjoint operator whose spectrum consists of a discrete sequence of eigenvalues, each of finite multiplicity, which can accumulate only at zero. This implies that A_1^{-1} is a compact operator which in turn implies that the injection into H_1 of $D(A_1)$, endowed with the graph norm of A, is compact. Since, moreover, $\dim E < \infty$, the desired conclusion follows.

Conversely, let us suppose the injection is compact. Then $\dim E < \infty$. Assume for the moment that $Rg(A)$ is closed. Then A_1^{-1} exists as a bounded operator with range in $D(A_1)$. The compactness of the injection shows that A_1^{-1} is compact. It follows that $(\lambda - A_1)^{-1}$ exists as a compact operator for all λ except a discrete sequence of eigenvalues of A_1, each of finite multiplicity, having no finite accumulation point.

It therefore remains to show that $Rg(A)$ is closed. Since A is closed, it suffices to prove that for all $u \in D(A_1)$,

$$|u| \leqq C|A_1 u|$$

with some constant C not depending on u. If this inequality is not true, there is a sequence $\{u_n\}$ in $D(A_1)$ such that $|u_n| = 1$ and $A_1 u_n \to 0$. $\{|u_n| + |Au_n|\}$ is therefore bounded and so, by choosing an appropriate subsequence if necessary, we may suppose $\{u_n\}$ converges in H_1. Since $A_1 u_n \to 0$ we conclude that $\{u_n\}$ also converges in $D(A_1)$. The limit must be zero since A_1 maps $D(A_1)$ (with the graph norm of A_1) boundedly into H_1 and $A_1 u_n \to 0$. But this contradicts $|u_n| = 1$.

REFERENCES

[1] D. E. Amos, *On half-space solutions of a modified heat equation*, Quart. Appl. Math., 27 (1969), pp. 359–369.

[2] G. Barenblat, I. Zheltov and I. Kochiva, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech., 24 (1960), pp. 1286–1303.

[3] F. Browder, *Estimates and existence theorems for elliptic boundary value problems*, Proc. Nat. Acad. Sci. U.S.A., 45 (1959), pp. 355–372.

[4] P. J. Chen and M. E. Gurtin, *On a theory of heat conduction involving two temperatures*, Z. Angew. Math. Phys., 19 (1968), pp. 614–627.

[5] B. D. Coleman, R. J. Duffin and V. J. Mizel, *Instability, uniqueness and nonexistence theorems for the equation $u_t + u_{xtx} = u_{xx}$ on a strip*, Arch. Rational Mech. Anal., 19 (1965), pp. 100–116.

[6] B. D. Coleman and W. Noll, *An approximation theorem for functionals, with applications to continuum mechanics*, Ibid. 6 (1960), pp. 355–370.

[7] N. Dunford and J. Schwaritz, *Linear Operators, Part II*, John Wiley, New York, 1963.

[8] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966.

[9] J. Lagnese, *General boundary value problems for differential equations of Sobolev type*, this Journal, 3 (1972), pp. 105–119.

[10] ———, *Exponential stability of solutions of differential equations of Sobolev type*, this Journal, 3 (1972), pp. 625–636.

[11] R. E. Showalter, *Well-posed problems for a partial differential equation of order* $2m + 1$, this Journal, 1 (1970), pp. 214–231.

[12] R. E. Showalter and T. W. Ting, *Pseudoparabolic partial differential equations*, this Journal, 1 (1970), pp. 1–26.

[13] ———, *Asymptotic behavior of solutions of pseudoparabolic partial differential equations*, Ann. Mat. Pura Appl., 90 (1971), pp. 241–258.

[14] D. W. Taylor, *Research on Consolidation of Clays*, MIT Press, Cambridge, 1952.

[15] T. W. Ting, *Certain nonsteady flows of second order fluids*, Arch. Rational Mech. Anal., 14 (1963), pp. 1–26.

[16] M. I. Visik, *The Cauchy problem for equations with operator coefficients; mixed boundary value problems for systems of differential equations and approximation methods for their solutions*, Mat. Sb., (81) 39 (1956), pp. 51–148; English transl., Amer. Math. Soc. Transl. (2), 24 (1963), pp. 173–278.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

LIE THEORY AND THE APPELL FUNCTIONS F_1*

WILLARD MILLER, JR.†

Abstract. It is shown that $sl(5, \mathbb{C})$ is the dynamical symmetry algebra of the Appell functions F_1. This permits use of representation theory and the techniques of Weisner and Vilenkin to derive systematically a variety of addition theorems, generating functions and Mellin–Barnes integrals for the F_1.

Introduction. In a recent paper Ciftan [7] has shown that Appell functions F_1 arise naturally in the study of irreducible representations of the unitary groups $U(n)$. However, the significance of this relationship is left somewhat unclear in Ciftan's work. Here we show that, in fact, $sl(5, \mathbb{C})$ is the dynamical symmetry algebra for the F_1 and that the F_1 are basis vectors for models of irreducible representations of $sl(5, \mathbb{C})$.

By the *dynamical symmetry algebra* for the F_1 we mean the maximal Lie algebra generated by all differential recurrence relations obeyed by this family. (In this sense $\mathcal{G}_{p,q}$ is the dynamical symmetry algebra for the generalized hypergeometric functions ${}_pF_q$, (see [1]), and $sl(4, \mathbb{C})$ is the algebra for the Gaussian hypergeometric functions ${}_2F_1$ (see [8]).) In [9] it is shown in some detail how one constructs these algebras for all Lauricella functions, including the Appell functions.

Alternatively, one can, in the sense of [8], define the dynamical symmetry algebra for the F_1 as the maximal algebra of Lie derivatives L which map all solutions f of the simultaneous partial differential equations $Cf = C'f = 0$, (1.6), into other solutions Lf. As indicated in [8], both definitions lead to the same algebra.

In § 1 we show that the F_1 transform as canonical basis vectors under certain irreducible representations of $sl(5, \mathbb{C})$. Computing the matrix elements of the associated local group representations, we find that some of these too can be expressed in terms of the F_1. In § 2 we demonstrate that the $sl(5, \mathbb{C})$ symmetry can be used to derive all the transformation formulas and large numbers of generating functions for the F_1. Finally, in § 3 we apply Vilenkin's integral transform method and obtain the F_1 as kernel functions associated with certain integral operators.

It follows from our work that the machinery of group representation theory and harmonic analysis can successfully be applied to the study of Appell functions. Indeed, the majority of known properties of these functions can be obtained most easily using group theory and this theory is well-suited to the derivation of new properties.

In future papers the group theoretic method will be simplified and applied to all Lauricella functions.

---

* Received by the editors February 28, 1972, and in revised form August 25, 1972.

† School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455. This research was supported in part by the National Science Foundation under Grant GP-20871.

1. The dynamical symmetry group $SL(5, \mathbb{C})$. The Appell function F_1 is defined by the series

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!}, \tag{1.1}$$

where α, β, β', γ are complex numbers and γ is not a negative integer. Here,

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) \tag{1.2}$$

is Pochhammer's symbol. The series (1.1) converges for $|x| < 1$, $|y| < 1$. Following the method described in [1] we search for the differential recurrence relations satisfied by F_1 and use them to construct the dynamical symmetry algebra for this class of functions. The results are as follows: The symmetry algebra is 24-dimensional with basis

$$\begin{aligned}
E_{\alpha\beta\gamma} &= sut\partial_x, \\
E_{\alpha\beta'\gamma} &= svt\partial_y, \\
E_{\alpha} &= s(x\partial_x + y\partial_y + s\partial_s), \\
E_{\beta} &= u(x\partial_x + u\partial_u), \\
E_{\beta'} &= v(y\partial_y + v\partial_v), \\
E_{-\gamma} &= -t^{-1}(x\partial_x + y\partial_y + t\partial_t - 1), \\
E_{\alpha\gamma} &= st((1 - x)\partial_x + (1 - y)\partial_y - s\partial_s), \\
E_{\beta'-\beta} &= (v/u)((y - x)\partial_y + v\partial_v), \\
E_{\beta,-\beta'} &= (u/v)((x - y)\partial_x + u\partial_u), \\
E_{\beta\gamma} &= ut((x - 1)\partial_x + u\partial_u), \\
E_{\beta'\gamma} &= vt((y - 1)\partial_y + v\partial_v), \\
E_{\gamma} &= t((1 - x)\partial_x + (1 - y)\partial_y + t\partial_t - s\partial_s - u\partial_u - v\partial_v), \\
E_{-\alpha} &= s^{-1}(x(1 - x)\partial_x + y(1 - y)\partial_y + t\partial_t - s\partial_s - xu\partial_u - yv\partial_v), \\
E_{-\beta} &= u^{-1}(x(1 - x)\partial_x + x(1 - y)\partial_y + t\partial_t - u\partial_u - v\partial_v - xs\partial_s), \\
E_{-\beta'} &= v^{-1}(y(1 - x)\partial_x + y(1 - y)\partial_y + t\partial_t - u\partial_u - v\partial_v - ys\partial_s), \\
E_{-\alpha,-\gamma} &= t^{-1}s^{-1}(x(1 - x)\partial_x + y(1 - y)\partial_y - xu\partial_u - yv\partial_v + t\partial_t - 1), \\
E_{-\alpha,-\beta,-\gamma} &= -u^{-1}t^{-1}s^{-1}(x(1 - x)\partial_x + y(1 - y)\partial_y + t\partial_t - xu\partial_u \\
&\qquad - yv\partial_v - xs\partial_s + x - 1), \\
E_{-\alpha,-\beta',-\gamma} &= -v^{-1}t^{-1}s^{-1}(y(1 - y)\partial_y + x(1 - x)\partial_x + t\partial_t - yv\partial_v \\
&\qquad - xu\partial_u - ys\partial_s + y - 1), \\
E_{-\beta,-\gamma} &= u^{-1}t^{-1}(x(1 - x)\partial_x + y(1 - x)\partial_y + t\partial_t - xs\partial_s - 1),
\end{aligned} \tag{1.3}$$

$$E_{-\beta',-\gamma} = -v^{-1}t^{-1}(y(1-y)\partial_y + x(1-y)\partial_x + t\partial_t - ys\partial_s - 1),$$
$$J_\alpha = s\partial_s - \tfrac{1}{2}t\partial_t,$$
$$J_{\beta'} = v\partial_v + \tfrac{1}{2}(u\partial_u - t\partial_t),$$
$$J_{\beta,-\beta'} = \tfrac{1}{2}(u\partial_u - v\partial_v),$$
$$J_\gamma = t\partial_t - \tfrac{1}{2}(s\partial_s + u\partial_u + v\partial_v + 1).$$

Setting

$$f_{\alpha\beta\beta'\gamma}(s,u,v,t,x,y) = \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\gamma)} F_1(\alpha,\beta,\beta';\gamma;x,y)s^\alpha u^\beta v^{\beta'} t^\gamma, \tag{1.4}$$

we can verify directly that the action of the operators (1.3) on the basis vectors (1.4) is

$$E_{\pm\alpha} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\gamma-\alpha-1\\ \alpha-1\end{bmatrix} f_{\alpha\pm1,\beta\beta'\gamma},$$
$$E_{\pm\beta} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta\\ \alpha-\beta-\gamma\end{bmatrix} f_{\alpha\beta\pm1,\beta'\gamma},$$
$$E_{\pm\beta'} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta'\\ \alpha-\beta'-\gamma\end{bmatrix} f_{\alpha\beta\beta'\pm1,\gamma},$$
$$E_{\pm\gamma} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\gamma-\beta-\beta'\\ \alpha-\gamma+1\end{bmatrix} f_{\alpha\beta\beta'\gamma\pm1},$$
$$E_{\pm\beta',\mp\beta} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta-1\\ \beta'-1\end{bmatrix} f_{\alpha\beta\mp1,\beta'\pm1,\gamma},$$
$$E_{\pm\alpha,\pm\beta,\pm\gamma} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta\\ -\alpha+1\end{bmatrix} f_{\alpha\pm1,\beta\pm1,\beta'\gamma\pm1},$$
$$E_{\pm\alpha,\pm\beta',\pm\gamma} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta'\\ -\alpha+1\end{bmatrix} f_{\alpha\pm1,\beta\beta'\pm1,\gamma\pm1}, \tag{1.5}$$
$$E_{\pm\alpha,\pm\gamma} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta+\beta'-\gamma\\ \alpha-1\end{bmatrix} f_{\alpha\pm1,\beta\beta'\gamma\pm1},$$
$$E_{\pm\beta,\pm\gamma} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta\\ \alpha-\gamma+1\end{bmatrix} f_{\alpha\beta\pm1,\beta'\gamma\pm1},$$
$$E_{\pm\beta',\pm\gamma} f_{\alpha\beta\beta'\gamma} = \begin{bmatrix}\beta\\ \alpha-\gamma+1\end{bmatrix} f_{\alpha\beta\beta'\pm1,\gamma\pm1},$$
$$J_\alpha f_{\alpha\beta\beta'\gamma} = \left(\alpha-\frac{\gamma}{2}\right) f_{\alpha\beta\beta'\gamma},$$
$$J_{\beta'} f_{\alpha\beta\beta'\gamma} = \left(\beta'+\frac{\beta}{2}-\frac{\gamma}{2}\right) f_{\alpha\beta\beta'\gamma},$$
$$J_{\beta,-\beta'} f_{\alpha\beta\beta'\gamma} = \tfrac{1}{2}(\beta-\beta') f_{\alpha\beta\beta'\gamma},$$
$$J_\gamma f_{\alpha\beta\beta'\gamma} = (\gamma - \tfrac{1}{2}(\alpha+\beta+\beta'+1)) f_{\alpha\beta\beta'\gamma}.$$

Here the upper factor in each bracket is associated with the plus sign and the lower with the minus sign.

By a tedious computation one can verify that the operators (1.3) form a basis for a 24-dimensional simple Lie algebra, necessarily isomorphic to $sl(5, \mathbb{C})$. It follows from relations (1.5) that the functions $f_{\alpha\beta\beta'\gamma}$ can be used to construct irreducible representations of $sl(5, \mathbb{C})$.

Let

$$C \equiv E_\alpha E_\beta - E_{\alpha\beta\gamma}E_{-\gamma}, \qquad (1.6)$$
$$C' \equiv E_\alpha E_{\beta'} - E_{\alpha\beta'\gamma}E_{-\gamma}.$$

It is easy to check that the solution f of the simultaneous equations

$$J_\alpha f = \left(\alpha - \frac{\gamma}{2}\right)f, \qquad J_{\beta'} f = \left(\beta' + \frac{\beta}{2} - \frac{\gamma}{2}\right)f,$$
$$J_{\beta,-\beta'} f = \tfrac{1}{2}(\beta - \beta')f, \qquad J_\gamma f = \tfrac{1}{2}(2\gamma - \alpha - \beta - \beta' - 1)f, \qquad (1.7)$$
$$Cf = 0, \qquad C'f = 0,$$

analytic in a neighborhood of $x = y = 0$ in the complex x, y-plane, is

$$f = F_1(\alpha; \beta, \beta'; \gamma; x, y)s^\alpha u^\beta v^{\beta'} t^\gamma, \qquad (1.8)$$

unique to within a multiplicative constant. Indeed, the first four equations imply

$$f = F(x, y)s^\alpha u^\beta v^{\beta'} t^\gamma$$

and the last two imply

$$\{(x\partial_x + y\partial_y + \alpha)(x\partial_x + \beta) - \partial_x(x\partial_x + y\partial_y + \gamma - 1)\}F = 0, \qquad (1.9)$$
$$\{(x\partial_x + y\partial_y + \alpha)(y\partial_y + \beta') - \partial_y(x\partial_x + y\partial_y + \gamma - 1)\}F = 0,$$

which are the standard partial differential equations for F_1 (see [2]). Although the operators C and C' do not commute with all of the generators of $sl(5, \mathbb{C})$, it is not difficult to show that each generator maps a solution f of the equations $Cf = 0$, $C'f = 0$ into another solution. Similarly, the group operators obtained by exponentiating the Lie derivatives (1.3) also map solutions into solutions.

If $f(s, u, v, t, x, y)$ is a solution of $Cf = C'f = 0$ which has a Laurent expansion

$$f = \sum_{\alpha,\beta,\beta'\gamma} g_{\alpha\beta\beta'\gamma}(x, y)s^\alpha u^\beta b^{\beta'} t^\gamma \qquad (1.10)$$

and if f is analytic at $x = y = 0$, then it follows from the above remarks that

$$g_{\alpha\beta\beta'\gamma} = k(\alpha\beta\beta'\gamma)F_1(\alpha, \beta, \beta'; \gamma; x, y), \qquad (1.11)$$

where $k(\alpha\beta\beta'\gamma)$ is a constant.

To determine the group action of $sl(5, \mathbb{C})$ we note that each of the triples

$$
\begin{aligned}
&\{J^+, J^-, J^3\} \equiv \{E_\alpha, E_{-\alpha}, J_\alpha\}, \qquad \{E_{\beta'}, E_{-\beta'}, J_{\beta'}\}, \qquad \{E_\gamma, E_{-\gamma}, J_\gamma\},\\
&\{E_{\alpha\beta\gamma}, E_{-\alpha,-\beta,-\gamma}, J_\alpha + J_\gamma + J_{\beta'} + J_{\beta,-\beta'}\},\\
&\{E_{\alpha\beta'\gamma}, E_{-\alpha,-\beta',-\gamma}, J_\alpha + J_{\beta'} + J_\gamma\}, \qquad \{E_{\alpha\gamma}, E_{-\alpha,-\gamma}, J_\alpha + J_\gamma\},\\
&\{E_{\beta\gamma}, E_{-\beta,-\gamma}, J_{\beta'} + J_{\beta,-\beta'} + J_\gamma\},\\
&\{E_{\beta'\gamma}, E_{-\beta',-\gamma}, J_{\beta'} + J_\gamma\}, \qquad \{E_{\beta,-\beta'}, E_{-\beta,\beta'}, J_{\beta,-\beta'}\}
\end{aligned}
\tag{1.12}
$$

satisfies the commutation relations

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^3$$

and forms a basis for a subalgebra of $sl(5, \mathbb{C})$ isomorphic to $sl(2, \mathbb{C})$. Thus each such triplet generates a subgroup of $SL(5, \mathbb{C})$ isomorphic to $SL(2, \mathbb{C})$ and the ten subgroups so obtained suffice to generate the full group $SL(5, \mathbb{C})$. The group action of $SL(2, \mathbb{C})$ is given in terms of the Lie algebra action by

$$\mathbf{T}(A) = \exp\left(-\frac{b}{d}J^+\right)\exp(-cdJ^-)\exp(\tau J^3), \qquad e^{\tau/2} = d^{-1}, \tag{1.13}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}), \qquad ad - bc = 1. \tag{1.14}$$

The computation of the group action from the Lie derivatives (1.3) is now routine [3]. The results are

$$
\begin{aligned}
\mathbf{T}_1(A)f(s, u, v, t, x, y) = f\Bigg(&\frac{as + c}{d + bs}, \frac{u(as + c)}{as + c(1 - x)}, \frac{v(as + c)}{as + c(1 - y)},\\
&\frac{ts}{as + c}, \frac{xs}{(d + bs)(as - cx + c)}, \frac{ys}{(d + bs)(as - cy + c)}\Bigg)
\end{aligned}
\tag{1.15}
$$

for the triplet $\{E_\alpha, E_{-\alpha}, J_\alpha\}$,

$$
\begin{aligned}
\mathbf{T}_2(A)f = f\Bigg(&\frac{s(av + c)}{av + c(1 - y)}, \frac{u}{v}(av + c), \frac{av + c}{d + bv}, \frac{vt}{av + c},\\
&\frac{avx + c(x - y)}{av + c(1 - y)}, \frac{yv}{(d + bv)(av - cy + c)}\Bigg)
\end{aligned}
\tag{1.16}
$$

for the triplet $\{E_{\beta'}, E_{-\beta'}, J_{\beta'}\}$,

$$
\begin{aligned}
\mathbf{T}_3(A)f = f\Bigg(&\frac{s(au + c)}{au + c(1 - x)}, \frac{au + c}{d + bu}, \frac{v}{u}(au + c), \frac{ut}{au + c},\\
&\frac{xu}{(d + bu)(au - cx + c)}, \frac{auy + c(y - x)}{au + c(1 - x)}\Bigg)
\end{aligned}
\tag{1.17}
$$

for the triplet $\{E_\beta, E_{-\beta}, J_{\beta'} J_{\beta'} + J_{\beta,-\beta'}\}$,

$$\mathbf{T}_4(A)f = f\Bigg(s(d+bt), u(d+bt), v(d+bt), \frac{at+c}{d+bt}, \tag{1.18}$$
$$[dx - bt(1-x)]\left(a+\frac{c}{t}\right), [dy - bt(1-y)]\left(a+\frac{c}{t}\right)\Bigg)\left(a+\frac{c}{t}\right)^{-1}$$

for the triplet $\{E_\gamma, E_{-\gamma}, J_\gamma\}$,

$$\mathbf{T}_5(A)f = f\Bigg(as - \frac{cx}{ut}, au - \frac{cx}{st}, v\left(\frac{asut - cx}{asut + c(y-x)}\right), \tag{1.19}$$
$$\frac{t(asut + c - cx)}{asut - cx}, (xd - bsut)(a + c - cx), \frac{y(astu + c - cx)}{astu + c(y-x)}\Bigg)$$
$$\cdot\left(a + \frac{c - cx}{uts}\right)^{-1}$$

for the triplet $\{E_{\alpha\beta\gamma}, E_{-\alpha,-\beta,-\gamma}, J_\alpha + J_{\beta'} + J_{\beta-\beta'} + J_\gamma\}$,

$$\mathbf{T}_6(A)f = f\Bigg(as - \frac{cy}{vt}, u\left(\frac{asvt - cy}{asvt + c(x-y)}\right), av - \frac{cy}{st}, \tag{1.20}$$
$$\frac{t(asvt + c - cy)}{asvt - cy}, \frac{x(astv + c - cy)}{astv + c(x-y)}, (yd - bsvt)(a + c - cy)\Bigg)$$
$$\cdot\left(a + \frac{c - cy}{vts}\right)^{-1}$$

for the triplet $\{E_{\alpha\beta'\gamma}, E_{-\alpha,-\beta',-\gamma}, J_\alpha + J_{\beta'} + J_\gamma\}$,

$$\mathbf{T}_7(A)f = f\Bigg(\frac{s}{d - bst}, \frac{ust}{ast - cx}, \frac{vst}{ast - cy}, at - \frac{c}{s}, \tag{1.21}$$
$$\frac{(dx - bst)(ast - c)}{(d - bst)(ast - cx)}, \frac{(dy - bst)(ast - c)}{(d - bst)(ast - cy)}\Bigg)\left(a - \frac{c}{st}\right)^{-1}$$

for the triplet $\{E_{\alpha\gamma}, E_{-\alpha,-\gamma}, J_\alpha + J_\gamma\}$,

$$\mathbf{T}_8(A)f = f\Bigg(\frac{sut}{aut + cx}, \frac{u}{d + but}, v, at + \frac{c}{u}, \tag{1.22}$$
$$\frac{(dx + but)(aut + c)}{(d + but)(aut + cx)}, \frac{y(aut + c)}{aut + cx}\Bigg)\left(a + \frac{c}{ut}\right)^{-1}$$

for the triplet $\{E_{\beta\gamma}, E_{-\beta,-\gamma}, J_{\beta'} + J_{\beta-\beta'} + J_\gamma\}$,

$$\mathbf{T}_9(A)f = f\Bigg(\frac{svt}{avt + cy}, u, \frac{v}{d + bvt}, at + \frac{c}{v}, \tag{1.23}$$
$$\frac{x(avt + c)}{avt + cy}, \frac{(dy + bvt)(avt + c)}{(d + bvt)(avt + cy)}\Bigg)\left(a + \frac{c}{vt}\right)^{-1}$$

for the triplet $\{E_{\beta'\gamma}, E_{-\beta',-\gamma}, J_{\beta'} + J_\gamma\}$, and

$$\mathbf{T}_{10}(A)f = f\left(s, \frac{uv}{dv + bu}, \frac{uv}{au + cv}, t, \frac{dxv + byu}{dv + bu}, \frac{ayu + cxv}{au + cv}\right) \tag{1.24}$$

for the triplet $\{E_{\beta,-\beta'}, E_{-\beta,\beta'}, J_{\beta,-\beta'}\}$. Each of the operators $\mathbf{T}_j(A)$ maps a solution f of $Cf = C'f = 0$ into another solution.

Let $\alpha_0, \beta_0, \beta'_0, \gamma_0$ be fixed complex numbers, not integers, and let $\alpha = \alpha_0 + n_1$. $\beta = \beta_0 + n_2$, $\beta' = \beta' = \beta'_0 + n_3$, $\gamma = \gamma_0 + n_4$, where the n_j run over all integers. Then the basis functions $\{f_{\alpha\beta\beta'\gamma}\}$, (1.4), and the operators (1.3) define an infinite-dimensional irreducible representation $\rho(\alpha_0\beta_0\beta'_0\gamma_0)$ of $sl(5, \mathbb{C})$. (The cases where some of the complex numbers are integers still lead to representations but will not be studied in this paper.) Using the operators (1.15)–(1.24) we can extend this Lie algebra representation to a local group representation of $SL(5, \mathbb{C})$. To determine the matrix elements of this representation with respect to the basis $\{f_{\alpha\beta\beta'\gamma}\}$, it is convenient to construct a simpler model of $\rho(\alpha_0\beta_0\beta'_0\gamma_0)$.

The following model is realized by Lie derivatives in only four complex variables s, u, v, t rather than six. The basis functions are

$$f_{\alpha\beta\beta'\gamma}(s, u, v, t) = s^\alpha u^\beta v^{\beta'} t^\gamma, \tag{1.25}$$

and the Lie derivatives are

$$\begin{aligned}
&E_\alpha = s(t\partial_t - s\partial_s - 1), \qquad E_{-\alpha} = s^{-1}(s\partial_s - 1),\\
&E_{\beta'} = v^2\partial_v, \qquad E_{-\beta'} = v^{-1}(t\partial_t - u\partial_u - v\partial_v),\\
&E_\beta = u^2\partial_u, \qquad E_{-\beta} = u^{-1}(t\partial_t - u\partial_u - v\partial_v),\\
&E_\gamma = t(t\partial_t - u\partial_u - v\partial_v), \qquad E_{-\gamma} = t^{-1}(s\partial_s - t\partial_t + 1),\\
&E_{\beta,-\beta'} = (u/v)(v\partial_v - 1), \qquad E_{\beta',-\beta} = (v/u)(u\partial_u - 1),\\
&E_{\alpha\gamma} = st(u\partial_u + v\partial_v - t\partial_t), \qquad E_{-\alpha,-\gamma} = s^{-1}t^{-1}(s\partial_s - 1),\\
&E_{\alpha\beta\gamma} = stu^2\partial_u, \qquad E_{-\alpha,-\beta,-\gamma} = s^{-1}u^{-1}t^{-1}(-s\partial_s + 1),\\
&E_{\alpha\beta',\gamma} = stv^2\partial_v, \qquad E_{-\alpha,-\beta',-\gamma} = s^{-1}v^{-1}t^{-1}(-s\partial_s + 1),\\
&E_{\beta\gamma} = tu^2\partial_u, \qquad E_{-\beta,-\gamma} = t^{-1}u^{-1}(s\partial_s - t\partial_t + 1),\\
&E_{\beta'\gamma} = tv^2\partial_v, \qquad E_{-\beta,-\gamma} = t^{-1}v^{-1}(s\partial_s - t\partial_t + 1),\\
&J_\alpha = s\partial_s - \tfrac{1}{2}t\partial_t, \qquad J_{\beta'} = v\partial_v + \tfrac{1}{2}(u\partial_u - t\partial_t),\\
&J_{\beta,-\beta'} = \tfrac{1}{2}(u\partial_u - v\partial_v), \qquad J_\gamma = t\partial_t - \tfrac{1}{2}(s\partial_s + u\partial_u + v\partial_v + 1).
\end{aligned} \tag{1.26}$$

It is easy to show that these operators and basis functions satisfy relations (1.5), so they define a model of $\rho(\alpha\beta\beta'\gamma)$. To extend this model to a representation of $SL(5, \mathbb{C})$ we compute the operators $\mathbf{T}_j(A)$ analogous to (1.15)–(1.24) for the previous model.

$$\mathbf{T}_1(A)f(s,u,v,t) = (d - bs)^{-1}\left(a - \frac{c}{s}\right)^{-1} f\left(\frac{as - c}{d - bs}, u, v, t(d - bs)\right),$$

$$\mathbf{T}_2(A)f = f\left(s, u\left(a + \frac{c}{v}\right), \frac{av + c}{d + bv}, \frac{tv}{av + c}\right),$$

$$\mathbf{T}_3(A)f = f\left(s, \frac{au + c}{d + bu}, v\left(a + \frac{c}{u}\right), \frac{tu}{au + c}\right),$$

$$\mathbf{T}_4(A)f = \left(a + \frac{c}{t}\right)^{-1} f\left(\frac{st}{at + c}, u(d + bt), v(d + bt), \frac{at + c}{d + bt}\right),$$

$$\mathbf{T}_5(A)f = \left(a + \frac{c}{sut}\right)^{-1} f\left(as + \frac{c}{ut}, \frac{u}{d + bust}, v, t\right),$$

$$\mathbf{T}_6(A)f = \left(a + \frac{c}{svt}\right)^{-1} f\left(as + \frac{c}{vt}, u, \frac{v}{d + bvst}, t\right), \tag{1.27}$$

$$\mathbf{T}_7(A)f = \left(a - \frac{c}{st}\right)^{-1} f\left(as - \frac{c}{t}, u(d - bst), v(d - bst), \frac{t}{d - bst}\right),$$

$$\mathbf{T}_8(A)f = \left(a + \frac{c}{ut}\right)^{-1} f\left(\frac{sut}{c + aut}, \frac{u}{d + but}, v, at + \frac{c}{u}\right),$$

$$\mathbf{T}_9(A)f = \left(a + \frac{c}{vt}\right)^{-1} f\left(\frac{svt}{c + avt}, u, \frac{v}{d + bvt}, at + \frac{c}{v}\right),$$

$$\mathbf{T}_{10}(A)f = \left(d - \frac{bu}{u}\right)^{-1}\left(a - \frac{cv}{v}\right)^{-1} f(s, au - cv, dv - bu, t).$$

Those matrix elements which correspond to a representation $\mathbf{T}(A)$ of $SL(2, \mathbb{C})$ induced by a triplet $\{J^+, J^-, J^3\}$ acting on a basis $\{f_m\}$ according to

$$J^{\pm}f_m = (-\omega \pm m)f_{m\pm 1}, \qquad J^3 f_m = mf_m, \qquad \omega \in \mathbb{C}, \tag{1.28}$$

are well known [3]. Setting

$$\mathbf{T}(A)f_{m_0+n} = \sum_{n'=-\infty}^{\infty} T_{n'n}(A)f_{m_0+n'}, \qquad n = 0, \pm 1, \pm 2, \tag{1.29}$$

we have

$$T_{n'n}(A) = a^{\omega+m_0+n'}d^{\omega-m_0-n}c^{n-n'}\frac{\Gamma(\omega + m_0 + n + 1)}{\Gamma(\omega + m_0 + n' + 1)}$$

$$\cdot \frac{{}_2F_1(-\omega - m_0 - n', -\omega + m_0 + n; n - n' + 1; bc/ad)}{\Gamma(n - n' + 1)}$$

for A in a sufficiently small neighborhood of the identity element.

From this it is easy to compute the matrix elements of each of the operators $\mathbf{T}_j(A)$. As an example we consider the triplet $\{E_{\beta,-\beta'}, E_{-\beta,\beta'}, J_{\beta,-\beta'}\}$. From (1.5) we make the identifications

$$\omega = 1 - \tfrac{1}{2}(\beta_0 + \beta_0'), \quad m_0 = \tfrac{1}{2}(\beta_0' - \beta_0), \quad \beta = \beta_0 - n, \quad \beta' = \beta_0' + n,$$

$n = 0, \pm 1, \pm 2, \cdots$. Then

$$\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\mathbf{T}_{10}(A) f_{\alpha,\beta_0-n,\beta_0'+n,\gamma} \\
&\qquad = F_1\left(\alpha, \beta_0 - n, \beta_0' + n; \alpha; \frac{dxv + byu}{dv + bu}, \frac{ayu + cxv}{au + cv}\right) \\
&\qquad \cdot s^\alpha u^{\beta_0 - n} v^{\beta_0' + n} t^\gamma \left(\frac{v}{dv + bu}\right)^{\beta_0 - n}\left(\frac{u}{au + cv}\right)^{\beta_0' + n} \\
&\qquad = \sum_{n'=-\infty}^{\infty} a^{1-\beta_0+n'} d^{1-\beta_0'-n} c^{n-n'} \frac{\Gamma(2-\beta_0+n)}{\Gamma(2-\beta_0+n')} \\
&\qquad \cdot \frac{{}_2F_1(\beta_0 - 1 - n', \beta_0' - 1 + n; n - n' + 1; bc/ad)}{\Gamma(n - n' + 1)} \\
&\qquad \cdot F_1(\alpha, \beta_0 - n', \beta_0' + n'; \gamma; x, y) s^\alpha u^{\beta_0 - n'} v^{\beta_0' + n} t^\gamma, \\
&\qquad\qquad |c/a| < |u/v| < |d/b|.
\end{aligned} \tag{1.30}$$

(This expression can be greatly simplified.) In the special cases where $\mathbf{T}_{10}(A)$ reduces to $\exp bE_{\beta,-\beta'}$ or $\exp bE_{\beta',-\beta}$, this identity simplifies to

$$\begin{aligned}
&F_1\left(\alpha, \beta, \beta'; \gamma; \frac{x - by}{1 - b}, y\right)(1 - b)^{-\beta} \\
&\quad = \sum_{k=0}^{\infty} \binom{\beta' - 1}{k} F_1(\alpha, \beta + k, \beta' - k; \gamma; x, y) b^k, \qquad |b| < 1,
\end{aligned} \tag{1.31}$$

with a similar identity obtained by interchanging β and β' as well as x and y. In the same way we can find identities corresponding to each of the operators (1.15)–(1.23).

To compute more complicated matrix elements we make use of our model (1.25)–(1.27). Consider the 5-dimensional complex Lie algebra with basis

$$\{J^+, J^-, J^3, E^+, E^-\}$$

and commutation relations

$$\begin{aligned}
&[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^3, \\
&[J^+, E^-] = -E^+, \qquad [J^-, E^+] = -E^-, \\
&[J^+, E^+] = [J^-, E^-] = 0, \\
&[J^3, E^\pm] = \pm\tfrac{1}{2}E^\pm, \qquad [E^+, E^-] = 0.
\end{aligned} \tag{1.32}$$

This is the Lie algebra of the matrix group $ISL(2)$ with elements

$$\{A, g\} = \begin{pmatrix} A & \mathbf{g} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & g_1 \\ c & d & g_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.33}$$

$$A \in SL(2, \mathbb{C}), \quad g_j \in \mathbb{C},$$

and multiplication law

$$\{A,\mathbf{g}\}\{A',\mathbf{g}'\} = \{AA', A\mathbf{g}' + \mathbf{g}\}. \tag{1.34}$$

The relation between the Lie algebra and the group can be chosen in the form

$$\begin{gathered}\{A,\mathbf{g}\} = \exp(g_2E^-)\exp(g_1E^+)\exp\left(-\frac{b}{d}J^+\right)\exp(-cdJ^-)\exp(\tau J^3),\\ e^{\tau/2} = d^{-1}.\end{gathered} \tag{1.35}$$

The Lie algebra $isl(2)$ can be embedded as a subalgebra of $sl(5,\mathbb{C})$ in many distinct ways. For example, the operators $\{E_\alpha, E_{-\alpha}, J_\alpha, E_{\alpha\gamma}, E_\gamma\}$ satisfy the commutation relations (1.32) as do the operators $\{E_\alpha, E_{-\alpha}, J_\alpha, E_{\alpha\beta\gamma}, E_{\beta\gamma}\}$. Indeed, each of the ten triplets (1.12) occurs in several nonconjugate ways as a $\{J^+, J^-, J^3\}$ in embeddings of $isl(2)$ in $sl(5,\mathbb{C})$.

Using the embedding $\{E_\alpha, E_{-\alpha}, J_\alpha, E_{\alpha\gamma}, E_\gamma\}$, we compute the action of $ISL(2)$ in our 4-variable model:

$$\begin{aligned}\mathbf{T}(A,\mathbf{g})f(s,u,v,t) &= \mathbf{T}(\exp g_2E_\alpha)\mathbf{T}(\exp g_1E_{\alpha\gamma})\mathbf{T}_1(A)f\\ &= (d-bs)^{-1}\left(a-\frac{c}{s}\right)^{-1} f\left(\frac{as-c}{d-bs}, u(1-(g_2+g_1s)t),\right.\\ &\qquad \left. v(1-(g_2+g_1s)t), \frac{t(d-bs)}{1-(g_2+g_1s)t}\right).\end{aligned} \tag{1.36}$$

Applying this operator to the basis vectors

$$f_{n_j}(s,u,v,t) = s^{\alpha_0+n_1}u^{\beta_0+n_2}v^{\beta_0'+n_3}t^{\gamma_0+n_4},$$

we find that

$$\mathbf{T}\{A,\mathbf{g}\}f_{n_j} = \sum_{n_k'=-\infty}^{\infty} \mathrm{T}(A,\mathbf{g})_{n_j}^{n_k'} f_{n_k'} \tag{1.37}$$

or

$$\begin{aligned}&\left(a-\frac{c}{s}\right)^{\alpha_0+n_1-1}(d-bs)^{-\alpha_0+n_4-n_1-1}(1-(g_2-g_1s)t)^{\beta_0+\beta_0'-\gamma_0+n_2+n_3-n_4}\\ &\quad\cdot s^{n_1}u^{n_2}v^{n_3}t^{n_4} = \sum_{n_k'=-\infty}^{\infty} T(A,\mathbf{g})_{n_j}^{n_k'} s^{n_1'}u^{n_2'}v^{n_3'}t^{n_4'}.\end{aligned} \tag{1.38}$$

Computing the coefficient of $s^{n_1'}u^{n_2'}v^{n_3'}t^{n_4'}$ on the left-hand side of this expression we find that

$$\begin{aligned}&T\{A,\mathbf{g}\}_{n_j}^{n_k'} = 0 \quad \text{unless } n_2 = n_2',\quad n_3 = n_3',\quad n_4' \geqq n_4,\\ &T\{A,\mathbf{g}\}_{n_1,n_2,n_3,n_4}^{n_1+m,n_2,n_3,n_4+k} = (-1)^{m+k}g_2^k d^{\gamma_0-\alpha_0+n_4-n_1-1}\\ &\quad\cdot a^{\alpha_0+n_1-m-1}c^{-m}\binom{\beta_0+\beta_0'-\gamma_0+n_2+n_3-n_4}{k}\\ &\quad\cdot\frac{1}{\Gamma(1-m)}F_1\left(-\alpha_0-n_1+1, \alpha_0-\gamma_0+n_1-n_4-1, -k;\right.\\ &\qquad\left. -m+1; \frac{bc}{ad}, \frac{cg_1}{ag_2}\right) \quad \text{if } k \geqq 0.\end{aligned} \tag{1.39}$$

The group multiplication property for $ISL(2)$ implies the addition theorem

$$T\{AB, A\mathbf{h} + \mathbf{g}\}_{n_j}^{n_k} = \sum_{m_l=-\infty}^{\infty} T\{A, \mathbf{g}\}_{m}^{n_k'} T\{B, \mathbf{h}\}_{n_j}^{m_l} \tag{1.40}$$

for the functions F_1. Now that we know the matrix elements we can use (1.37) and our six-variable model to compute other identities for the F_1. In the six-variable model we find that

$$\begin{aligned} \mathbf{T}\{A, \mathbf{g}\} f(s, u, v, t, x, y) = f\Bigg[& \frac{as(1 - g_2 t) + c(1 + g_1 st)}{d(1 + g_1 st) + bs(1 - g_2 t)}, \\ & \frac{u[as(1 - g_2 t) + c(1 + g_1 st)]}{as + c(1 - x)}, \frac{v[as(1 - g_2 t) + c(1 + g_1 st)]}{as + c(1 - y)}, \\ & \frac{st}{as(1 - g_2 t) + c(1 + g_1 st)}, \frac{s[x + g_2 t(1 - x) + g_1 st]}{[as(1 - g_2 t) + c(1 + g_1 st)][as + c(1 - x)]}, \\ & \frac{s[y + g_2 t(1 - y) + g_1 st]}{[as(1 - g_2 t) + c(1 + g_1 st)][as + c(1 - x)]} \Bigg]. \end{aligned} \tag{1.41}$$

Substitution of (1.4), (1.39) and (1.41) into (1.37) yields the desired identities. Note that the sum in (1.37) is a double sum over n_1' and n_4' in the general case. In a similar manner each embedding of $ISL(2)$ in $SL(5, \mathbb{C})$ yields a family of identities for the F_1.

2. Transformation formulas and generating functions. The transformation formulas for the F_1 are simple consequences of the $SL(5, \mathbb{C})$ symmetry. Let

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{C}). \tag{2.1}$$

From (1·4) and (1.15) we find that

$$\begin{aligned} \mathbf{T}_1(I) f_{\alpha\beta\beta'\gamma} = & \frac{(-1)^{\alpha+\gamma}\Gamma(\gamma - \alpha)\Gamma(\alpha)}{\Gamma(\gamma)} \\ & \cdot F_1\left(\alpha, \beta, \beta'; \gamma; \frac{x}{x - 1}, \frac{y}{y - 1}\right)(1 - x)^{-\beta}(1 - y)^{-\beta'} s^{-\alpha+\gamma} u^{\beta} v^{\beta'} t^{\gamma}. \end{aligned} \tag{2.2}$$

On the other hand, $\mathbf{T}_1(I) f_{\alpha\beta\beta'\gamma}$ is a simultaneous eigenfunction of $J_\alpha, J_{\beta'}, J_{\beta,-\beta'}, J_\gamma$ which is analytic at $x = y = 0$. Therefore,

$$\mathbf{T}_1(I) f_{\alpha\beta\beta'\gamma} = kF_1(\gamma - \alpha, \beta, \beta'; \gamma; x, y) s^{-\alpha+\gamma} u^{\beta} v^{\beta'} t^{\gamma}, \tag{2.3}$$

and the constant k can be determined by setting $x = y = 0$ in (2.2) and (2.3). This yields the transformation formula

$$(1 - x)^{-\beta}(1 - y)^{-\beta'} F_1\left(\alpha, \beta, \beta'; \gamma; \frac{x}{x - 1}, \frac{y}{y - 1}\right) = F_1(\gamma - \alpha, \beta, \beta'; \gamma; x, y) \tag{2.4}$$

(see [4]). Similarly, the expression $\mathbf{T}_2(I) f_{\alpha\beta\beta'\gamma}$ leads to

$$(1 - y)^{-\alpha} F_1\left(\alpha, \beta, \beta'; \gamma; \frac{y - x}{y - 1}, \frac{y}{y - 1}\right) = F_1(\alpha, \beta, \gamma - \beta - \beta'; \gamma; x, y), \tag{2.5}$$

and $\mathbf{T}_3(I)f_{\alpha\beta\beta'\gamma}$ leads to

$$(2.6)\quad (1-x)^{-\alpha}F_1\left(\alpha,\beta,\beta';\gamma;\frac{x}{x-1},\frac{x-y}{x-1}\right) = F_1(\alpha,\gamma-\beta-\beta',\beta';\gamma;x,y).$$

The remaining two transformation formulas can be obtained by composition of those derived here. Consideration of $\mathbf{T}_4(I)f_{\alpha\beta\beta'\gamma}$ leads to the result that

$$(2.7)\qquad F_1(\alpha;\beta,\beta';\alpha+\beta+\beta'-\gamma+1,1-x,1-y)$$

is a solution of equations (1.9), analytic at $x = y = 1$, consideration of $\mathbf{T}_7(I)f_{\alpha\beta\beta'\gamma}$ shows that

$$(2.8)\qquad x^{-\beta}y^{-\beta'}F_1(\beta+\beta'-\gamma+1;\beta,\beta';\beta+\beta'-\alpha+1,x^{-1},y^{-1})$$

is a solution of equations (1.9) and consideration of $\mathbf{T}_8(I)f_{\alpha\beta\beta'\gamma}$ shows that

$$(2.9)\qquad x^{-\alpha}F_1\left(\alpha;\alpha-\gamma+1,\beta';\alpha-\beta+1;\frac{1}{x},\frac{y}{x}\right)$$

is also a solution of (1.9). The operator $\mathbf{T}_9(I)$ yields an analogous result with the interchanges $\beta \leftrightarrow \beta'$, $x \leftrightarrow y$. By applying all possible combinations of these transformations to a basis of solutions of (1.9) near $x = y = 0$, one can obtain the full 60 solutions of (1.9) (see [5]).

We can obtain many more generating functions for the F_1 by considering expressions $\mathbf{T}_j(A)f_{\alpha\beta\beta'\gamma}$ where A is bounded away from the identity. As we have seen, for A far enough from the identity the matrix elements (1.29) may no longer be valid. As another example, consider the operator $\exp cE_{\alpha\gamma}$ in the six-variable model:

$$(2.10)\quad (\exp cE_{\alpha\gamma})f(s,u,v,t,x,y) = f\left(\frac{s}{1+cst},u,v,t,\frac{x+cst}{1+cst},\frac{y+cst}{1+cst}\right).$$

For $|c|$ sufficiently small, we have

$$(\exp cE_{\alpha\gamma})f_{\alpha\beta\beta'\gamma} = \sum_{k=0}^{\infty}\binom{\beta+\beta'-\gamma}{k}f_{\alpha+k,\beta\beta',\gamma+k}c^k,$$

or

$$(2.11)\quad \begin{aligned}&(1+c)^{-\alpha}F_1\left(\alpha,\beta,\beta';\gamma;\frac{x+c}{1+c},\frac{y+c}{1+c}\right)\\ &\quad = \sum_{k=0}^{\infty}\binom{\beta+\beta'-\gamma}{k}\frac{(\alpha)_k}{(\gamma)_k}F_1(\alpha+k,\beta,\beta';\gamma+k;x,y)c^k, \qquad |c|<1.\end{aligned}$$

If $c = 1$ and $|\tau| < 1$ where $\tau = s^{-1}t^{-1}$, then $\exp E_{\alpha\gamma}f_{\alpha\beta\beta'\gamma}$ is not analytic at $x = y = \tau = 0$. However, applying $\exp E_{\alpha\gamma}$ to the solution (2.7) and making use of (1.10), we obtain

$$(2.12)\quad \begin{aligned}&(1+\tau)^{-\alpha}F_1\left(\alpha,\beta,\beta';\alpha+\beta+\beta'-\gamma+1;\frac{\tau(1-x)}{1+\tau},\frac{\tau(1-y)}{1+\tau}\right)\\ &\quad = \sum_{k=0}^{\infty}h_kF_1(-k,\beta,\beta';\gamma-\alpha-k;x,y)\tau^k.\end{aligned}$$

Setting $x = y = 0$ we find that

$$(1+\tau)^{-\alpha}F_1\left(\alpha,\beta,\beta';\alpha+\beta+\beta'-\gamma+1;\frac{\tau}{1+\tau},\frac{\tau}{1+\tau}\right)$$
$$=(1+\tau)^{-\alpha}{}_2F_1\left(\alpha,\beta+\beta';\alpha+\beta+\beta'-\gamma+1;\frac{\tau}{1+\tau}\right)$$
$$=\sum_{k=0}^{\infty}h_k\tau^k,$$

or

$$\begin{aligned} h_k &= \binom{-\alpha}{k}{}_2F_1(-k,\beta+\beta';\alpha+\beta+\beta'-\gamma+1;1) \\ &= \binom{-\alpha}{k}\frac{(\alpha-\gamma+1)_k}{(\alpha+\beta+\beta'-\gamma+1)_k} \end{aligned} \tag{2.13}$$

from [3, p. 211] and Vandermonde's theorem.

For our second example, we expand $\mathbf{T}_1(A)f_{\alpha\beta\beta'\gamma}$ as a power series in $\tau = s^{-1}$:

$$\begin{aligned} &a^{\alpha-\gamma}b^{-\alpha}\left(1+\frac{c\tau}{a}\right)^{\alpha+\beta+\beta'-\gamma}\left(\frac{1+d\tau}{b}\right)^{-\alpha}\left[1+\frac{c\tau}{a}(1-x)\right]^{-\beta} \\ &\cdot\left[1+\frac{c\tau}{a}(1-y)\right]^{-\beta'}F_1\left(\alpha,\beta,\beta';\gamma;\frac{x\tau}{(b+d\tau)(a+c\tau(1-x))},\right. \\ &\left.\frac{y\tau}{(b+d\tau)(a+c\tau(1-y))}\right)=\sum_{k=0}^{\infty}h_kF_1(-k,\beta;\beta';\gamma;x,y)\tau^k. \end{aligned} \tag{2.14}$$

Setting $x = y = 0$ and using [3, p. 206, (5.124)] , we find that

$$h_k = \left(\frac{a}{b}\right)^{\alpha}a^{-\gamma-k}c^k\binom{-\gamma}{k}\cdot{}_2F_1\left(-k,\alpha;\gamma;\frac{-1}{bc}\right),\qquad ad-bc=1. \tag{2.15}$$

If $a = d = b = 1$, $c = 0$, this identity simplifies to

$$\begin{aligned} &(1+\tau)^{-\alpha}F_1\left(\alpha,\beta,\beta';\gamma;\frac{x\tau}{1+\tau},\frac{y\tau}{1+\tau}\right) \\ &\quad=\sum_{k=0}^{\infty}\binom{-\alpha}{k}F_1(-k,\beta,\beta';\gamma,-x,y)\tau^k,\qquad |\tau|<1, \end{aligned} \tag{2.16}$$

and if $a = c = 1$, $b = -\omega^{-1}$, it becomes

$$\begin{aligned} &(1+\tau)^{\alpha+\beta+\beta'-\gamma}[1+(1-\omega)\tau]^{-\alpha}[1+(1-x)\tau]^{-\beta}[1+(1-y)\tau]^{-\beta} \\ &\cdot F_1\left(\alpha,\beta,\beta';\gamma;-\frac{x\tau\omega}{[1+(1-\omega)\tau][1+(1-x)\tau]},\right. \\ &\left.\cdot\frac{-y\tau\omega}{[1+(1-\omega)\tau][1+(1-y)\tau]}\right)=\sum_{k=0}^{\infty}\binom{-\gamma}{k}{}_2F_1(-k,\alpha;\gamma;\omega) \\ &\cdot F_1(-k,\beta,\beta';\gamma;x,y)\tau^k, \\ &\qquad\qquad |\tau|<\min(1,|1-x|^{-1},|1-y|^{-1},|1-\omega|^{-1}). \end{aligned} \tag{2.17}$$

Finally we discuss a more general method for computing generating functions (basically Weisner's method) which includes all of the above as special cases. Here we characterize a solution of $Cf = C'f = 0$ by requiring that f be analytic at $x = y = 0$ and that it will be a simultaneous eigenfunction of four independent operators constructed from the Lie algebra $sl(5, \mathbb{C})$. Such a characterization of the $f_{\alpha\beta\beta'\gamma}$ is given by expressions (1.7), (1.8). (A more detailed description of this method can be found in [1].)

We illustrate the method with another example. Consider the solution $f(s, u, v, t, x, y)$ of the simultaneous equations

$$E_\alpha f = f, \quad J_{\beta'} f = \left(\beta' + \frac{\beta}{2} - \frac{\gamma}{2}\right) f,$$

$$J_{\beta,-\beta'} f = \frac{1}{2}(\beta - \beta')f, \quad \left(J_\gamma + \frac{1}{2}J_\alpha\right) f = \left(\frac{3\gamma}{4} - \frac{1}{2}(\beta + \beta' + 1)\right) f, \tag{2.18}$$

$$Cf = 0, \quad C'f = 0,$$

which is analytic at $x = y = 0$. The first four equations have the general solution

$$f = h\left(\frac{x}{s}, \frac{y}{s}\right) \exp(-s^{-1}) u^\beta v^{\beta'} t^\gamma,$$

where h is arbitrary. Substituting this expression into the remaining two equations we find that

$$\begin{aligned} h(x, y) &= \Phi(\beta, \beta'; \gamma; x, y) \\ &= \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m!\,n!} \\ &= \lim_{\alpha \to \infty} F_1\left(\alpha, \beta, \beta'; \gamma; \frac{x}{\alpha}, \frac{y}{\alpha}\right), \end{aligned} \tag{2.19}$$

unique to within a constant multiple. Writing $\mathbf{T}_1(A)f$ as a power series in $\tau = s^{-1}$, we find that

$$\begin{aligned} &\exp\left[-\left(\frac{d\tau + b}{a + c\tau}\right)\right](a + ct\tau)^{\beta+\beta'-\gamma}[a + c\tau(1 - x)]^{-\beta} \\ &\cdot [a + c\tau(1 - y)]^{-\beta'} \Phi\left(\beta, \beta'; \gamma; \frac{x\tau}{(a + c\tau)[a + c\tau(1 - x)]},\right. \\ &\left.\cdot \frac{y\tau}{(a + c\tau)[a + c\tau(1 - y)]}\right) = \sum_{k=0}^{\infty} h_k F_1(-k, \beta, \beta'; \gamma; x, y)\tau^k, \\ &ad - bc = 1. \end{aligned} \tag{2.20}$$

Setting $x = y = 0$ and making use of a generating function for the generalized Laguerre polynomials [3, p. 190, (5.101)], we find that

$$h_k = a^{-\gamma} e^{-b/a} \left(\frac{c}{a}\right)^k L_k^{(\gamma-1)}\left(\frac{1}{ac}\right), \tag{2.21}$$

where $L_n^{(\alpha)}(x)$ is a generalized Laguerre polynomial. If $b = c = 0$, $a = d = 1$, this identity becomes

$$\exp(-\tau)\Phi(\beta, \beta'; \gamma; x\tau, y\tau) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} F_1(-k, \beta, \beta'; \gamma; x, y)\tau^k. \tag{2.22}$$

If $a = c = d^{-1} = \omega^{-1/2}$, $b = 0$, there results

$$\begin{aligned} &\exp\left[\frac{-\omega\tau}{1+\tau}\right](1+\tau)^{\beta+\beta'-\gamma}[1+\tau(1-x)]^{-\beta}[1+\tau(1-y)]^{-\beta'} \\ &\cdot \Phi\left(\beta, \beta'; \gamma; \frac{x\omega\tau}{(1+\tau)[1+\tau(1-x)]}, \frac{y\omega\tau}{(1+\tau)[1+\tau(1-y)]}\right) \\ &= \sum_{k=0}^{\infty} L_k^{(\gamma-1)}(\omega) F_1(-k, \beta, \beta'; \gamma; x, y)\tau^k, \\ &\qquad |\tau| < \min(1, |x-1|^{-1}, |y-1|^{-1}). \end{aligned} \tag{2.23}$$

Finally, if $b = -c = 1$, $a = d = 0$ we find for $\mathbf{T}_1(A)f$ the expression

$$e^s(1-x)^{-\beta}(1-y)^{-\beta'} s^\gamma \Phi\left(\beta, \beta'; \gamma; \frac{xs}{1-x}, \frac{ys}{1-y}\right) u^\beta v^{\beta'} t^\gamma.$$

Expanding as usual in powers of s, we obtain

$$e^s(1-x)^{-\beta}(1-y)^{-\beta'}\Phi\left(\beta, \beta'; \gamma; \frac{xs}{1-x}, \frac{ys}{1-y}\right) = \sum_{k=0}^{\infty} \frac{s^k}{k!} F_1(\gamma + k, \beta, \beta'; \gamma; x, y).$$

Using similar techniques one can derive a large number of such generating functions.

3. Mellin–Barnes integrals. We now apply Vilenkin's method [6] for computing Mellin–Barnes integral identities to the F_1. For this purpose the four-variable model (1.27) is most useful. To show how the method works, we consider a modification of the representation of $ISL(2)$ defined by the operator $\mathbf{T}(A, \mathbf{g})$ (see (1.36)). Let $ISL(2, R)$ be the subgroup of $ISL(2)$ consisting of all $\{A, \mathbf{g}\}$ with real parameters, and let $D(\mu, \omega)$ be the space of all complex-valued C^∞-functions $f(s, t)$ in the real variables s, t such that

$$\hat{f}(s, t) = |s|^\mu f\left(\frac{1}{s}, t\right), \qquad \tilde{f}(s, t) = |t|^\omega f\left(s, \frac{1}{t}\right) \tag{3.1}$$

are also C^∞. Here μ and ω are fixed complex constants. The operators

$$\mathbf{B}(A, \mathbf{g})f(s, t) = |1 - (g_2 + g_1 s)t|^\omega |d - bs|^\mu f\left[\frac{as - c}{d - bs}, \frac{t(d - bs)}{1 - (g_2 + g_1 s)t}\right] \tag{3.2}$$

define a global representation of $ISL(2, R)$ on $D(\mu, \omega)$. With each $f \in D(\mu, \omega)$, we associate the vector-valued function

$$\mathbf{F}(\lambda, \psi) = (F_{++}(\lambda, \psi), F_{+-}(\lambda, \psi), F_{-+}(\lambda, \psi), F_{--}(\lambda, \psi)$$

defined by

$$
\begin{aligned}
F_{++}(\lambda,\psi) &= \int_0^\infty ds \int_0^\infty dt s^{\lambda-1} t^{\psi-1} f(s,t) \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty ds\, dt (s)_+^{\lambda-1} (t)_+^{\psi-1} f(s,t), \\
F_{+-}(\lambda,\psi) &= \int_0^\infty ds \int_{-\infty}^0 dt s^{\lambda-1} (-t)^{\psi-1} f(s,t) \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty ds\, dt (s)_+^{\lambda-1} (t)_-^{\psi-1} f(s,t), \\
F_{-+}(\lambda,\psi) &= \int_{-\infty}^\infty \int_{-\infty}^\infty ds\, dt (s)_-^{\lambda-1} (t)_+^{\psi-1} f(s,t), \\
F_{--}(\lambda,\psi) &= \int_{-\infty}^\infty \int_{-\infty}^\infty ds\, dt (s)_-^{\lambda-1} (t)_-^{\psi-1} f(s,t).
\end{aligned}
\tag{3.3}
$$

These integrals converge absolutely in the region $0 < \operatorname{Re} \lambda < -\operatorname{Re} \mu$, $0 < \operatorname{Re} \psi < -\operatorname{Re} \omega$. It follows from the inversion formula for the Mellin transform that

$$
f(s,t) = \begin{cases}
\dfrac{1}{2\pi i} \displaystyle\int_{\nu-i\infty}^{\nu+i\infty} d\lambda \int_{\xi-i\infty}^{\xi+i\infty} d\psi F_{++}(\lambda,\psi) s^{-\lambda} (t)^{-\psi}, & s,t > 0, \\
\dfrac{1}{2\pi i} \displaystyle\int_{\nu-i\infty}^{\nu+i\infty} d\lambda \int_{\xi-i\infty}^{\xi+i\infty} d\psi F_{+-}(\lambda,\psi) s^{-\lambda} (-t)^{-\psi}, & s > 0, t < 0, \\
\dfrac{1}{2\pi i} \displaystyle\int_{\nu-i\infty}^{\nu+i\infty} d\lambda \int_{\xi-i\infty}^{\xi+i\infty} d\psi F_{-+}(\lambda,\psi) (-s)^{-\lambda} (t)^{-\psi}, & s < 0, t > 0, \\
\dfrac{1}{2\pi i} \displaystyle\int_{\nu-i\infty}^{\nu+i\infty} d\lambda \int_{\xi-i\infty}^{\xi+i\infty} d\psi F_{--}(\lambda,\psi) (-s)^{-\lambda} (-t)^{-\psi}, & s,t < 0,
\end{cases}
\tag{3.4}
$$

where $0 < \nu < -\operatorname{Re} \mu$, $0 < \xi < -\operatorname{Re} \omega$.

The operators $\mathbf{B}(A, \mathbf{g})$ induce a representation of $ISL(2, R)$ on the Mellin-transform space:

$$
[\mathbf{B}(A,\mathbf{g})F]_{\pm,\pm}(\lambda,\psi) = \int_{-\infty}^\infty \int_{-\infty}^\infty ds\, dt (s)_\pm^{\lambda-1} (t)_\pm^{\psi-1} \mathbf{B}(A,\mathbf{g}) f(s,t).
\tag{3.5}
$$

If $f(s, t)$ in (3.5) is expressed in terms of $F_{\pm,\pm}(\lambda, \psi)$ by (3.4) and if it is permissible to interchange the order of integration, one obtains

$$
[\mathbf{B}(A,\mathbf{g})\mathbf{F}](\lambda,\psi) = \int_{\nu-i\infty}^{\nu+i\infty} d\rho \int_{\xi-i\infty}^{\xi+i\infty} d\eta \mathbf{K}(\lambda,\psi;\rho,\eta;A,\mathbf{g}) \mathbf{F}(\rho,\eta),
\tag{3.6}
$$

where $\mathbf{K}$ is the 4×4 matrix

$$\mathbf{K} = \begin{pmatrix} K_{++}^{++} & K_{++}^{+-} & K_{++}^{-+} & K_{++}^{--} \\ K_{+-}^{++} & K_{+-}^{+-} & K_{+-}^{-+} & K_{+-}^{--} \\ K_{-+}^{++} & K_{-+}^{+-} & K_{-+}^{-+} & K_{-+}^{--} \\ K_{--}^{++} & K_{--}^{+-} & K_{--}^{-+} & K_{--}^{--} \end{pmatrix} \tag{3.7}$$

defined by

$$\begin{aligned} &K_{\alpha_3\alpha_4}^{\alpha_1\alpha_2}(\lambda, \psi; \rho, \eta; A, \mathbf{g}) \\ &= \frac{-1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ds\, dt |a + bs|^{-\mu-2} |1 + t(g_2(a + bs) + g_1(ds + c))|^{-\omega-2} \\ &\cdot \left[\frac{ds + c}{a + bs}\right]_{\alpha_3}^{\lambda-1} \left[\frac{(a + bs)\tau}{1 + \tau(g_2(a + bs) + g_1(ds + c))}\right]_{\alpha_4}^{\psi-1} (s)_{\alpha_1}^{-\rho} (t)_{\alpha_2}^{-\psi}, \end{aligned} \tag{3.8}$$

where $\alpha_j = \pm$. The domain of validity of (3.6) depends on $\{A, \mathbf{g}\}$. These matrix elements can all be expressed in terms of the functions F_1 and their limiting cases. For example, if $a, b, c, d, g_1, g_2 > 0$, we find that

$$\begin{aligned} K_{++}^{++} &= \frac{-1}{4\pi^2} a^{\psi-\rho-\mu-\lambda-1} b^{\rho-1} c^{\lambda-1} (g_2 a + g_1 c)^{\eta-\psi} \\ &\cdot \frac{\Gamma(\omega + \eta + 1)\Gamma(\psi - \eta)}{\Gamma(\omega + \psi + 1)} \\ &\cdot F_1\left(1 - \rho, 1 - \lambda, \psi - \eta; \mu - \eta + 3; \frac{-1}{bc}, \frac{-g_2}{b(g_1 a + g_2 c)}\right) \end{aligned} \tag{3.9}$$

for $\operatorname{Re} \rho < 1$, $\operatorname{Re}(\psi - \eta) > 0$, $\operatorname{Re}(\omega + \eta + 1) > 0$, $\operatorname{Re}(\mu + \rho - \eta + 2) > 0$. For $\mathbf{g} = \mathbf{0}$ the matrix elements are hypergeometric functions ${}_2F_1$, exactly as computed in [6, Chap. 7]. For A the identity matrix, the elements are products of gamma functions. The group multiplication law for $ISL(2, R)$ implies the matrix identity

$$\begin{aligned} &\mathbf{K}(\lambda, \psi, \rho, \eta; A_1A_2, A_1\mathbf{g}_2 + \mathbf{g}_1) \\ &= \int_{\nu-i\infty}^{\nu+i\infty} d\alpha \int_{\xi-i\infty}^{\xi+i\infty} d\beta \mathbf{K}(\lambda, \psi; \alpha, \beta; A_1, \mathbf{g}_1)\mathbf{K}(\alpha, \beta; \rho, \eta; A_2, \mathbf{g}_2). \end{aligned} \tag{3.10}$$

Each of these 16 equations is valid exactly when both sides of the equation are defined. Choosing $\{A_1, \mathbf{g}_1\}$ and $\{A_2, \mathbf{g}_2\}$ in an appropriate fashion, one can obtain an enormous variety of identities for the F_1 which contain the results of Vilenkin [6, Chap. 7] as a special case. Similarly, by choosing subgroups of $SL(5, R)$ other than $ISL(2, R)$, one can obtain new identities of the form (3.10). All of these computations are straightforward but tedious.

REFERENCES

[1] W. MILLER, JR., *Lie theory and generalized hypergeometric functions*, this Journal, 3 (1972), pp. 31–44.

[2] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. TRICOMI, *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1953.

[3] W. MILLER, JR., *Lie Theory and Special Functions*, Academic Press, New York, 1968.

[4] W. BAILEY, *Generalized Hypergeometric Series*, 2nd ed., Cambridge University Press, Cambridge, England, 1964.

[5] P. APPELL AND J. KAMPÉ DE FÉRIET, *Fonctions hypergeometriques et hyperspheriques*, Gauthier-Villars, Paris, 1926.

[6] N. VILENKIN, *Special Functions and the Theory of Group Representations*, Americal Mathematical Society Translations, Providence, R.I., 1968.

[7] M. CIFTAN, *On the combinatorial structure of state vectors in U(n). II. The Generalization of hypergeometric functions on U(n) states*, J. Mathematical Phys., 10 (1969), pp. 1635–1646.

[8] W. MILLER, JR., *Symmetries of differential equations. The hypergeometric and Euler–Darboux equations*, this Journal, 4 (1973), pp. 314–328.

[9] ———, *Lie theory and generalizations of the hypergeometric functions*, SIAM-Drexel Conference on Lie Algebras, SIAM J. Appl. Math., 25 (1973).

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

HILBERT TRANSFORMS, PLEMELJ RELATIONS, AND FOURIER TRANSFORMS OF DISTRIBUTIONS*

MARION ORTON†

Abstract. Using the theory of analytic representations, it is shown that a generalized Hilbert transformation may be defined on the space $\mathscr{D}'$ of Schwartz distributions. It yields extended Plemelj and dispersion relations for all distributions in $\mathscr{D}'$. Distributions which are limits of functions analytic in the upper or lower half-plane are seen to be completely characterized by a certain Fourier transform property. Corresponding results were previously obtained for distributions in the subspaces $\mathscr{S}'$ and O'_α of $\mathscr{D}'$.

1. Introduction. Classically the Hilbert transform of a function f is defined as the Cauchy principal value of the integral

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,dt,$$

provided it exists. In this work it is shown that the Hilbert transformation can be extended to the space $\mathscr{D}'$ of Schwartz distributions defined on the set of real numbers. Using the Hilbert transform, extended Plemelj and dispersion relations are proved to be valid in $\mathscr{D}'$, in analogy to the classical case [9], [28]. Furthermore, some Fourier transform properties are derived which generalize results that previously were known to hold in L_p-spaces, for $p \geqq 1$ [8], and for tempered distributions [6], [29]. Some theorems on products of certain distributions are included to facilitate the application of our results to distributional Hilbert problems, that will be discussed in another work.

Hilbert transforms for distributions in various subspaces of $\mathscr{D}'$ were investigated by a number of authors in [1], [3]–[5], [11]–[16], [18]–[21], [23], and [10, vol. I]. To this author's knowledge, there is, to date, no comprehensive treatment of the Hilbert transformation for arbitrary distributions in $\mathscr{D}'$. However, many of the results established here are closely related to and extend those discussed by Beltrami and Wohlers [1], [3], [4], Lauwerier [18], Bremermann [6], and Vladimirov [29], all of which apply to distributions in $\mathscr{S}'$, as well as those results obtained by Mitrović in [19]–[21], which pertain to distributions in the spaces O'_α ([5]).

In agreement with much of the literature, we define the distributional Hilbert transformation in terms of analytic representations of distributions, which take the place of the Cauchy integrals used in the classical theory. Since extensive use will be made of certain fundamental results from the theory of analytic representations, we recall briefly those definitions that will be needed.

All distributions in $\mathscr{D}'$ and test functions in $\mathscr{D}$ are assumed to be real-valued unless otherwise indicated.

* Received by the editors October 26, 1971.
† Department of Mathematics, Oakland University, Rochester, Michigan 48063.

DEFINITION 1.1. Let $f \in \mathscr{D}'$. An *analytic representation* of f is any function $\hat{f}(z)$ defined and analytic on the complement of the support of f such that for all test functions $\varphi \in \mathscr{D}$,

$$\lim_{y \to 0^+} \int_{-\infty}^{\infty} [\hat{f}(x + iy) - \hat{f}(x - iy)]\varphi(x)\,dx = \langle f(x), \varphi(x) \rangle.$$

THEOREM 1.1. *Let $f \in \mathscr{D}'$ be a distribution of compact support. Then an analytic representation of f is given by the generalized Cauchy integral*

$$\hat{f}(z) = \frac{1}{2\pi i} \left\langle f(t), \frac{1}{t - z} \right\rangle, \tag{1.1}$$

where $z = x + iy$, $y \neq 0$. For given f, $\hat{f}(z)$ is determined uniquely within the space of functions which are analytic for $y \neq 0$ and satisfy an inequality of the form

$$|\hat{f}(z)| \leqq M|y|^{-n} \quad \text{for } 0 < |y| < 1, \tag{1.2}$$

for some integer n and a constant M.

A proof of this theorem is found in [6] or in [24].

THEOREM 1.2. *Let f be a tempered distribution, that is let $f \in \mathscr{S}'$. Then there exist analytic representations of f. These may be chosen within the space of functions which are analytic for $y \neq 0$ and satisfy an inequality of the form*

$$|\hat{f}(z)| \leqq M(1 + |z|^2)^m |y|^{-n} \quad \text{for } 0 < |y| < 1, \tag{1.3}$$

for some integers m, n, and a constant M. For given $f \in \mathscr{S}'$, its analytic representations are determined by (1.3) *modulo a polynomial of degree $\leqq 2m$.*

This theorem is proved in [27].

THEOREM 1.3. *Every distribution $f \in \mathscr{D}'$ has an analytic representation $\hat{f}(z)$ such that*

$$|\hat{f}(z)| \leqq M(|z|)|y|^{-n(|z|)} \text{ for } 0 < |y| < 1, \tag{1.4}$$

for some continuous, monotonely increasing functions M and n. Any two analytic representations of f differ by an entire analytic function which obeys (1.4).

Proof. A complete proof is found in [27] and [5]. Parts of the proof are repeated here, since they will be used later. Let $f \in \mathscr{D}'$ be given. Let $I_k = (k - 1, k + 1)$ for $k = 0, \pm 1, \pm 2, \cdots$. Then by [10, vol. I], there exists a partition of unity $\{\alpha_k\}_{k=-\infty}^{\infty}$ such that $\alpha_k \in \mathscr{D}$, $\alpha_k(x) = 0$ if $x \notin I_k$ for $k = 0, \pm 1, \cdots$, and $\sum_{k=-\infty}^{\infty} \alpha_k(x) = 1$. For each k, let $f_k = \alpha_k f$. Then $f = \sum_{k=-\infty}^{\infty} f_k$. Let $\hat{f}_k(z)$ be an analytic representation of f_k, as given by Theorem 1.1. Then $\hat{f}_k(z)$ is analytic for $|z| \leqq |k| - 1$. Hence there exist polynomials $h_k(z)$ for $k = \pm 1, \pm 2, \cdots$ such that

$$|\hat{f}_k(z) - h_k(z)| \leqq 2^{-k} \quad \text{for } |z| \leqq |k| - 1.$$

Let

$$g(z) = \hat{f}_0(z) + \sum_{0 < |k| < \infty} [\hat{f}_k(z) - h_k(z)].$$

Then $g(z)$ is an analytic representation of f and satisfies an inequality of the form (1.4). Every analytic representation $\hat{f}(z)$ of f is of the form $\hat{f}(z) = g(z) + e(z)$, where $e(z)$ is an entire analytic function and $\hat{f}(z)$ satisfies (1.4).

2. Definition and basic properties of the distributional Hilbert transformation. The following lemma will provide the basis for the definition of the Hilbert transformation for arbitrary distributions.

LEMMA 2.1. *Let $f \in \mathscr{D}'$ and let $\hat{f}(z)$ be an analytic representation of f. Then the limit*

$$\lim_{y \to 0} \int_{-\infty}^{\infty} i[\hat{f}(x + iy) + \hat{f}(x - iy)]\varphi(x)\, dx$$

exists for every $\varphi \in \mathscr{D}$, and defines a continuous linear functional on $\mathscr{D}$.

Proof. Define u and v for $y > 0$ by

$$u(x, y) = \hat{f}(x + iy) - \hat{f}(x - iy)$$

and

$$v(x, y) = -i[\hat{f}(x + iy) + \hat{f}(x - iy)].$$

It is easily seen that u satisfies Laplace's equation for $y > 0$. Moreover, v is a harmonic conjugate of u for $y > 0$, that is u and v satisfy the Cauchy–Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{for } y > 0.$$

Let $g(\varphi; y)$ be a function of y defined for $y \geqq 0$ by

$$g(\varphi; y) = -\int_y^a dt \int_{-\infty}^{\infty} u_x(x, t)\varphi(x)\, dx \quad \text{for } y > 0,$$

$$g(\varphi; 0) = \lim_{y \to 0^+} -\int_y^a dt \int_{-\infty}^{\infty} u_x(x, t)\varphi(x)\, dx \quad \text{for } y = 0,$$

where $a > 0$. Note that $\lim_{t \to 0^+} \int_{-\infty}^{\infty} u_x(x, t)\varphi(x)\, dx = \langle f^{(1)}(x), \varphi(x)\rangle$.

For $y > 0$ we obtain from the Cauchy–Riemann equations

$$g(\varphi; y) = \int_{-\infty}^{\infty} v(x, y)\varphi(x)\, dx - \int_{-\infty}^{\infty} v(x, a)\varphi(x)\, dx.$$

Since $g(\varphi; y)$ is a continuous function of y for $y \geqq 0$, the limit

$$\lim_{y \to 0^+} \int_{-\infty}^{\infty} v(x, y)\varphi(x)\, dx = g(\varphi; 0) + \int_{-\infty}^{\infty} v(x, a)\varphi(x)\, dx \tag{2.1}$$

exists for $a > 0$. This limit is, in fact, independent of the choice of a, since for $a > 0$,

$$\begin{aligned} &\frac{d}{da}\left[g(\varphi; 0) + \int_{-\infty}^{\infty} v(x, a)\varphi(x)\, dx\right] \\ &= -\int_{-\infty}^{\infty} u_x(x, a)\varphi(x)\, dx + \int_{-\infty}^{\infty} v_a(x, a)\varphi(x)\, dx \\ &= 0. \end{aligned}$$

Hence the limit (2.1) is a constant function of a for $a > 0$, and therefore it is well-defined. Now let

$$\langle \mathscr{H}\hat{f}(x), \varphi(x)\rangle = \lim_{y\to 0^+} \int_{-\infty}^{\infty} v(x, y)\varphi(x)\, dx.$$

Then $\mathscr{H}\hat{f}(x)$ is a distribution in $\mathscr{D}'$ by [10, vol. I, p. 147]. This completes the proof.

It should be mentioned that the statement of Lemma 2.1 may be inferred from the work of Tillmann [27]. We gave another proof here to provide us with a method that has applications in particular to the theory of distributional boundary value problems, which will be discussed in another work.

In order to simplify the notation, we introduce the following definition.

DEFINITION 2.1. Let $u(x, y)$ be a function defined for $y \neq 0$. A distribution $f \in \mathscr{D}'$ is called the *distributional limit* of u at 0, written as $f(x) = \lim_{y\to 0} u(x, y)$, if for every test function $\varphi \in \mathscr{D}$,

$$\lim_{y\to 0} \int_{-\infty}^{\infty} u(x, y)\varphi(x)\, dx = \langle f(x), \varphi(x)\rangle.$$

We are now in a position to make the following definition.

DEFINITION 2.2. Let $f \in \mathscr{D}'$ and let $\hat{f}(z)$ be an analytic representation of f. Then the *Hilbert transform of f relative to* $\hat{f}(z)$ is defined to be the distributional limit

$$\mathscr{H}\hat{f}(x) = \lim_{y\to 0} i[\hat{f}(x + iy) + \hat{f}(x - iy)].$$

To justify the nomenclature, consider a distribution f of compact support. An analytic representation of f is given by (1.1) and yields the distributional limit

$$\begin{aligned} \mathscr{H}\hat{f}(x) &= \lim_{y\to 0} i[\hat{f}(x + iy) + \hat{f}(x - iy)] \\ &= \lim_{y\to 0} -\frac{1}{\pi i}\left\langle f(t), \frac{x - t}{(x - t)^2 + y^2}\right\rangle \\ &= \lim_{y\to 0} -\frac{1}{\pi} f(x) * \frac{x}{x^2 + y^2} = -\frac{1}{\pi} f(x) * pv\frac{1}{x}, \end{aligned} \tag{2.2}$$

by [10, vol. I, p. 95]. This agrees with the usual definition of the Hilbert transform for distributions of compact support [5].

Suppose now that $\hat{f}(z)$ and $\hat{F}(z)$ are analytic representations of the same distribution $f \in \mathscr{D}'$. It follows from Lemma 1.3 that $\hat{f}(z) - \hat{F}(z) = e(z)$ for some entire function $e(z)$, which satisfies an inequality of the form (1.4). Thus $\mathscr{H}\hat{f}(x)$ differs from $\mathscr{H}\hat{F}(x)$ by a term $\mathscr{H}\hat{f}(x) - \mathscr{H}\hat{F}(x) = 2i\, e(x)$. This shows that the Hilbert transform $\mathscr{H}\hat{f}(x)$ contains an entire function that depends on $\hat{f}(z)$, but not on f. More generally we have the following.

LEMMA 2.2. *The Hilbert transformation defines a mapping of $\mathscr{D}'$ onto itself. The Hilbert transform of a distribution $f \in \mathscr{D}'$ relative to any analytic representation $\hat{f}(z)$ agrees with an entire analytic function on $\mathbb{R}$ if and only if f agrees with an entire analytic function.*

Proof. Let $f \in \mathscr{D}'$, $\hat{f}(z)$ be an analytic representation of f and $g(x) = \mathscr{H}\hat{f}(x)$. Then $\hat{g}(z) = -i \operatorname{sgn} y \hat{f}(z)$ is an analytic representation of g and $f(x) = \mathscr{H}\hat{g}(x)$. This shows that the Hilbert transform maps $\mathscr{D}'$ onto itself. Now let f be a distribution in $\mathscr{D}'$ which agrees with an entire function e. Let $e(z)$ be the analytic continuation of e to the complex plane. Then every analytic representation of f is of the form $\hat{f}(z) = (1/2) \operatorname{sgn} y\, e(z) + e_0(z)$ for some entire function e_0. Hence $\mathscr{H}\hat{f}(x) = 2i\, e_0(x)$, and is entire. Conversely, suppose that for some analytic representation $\hat{f}(z)$, $\mathscr{H}\hat{f}(x)$ agrees with an entire function e on $\mathbb{R}$. Then $\hat{e}(z) = (1/2) \operatorname{sgn} y\, e(z)$ and $\hat{h}(z) = i \operatorname{sgn} y \hat{f}(z)$ are both analytic representations of $\mathscr{H}\hat{f}(x)$. Therefore, there is an entire function $e_1(z)$ such that $\hat{f}(z) = -(1/2)\, e(z) - i \operatorname{sgn} y\, e_1(z)$. Since

$$f(x) = \lim_{y \to 0^+} [\hat{f}(x + iy) - \hat{f}(x - iy)] = -2i\, e_1(x),$$

f agrees with an entire function.

3. Extended Plemelj and dispersion relations. Analogous to the case of ordinary functions, the distributional Hilbert transformation enables us to derive extended Plemelj and dispersion relations.

LEMMA 3.1 (Plemelj relations). *Let $f \in \mathscr{D}'$. Then the distributional limits*

$$\lim_{y \to 0^+} \hat{f}(x + iy) = \hat{f}_+(x) \quad \textit{and} \quad \lim_{y \to 0^-} \hat{f}(x + iy) = \hat{f}_-(x)$$

exist for every analytic representation $\hat{f}(z)$ of f and satisfy the Plemelj relations

$$\hat{f}_+(x) = \tfrac{1}{2} f(x) + \frac{1}{2i} \mathscr{H}\hat{f}(x), \tag{3.1a}$$

$$\hat{f}_-(x) = -\tfrac{1}{2} f(x) + \frac{1}{2i} \mathscr{H}\hat{f}(x). \tag{3.1b}$$

Let $\Phi_+(z)$ and $\Phi_-(z)$ be any two functions analytic for $\operatorname{Im} z > 0$ and for $\operatorname{Im} z < 0$, respectively. Suppose that for some $f \in \mathscr{D}'$ and all $\varphi \in \mathscr{D}$ we have

$$\langle f(x), \varphi(x) \rangle = \lim_{y \to 0^+} \int_{-\infty}^{\infty} \Phi_+(x + iy)\varphi(x)\, dx - \lim_{y \to 0^-} \int_{-\infty}^{\infty} \Phi_-(x + iy)\varphi(x)\, dx. \tag{3.2}$$

Define

$$\hat{f}(z) = \begin{cases} \Phi_+(z) & \text{for } \operatorname{Im} z > 0, \\ \Phi_-(z) & \text{for } \operatorname{Im} z < 0. \end{cases}$$

Then $\hat{f}(z)$ is an analytic representation of f. Let $\hat{f}_+$ and $\hat{f}_-$ be defined as in Lemma 3.1. The pair $(\hat{f}_+, \hat{f}_-)$ will be called a *Hilbert decomposition* of f. From the above argument it follows that every Hilbert decomposition of f is given in terms of an analytic representation.

Suppose now that $(\hat{f}_+, \hat{f}_-)$ and $(\hat{F}_+, \hat{F}_-)$ are two Hilbert deeompositions of f. Then it follows from Lemma 1.3 that

$$\hat{F}_+(x) - \hat{f}_+(x) = \hat{F}_-(x) - \hat{f}_-(x) = e(x) \tag{3.3}$$

for some entire analytic function $e(z)$. We note also that the Hilbert transform $\mathscr{H}\hat{f}$ has the decomposition

$$\mathscr{H}\hat{f}(x) = i[\hat{f}_+(x) + \hat{f}_-(x)]. \tag{3.4}$$

DEFINITION 3.1. The *space* $\mathscr{D}'_+$ is the space of all distributions $f \in \mathscr{D}'(\mathbb{R})$ such that $f(x) = \hat{f}_+(x)$ for some analytic representation $\hat{f}(z)$ of f. The *space* $\mathscr{D}'_-$ is the space of all distributions $f \in \mathscr{D}'(\mathbb{R})$ for which $f(x) = -\hat{f}_-(x)$ for some analytic representation $\hat{f}(z)$.

Distributions in $\mathscr{D}'_+$ are characterized by the following lemma.

LEMMA 3.2 (dispersion relations). *Let $f \in \mathscr{D}'$ be complex-valued and suppose that $f_1 = \operatorname{Re} f$, and $f_2 = \operatorname{Im} f$. Then $f \in \mathscr{D}'_+$ if and only if there exist analytic representations $\hat{f}_1(z)$ and $\hat{f}_2(z)$ of f_1 and f_2 such that the dispersion relations hold; that is*

(i) $f_1(x) = \mathscr{H}\hat{f}_2(x)$,

(ii) $f_2(x) = -\mathscr{H}\hat{f}_1(x)$.

If such a pair $\hat{f}_1(z), \hat{f}_2(z)$ exists, then it is unique.

Proof. Let us first prove the uniqueness. Suppose that $\hat{f}_1(z)$ and $\hat{F}_1(z)$ are analytic representations of f_1 for which (ii) holds. Then there exists an entire function $e_1(z)$, such that

$$\hat{f}_1(z) = \hat{F}_1(z) - e_1(z).$$

From (ii) we have

$$-\mathscr{H}\hat{f}_1(x) = f_2(x) = -\mathscr{H}\hat{f}_1(x) = -\mathscr{H}\hat{f}_1(x) + 2ie_1(x),$$

so that $e_1(x) = 0$ almost everywhere on $\mathbb{R}$. Since e_1 is entire, this implies that $e_1(z) \equiv 0$ by the uniqueness of analytic continuations. Hence $\hat{f}_1(z) = \hat{F}_1(z)$. Similarly we obtain that if $\hat{f}_2(z)$ and $\hat{F}_2(z)$ are two analytic representations of f_2 such that (i) holds, then $\hat{f}_2(z) = \hat{F}_2(z)$.

Now let $f \in \mathscr{D}'_+$. Then there exist analytic representations $\hat{f}_1(z)$ and $\hat{f}_2(z)$ of f_1 and f_2 such that for $\hat{f}(z) = \hat{f}_1(z) + i\hat{f}_2(z)$ we have $f(x) = \hat{f}_+(x)$. It follows from the Plemelj relations in Lemma 3.1 that

$$\begin{aligned} f(x) = \hat{f}_+(x) &= \tfrac{1}{2}f(x) + \frac{1}{2i}\mathscr{H}\hat{f}(x) \\ &= \tfrac{1}{2}[f_1(x) + if_2(x)] + \frac{1}{2i}[\mathscr{H}\hat{f}_1(x) + i\mathscr{H}\hat{f}_2(x)] \\ &= \tfrac{1}{2}f_1(x) + \tfrac{1}{2}\mathscr{H}\hat{f}_2(x) + i[\tfrac{1}{2}f_2(x) - \tfrac{1}{2}\mathscr{H}\hat{f}_1(x)]. \end{aligned}$$

Therefore we have

$$\operatorname{Re} f(x) = \tfrac{1}{2}f_1(x) + \tfrac{1}{2}\mathscr{H}\hat{f}_2(x) = f_1(x),$$
$$\operatorname{Im} f(x) = \tfrac{1}{2}f_2(x) - \tfrac{1}{2}\mathscr{H}\hat{f}_1(x) = f_2(x),$$

and hence

$$f_1(x) = \mathscr{H}\hat{f}_2(x),$$
$$f_2(x) = -\mathscr{H}\hat{f}_1(x).$$

Conversely, suppose that $f \in \mathscr{D}'$ and for $f_1 = \operatorname{Re} f$, $f_2 = \operatorname{Im} f$, there exist analytic representations $\hat{f}_1(z)$ and $\hat{f}_2(z)$, respectively, such that (i) and (ii) hold. Let $\hat{f}(z) = \hat{f}_1(z) + i\hat{f}_2(z)$. Then $\hat{f}(z)$ is an analytic representation of f and

$$\begin{aligned}
\lim_{y \to 0^+} \hat{f}(x + iy) &= \tfrac{1}{2} f(x) + \frac{1}{2i} \mathscr{H}\hat{f}(x) \\
&= \tfrac{1}{2}[f_1(x) + if_2(x)] + \frac{1}{2i}[\mathscr{H}\hat{f}_1(x) + i\mathscr{H}\hat{f}_2(x)] \\
&= \tfrac{1}{2} f_1(x) + \tfrac{1}{2}\mathscr{H}\hat{f}_2(x) + i[\tfrac{1}{2} f_2(x) - \tfrac{1}{2}\mathscr{H}\hat{f}_1(x)] \\
&= f_1(x) + if_2(x) \\
&= f(x).
\end{aligned}$$

This completes the proof.

From Lemma 3.2 there follow these corollaries.

COROLLARY 3.3. *Let $f \in \mathscr{D}'$, $f_1 = \operatorname{Re} f$, $f_2 = \operatorname{Im} f$. Then $f \in \mathscr{D}'_-$ if and only if there exist analytic representations $\hat{f}_1(z)$ and $\hat{f}_2(z)$ such that*

(i) $f_1(x) = -\mathscr{H}\hat{f}_2(x)$,

(ii) $f_2(x) = \mathscr{H}\hat{f}_1(x)$.

If such $\hat{f}_1(z)$ and *$\hat{f}_2(z)$ exist, they are unique.*

COROLLARY 3.4. *Let $f \in \mathscr{D}'_+$. If f is real-valued, then f agrees with a real entire function.*

Proof. If $f = f_1 + if_2$ is real-valued then $f_2 = 0$. By Lemma 3.2, f_1 is the Hilbert transform of an entire function (namely of zero). Thus f_1 is entire by Lemma 2.2.

The following is an extension of a theorem given in [4, § 3] for distributions in $\mathscr{S}'_+ = \mathscr{S}' \cap \mathscr{D}'_+$.

LEMMA 3.5. *Let $\hat{f}(z)$ be an analytic representation of $f \in \mathscr{D}'_+$. Let Ω be an open subset of $\mathbb{R}$. Then $\hat{f}(z)$ may be continued analytically across Ω if and only if the real part of f agrees with an analytic function on Ω.*

Proof. Without loss of generality it may be assumed that Ω is an interval. Suppose first that $f_1 = \operatorname{Re} f$ agrees with an analytic function on Ω. Let $\hat{f}(z)$ be an analytic representation of f, with $u(x, y) = \operatorname{Re} \hat{f}(x + iy)$, $v(x, y) = \operatorname{Im} \hat{f}(x + iy)$ and $\lim_{y \to 0^+} \hat{f}(x + iy) = f(x)$. Then $u(x, y)$ has the distributional limit $\lim_{y \to 0^+} u(x, y) = f_1(x)$. Define

$$\begin{aligned}
G(x + iy) &= (1/2)[u(x, y) + iv(x, y)] \quad \text{for } y > 0, \\
G(x + iy) &= -\overline{G(x - iy)} \quad \text{for } y < 0.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{y \to 0^+} [G(x + iy) - G(x - iy)] &= \lim_{y \to 0^+} [G(x + iy) + \overline{G(x + iy)}] \\
&= \lim_{y \to 0^+} u(x, y) = f_1(x).
\end{aligned}$$

Since $G(z)$ is analytic for $y \neq 0$, it is therefore an analytic representation of f_1. Thus we may write $G(z) = \hat{f}_1(z)$. It follows that $\hat{f}(z) = 2\hat{f}_1(z)$ for $y > 0$.

Let K be a compact subset of Ω. Choose $\alpha \in \mathscr{D}$ such that $\alpha(x) = 1$ if $x \in K$ and $\alpha(x) = 0$ if $x \notin \Omega$. Let $g(x) = \alpha(x)f_1(x)$ and $h(x) = [1 - \alpha(x)]f_1(x)$. Then g agrees with an analytic function on K, whereas $h(x) = 0$ for $x \in K$. Let $\hat{g}(z)$ and $\hat{h}(z)$ be analytic representations of g and h such that $\hat{f}_1(z) = \hat{g}(z) + \hat{h}(z)$ with $\hat{f}_1(z)$ as above. By [22, p. 38], $\hat{g}(x + iy)$ and $\hat{h}(x + iy)$ converge uniformly on K to $\hat{g}_+(x)$ and $\hat{h}_+(x)$, respectively, as $y \to 0^+$, since both g and h agree with an analytic function on K. $\hat{g}_+(x)$ and $\hat{h}_+(x)$ are analytic on K. Define $F(z)$ by

$$F(z) = \begin{cases} \hat{f}(z) & \text{for } \operatorname{Im} z > 0, \\ \hat{f}(\bar{z}) & \text{for } \operatorname{Im} z < 0. \end{cases}$$

Then $F(z) = 2[\hat{g}(z) + \hat{h}(z)]$ for $\operatorname{Im} z > 0$. Hence $F(x + iy)$ converges uniformly on K as $y \to 0^+$ and as $y \to 0^-$ to $2[\hat{g}_+(x) + \hat{h}_+(x)]$ and agrees with an analytic function on K. This proves that $F(z)$ and thus $\hat{f}(z)$ have an analytic continuation across K from $\operatorname{Im} z > 0$ to $\operatorname{Im} z < 0$. Since K was an arbitrary compact subset of Ω, it follows that $\hat{f}(z)$ may be continued analytically across Ω. Any other analytic representation of f differs from $\hat{f}(z)$ by an entire function, and may therefore be continued analytically across Ω as well.

Conversely, suppose that f has an analytic representation $\hat{f}(z)$ that may be continued analytically across Ω. Since $f \in \mathscr{D}'_+$, $f(x) = \hat{f}_+(x) + e(x)$ for some entire analytic function e. This shows that f agrees with an analytic function on Ω, and so does $f_1 = \operatorname{Re} f$.

4. The Fourier transforms of $\hat{f}_+$, $\hat{f}_-$, and $\mathscr{H}\hat{f}$. In this section we will derive the Fourier transforms of the distributions $\hat{f}_+ \hat{f}_-$, and $\mathscr{H}\hat{f}$, for arbitrary $f \in \mathscr{D}'(\mathbb{R})$ and any analytic representations $\hat{f}(z)$ of f. As expected, the results are generalizations of the classical theorems and extend earlier results of other authors for distributions in subspaces of $\mathscr{D}'(\mathbb{R})$.

We will need the corresponding Fourier transforms pertaining to distributions of compact support in order to derive the identities for the general case. Let f be a distribution of compact support. By Theorem 1.1, an analytic representation of f is given by

$$\hat{f}(z) = \frac{1}{2\pi i}\left\langle f(t), \frac{1}{t - z}\right\rangle.$$

From (2.2) we have

$$\mathscr{H}\hat{f}(x) = -\frac{1}{\pi} f(x) * pv\frac{1}{x}.$$

Applying the convolution theorem for the distributional Fourier transform, we obtain

$$\begin{aligned} \mathscr{F}(\mathscr{H}\hat{f})(s) &= \mathscr{F}\left(-\frac{1}{\pi}pv\frac{1}{x}\right)(s)\mathscr{F}f(s) \\ &= -i \operatorname{sgn} s\mathscr{F}f(s). \end{aligned} \tag{4.1}$$

Using the Plemelj relations (3.1), we have

$$\begin{aligned} \mathscr{F}\hat{f}_+(s) &= \tfrac{1}{2}\mathscr{F}f(s) + \frac{1}{2i}\mathscr{F}(\mathscr{H}\hat{f})(s) \\ &= \tfrac{1}{2}\mathscr{F}f(s) - \tfrac{1}{2}\operatorname{sgn} s\mathscr{F}f(s) \\ &= H(-s)\mathscr{F}f(s). \end{aligned} \tag{4.2}$$

Here $H(s)$ denotes Heaviside's unit step function. Similarly we obtain

$$\mathscr{F}f_-(s) = -H(s)\mathscr{F}f(s). \tag{4.3}$$

We now turn to the general case of distributions in $\mathscr{D}'$.

THEOREM 4.1. *Let $f \in \mathscr{D}'$ and let $\hat{f}(z)$ be an analytic representation of f. Then*

$$\mathscr{F}\hat{f}_+(s) = H(-s)\mathscr{F}f(s) + \sum_{j=0}^{\infty} a_j\delta^{(j)}(s), \tag{4.4}$$

$$\mathscr{F}\hat{f}_-(s) = -H(s)\mathscr{F}f(s) + \sum_{j=0}^{\infty} a_j\delta^{(j)}(s), \tag{4.5}$$

$$\mathscr{F}(\mathscr{H}\hat{f})(s) = -i \operatorname{sgn} s\mathscr{F}f(s) + 2i\sum_{j=0}^{\infty} a_j\delta^{(j)}(s), \tag{4.6}$$

where the a_j are constants such that $\sum_{j=0}^{\infty} a_j(iz)^j$ is an entire analytic function depending only on $\hat{f}(z)$ but not on f.

Remark. The products on the right sides of equations (4.4)–(4.6) are not well-defined. They are to be interpreted in the following sense.

For suitably chosen distributions f_n of compact support such that $f = \sum_{n=1}^{\infty} f_n$, there exist constants a_{nj}, $n = 1, 2, \cdots, j = 0, 1, \cdots, k_n$, such that

$$\mathscr{F}\hat{f}_+(s) = \sum_{n=1}^{\infty}\left[H(-s)\mathscr{F}f_n(s) - \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s)\right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s). \tag{4.7}$$

Here the c_j, $j = 0, 1, \cdots$, are constants such that $\sum_{j=0}^{\infty} c_j(iz)^j$ is an entire analytic function. For the same f_n's,

$$\mathscr{F}\hat{f}_-(s) = \sum_{n=1}^{\infty}\left[-H(s)\mathscr{F}f_n(s) - \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s)\right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s), \tag{4.8}$$

$$\begin{aligned}\mathscr{F}(\mathscr{H}\hat{f})(s) = {} & \sum_{n=1}^{\infty}\left[-i \operatorname{sgn} s\mathscr{F}f_n(s) - 2i\sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s)\right] \\ & + 2i\sum_{j=0}^{\infty} c_j\delta^{(j)}(s).\end{aligned} \tag{4.9}$$

Theorem 4.1 will be proved using this interpretation.

Proof of Theorem 4.1. Let $f \in \mathscr{D}'$ and let $\hat{f}(z)$ be given. By Theorem 1.3 there exists an entire function $e(z)$ such that

$$\hat{f}(z) = \hat{f}_0(z) + \sum_{0<|v|<\infty} [\hat{f}_v(z) - h_v(z)] + e(z),$$

where $f_0, f_v, \hat{f}_v(z)$ and $h_v(z)$ for $v = \pm 1, \pm 2, \cdots$, are defined as in Theorem 1.3. Let $\psi \in Z$ be given by $\psi = \mathscr{F}\varphi$ for $\varphi \in \mathscr{D}$. Then there exists an integer N such that supp $\varphi \subset (-N+1, N-1)$. We note that for $|v| \geqq N$, $\hat{f}_v(z)$ is analytic for $|z| \leqq N - 1$ and $\sum_{N \leqq |v| < \infty} [\hat{f}_v(z) - h_v(z)]$ converges uniformly to a function analytic for $|z| \leqq N - 1$. By the definition of the distributional Fourier transform, we have

$$\begin{aligned}
\langle \mathscr{F}\hat{f}_+(s), \psi(s)\rangle &= \frac{1}{2\pi}\langle \hat{f}_+(x), \varphi(x)\rangle = \frac{1}{2\pi}\lim_{y\to 0^+}\langle \hat{f}(x+iy), \varphi(x)\rangle \\
&= \frac{1}{2\pi}\lim_{y\to 0^+}\left\{\left\langle \hat{f}_0(x+iy) + \sum_{0<|\nu|<\infty}[\hat{f}_\nu(x+iy) - h_\nu(x+iy)], \varphi(x)\right\rangle \right. \\
&\qquad \left. + \langle e(x+iy), \varphi(x)\rangle\right\} \\
&= \frac{1}{2\pi}\left\langle \hat{f}_{0+}(x) + \sum_{0<|\nu|<N}[\hat{f}_{\nu+}(x) - h_\nu(x)], \varphi(x)\right\rangle \\
&\quad + \frac{1}{2\pi}\lim_{y\to 0^+}\left\langle \sum_{N\leqq|\nu|<\infty}[\hat{f}_\nu(x+iy) - h_\nu(x+iy)], \varphi(x)\right\rangle \\
&\quad + \frac{1}{2\pi}\langle e(x), \varphi(x)\rangle.
\end{aligned}$$

We may apply the results for distributions of compact support to the first finite sum. For the infinite series we observe that the series converges uniformly on the support of φ to an analytic function. Each term converges uniformly as $y \to 0^+$ to $\hat{f}_{\nu+}(x) - h_\nu(x)$ by [22, p. 38]. Thus, by the continuity of the functionals involved we may interchange limits, sums, and integrals to obtain

$$\begin{aligned}
\langle \mathscr{F}\hat{f}_+(s), \psi(s)\rangle &= \left\langle H(-s)\mathscr{F}f_0(s) + \sum_{0<|\nu|<N}\left[H(-s)\mathscr{F}f_\nu(s) - \sum_{j=0}^{k_\nu} a_{\nu_j}\delta^{(j)}(s)\right], \psi(s)\right\rangle \\
&\quad + \sum_{N\leqq|\nu|<\infty}\frac{1}{2\pi}\langle \hat{f}_{\nu+}(x) - h_\nu(x), \varphi(x)\rangle \\
&\quad + \frac{1}{2\pi}\langle e(x), \varphi(x)\rangle \\
&= \left\langle H(-s)\mathscr{F}f_0(s) + \sum_{0<|\nu|<\infty}\left[H(-s)\mathscr{F}f_\nu(s) - \sum_{j=0}^{k_\nu} a_{\nu_j}\delta^{(j)}(s)\right], \psi(s)\right\rangle \\
&\quad + \left\langle \sum_{j=0}^{\infty} c_j\delta^{(j)}(s), \psi(s)\right\rangle,
\end{aligned}$$

where the a_{ν_j}, $j = 0, 1, \cdots, k_\nu$; $\nu = \pm 1, \pm 2, \cdots$ are constants such that $h_\nu(z) = \sum_{j=0}^{k_\nu} a_{\nu_j}(iz)^j$ and c_j, $j = 0, 1, \cdots$, are constants such that $e(z) = \sum_{j=0}^{\infty} c_j(iz)^j$. With $a_{00} = 0$, $k_0 = 0$, we thus obtain, since ψ was arbitrary,

$$\begin{aligned}
\mathscr{F}\hat{f}_+(s) &= \sum_{\nu=-\infty}^{\infty}\left[H(-s)\mathscr{F}f_\nu(s) - \sum_{j=0}^{k_\nu} a_{\nu_j}\delta^{(j)}(s)\right] \\
&\quad + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s).
\end{aligned}$$

Similarly, for the same f_ν and constants $a_{\nu j}, c_j$,

$$\mathscr{F}\hat{f}_-(s) = \sum_{\nu=-\infty}^{\infty}\left[-H(s)\mathscr{F}f_\nu(s) - \sum_{j=0}^{k_\nu} a_{\nu j}\delta^{(j)}(s)\right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s),$$

$$\mathscr{F}(\mathscr{H}\hat{f})(s) = \sum_{\nu=-\infty}^{\infty}\left[-i\operatorname{sgn} s\mathscr{F}f_\nu(s) - 2i\sum_{j=0}^{k_\nu} a_{\nu j}\delta^{(j)}(s)\right] + 2i\sum_{j=0}^{\infty} c_j\delta^{(j)}(s).$$

Using Theorem 4.1 we find that distributions in $\mathscr{D}'_+$ are characterized by the following lemma.

LEMMA 4.2. *Let $f \in \mathscr{D}'$. Then $f \in \mathscr{D}'_+$ if and only if there exist distributions f_n, $n = 1, 2, \cdots$, of compact support such that $f = \sum_{n=1}^{\infty} f_n$ and*

$$\mathscr{F}f(s) = \sum_{n=1}^{\infty}\left[H(-s)\mathscr{F}f_n(s) - \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s)\right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s)$$

for some constants a_{nj}, $j = 0, 1, \cdots, k_n$; $n = 1, 2, \cdots$, and c_j, $j = 0, 1, \cdots$, such that $\sum_{j=0}^{\infty} c_j(iz)^j$ is entire.

Proof. Without loss of generality we may assume that all but a finite number of the f_n vanish on any compact subset of $\mathbb{R}$. For $i = 1, 2, \cdots$, let N_i be the smallest integer such that for all $n > N_i$ $f_n(x) = 0$ for $x \in (-i, i)$. Let $N_0 = 0$. Define

$$g_i(x) = \sum_{n=N_i+1}^{N_{i+1}} f_n(x), \qquad i = 0, 1, 2, \cdots.$$

Then g_i is a distribution of compact support for $i = 0, 1, \cdots$ and $g_i(x) = 0$ for $x \in (-i, i)$. Furthermore, $f = \sum_{i=0}^{\infty} g_i$. Define $\hat{g}_i(z)$ by

$$\hat{g}_i(z) = \frac{1}{2\pi i}\left\langle g_i(t), \frac{1}{t-z}\right\rangle \quad \text{for } i = 0, 1, \cdots.$$

Let $h_i(z)$ be polynomials such that for $i = 1, 2, \cdots$,

$$|\hat{g}_i(z) - h_i(z)| < 2^{-i} \quad \text{for } |z| \leqq i - 1,$$

and $h_0(z) \equiv 0$. Let $\hat{f}(z)$ be an analytic representation of f. Then there exists (analogous to Theorem 1.3) an entire function $e(z)$ for which

$$\hat{f}(z) = \sum_{i=0}^{\infty}[\hat{g}_i(z) - h_i(z)] + e(z).$$

As in Theorem 4.1 we obtain

$$\mathscr{F}\hat{f}_+(s) = \sum_{i=0}^{\infty} \left[H(-s)\mathscr{F}g_i(s) - \sum_{j=0}^{m_i} b_{ij}\delta^{(j)}(s) \right] + \sum_{j=0}^{\infty} d_j\delta^{(j)}(s),$$

where $\sum_{j=0}^{m_i} b_{ij}(iz)^j = h_i(z)$ and $\sum_{j=0}^{\infty} d_j(iz)^j = e(z)$.

Now consider $\mathscr{F}f(s)$. By assumption $\mathscr{F}f(s)$ has the form

$$\begin{aligned}
\mathscr{F}f(s) &= \sum_{n=1}^{\infty} \left[H(-s)\mathscr{F}f_n(s) - \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s) \right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s) \\
&= \sum_{i=0}^{\infty} \sum_{n=N_i+1}^{N_{i+1}} \left[H(-s)\mathscr{F}f_n(s) - \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s) \right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s) \\
&= \sum_{i=0}^{\infty} \left[H(-s)\mathscr{F}g_i(s) - \sum_{n=N_i+1}^{N_{i+1}} \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s) \right] + \sum_{j=0}^{\infty} c_j\delta^{(j)}(s).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathscr{F}f(s) - \mathscr{F}\hat{f}_+(s) = &\sum_{i=0}^{\infty} \left[\sum_{j=0}^{m_i} b_{ij}\delta^{(j)}(s) - \sum_{n=N_i+1}^{N_{i+1}} \sum_{j=0}^{k_n} a_{nj}\delta^{(j)}(s) \right] \\
&+ \sum_{j=0}^{\infty} (c_j - d_j)\delta^{(j)}(s).
\end{aligned}$$

Therefore,

$$f(x) - \hat{f}_+(x) = \sum_{i=0}^{\infty} [h_i(x) - p_i(x)] + [e_0(x) - e(x)],$$

where

$$p_i(z) = \sum_{n=N_i+1}^{N_{i+1}} \sum_{j=0}^{k_n} a_{nj}(iz)^j \quad \text{and} \quad e_0(z) = \sum_{j=0}^{\infty} c_j(iz)^j.$$

Now, in the sense of equality in $\mathscr{D}'$,

$$\begin{aligned}
\sum_{i=0}^{\infty} [h_i(x) - p_i(x)] &= \sum_{i=0}^{\infty} \left[\sum_{j=0}^{m_i} b_{ij}(ix)^j - \sum_{n=N_i+1}^{N_{i+1}} \sum_{j=0}^{k_n} a_{nj}(ix)^j \right] \\
&= \sum_{j=0}^{\infty} e_j x^j
\end{aligned}$$

for some constants e_j, since limits and sums are interchangeable in $\mathscr{D}'$. This series converges in $\mathscr{D}'$, since it is the inverse Fourier transform of a series which converges in Z'. This shows that $\{\sum_{i=0}^{n} [h_i(x) - p_i(x)]\}_{n=1}^{\infty}$ converges in $\mathscr{D}'$. However, convergence of a power series in the topology of $\mathscr{D}'$ implies the uniform convergence of the series on every compact subset of $\mathbb{R}$. Thus, $\sum_{i=0}^{\infty} [h_i(x) - p_i(x)]$ agrees with an entire analytic function on $\mathbb{R}$ and so does $f(x) - \hat{f}_+(x)$. Therefore, $f \in \mathscr{D}'_+$.

For $f \in \mathscr{S}'$, let $\hat{f}(z)$ be as in Theorem 1.2. Let $\mathscr{S}'_+ = \mathscr{S}' \cap \mathscr{D}'_+$. Then Lemma 4.2 implies that $f \in \mathscr{S}'_+$ if and only if $\mathscr{F}f(s)$ is a distribution in $\mathscr{S}'$ which has its support contained in $(-\infty, 0]$, or, equivalently, if and only if f is the Fourier transform of a distribution with support on the positive half-axis. This result is well known [29], [6].

5. Multiplication in $\mathscr{D}'_+$. As was mentioned above, the space $\mathscr{S}'_+ = \mathscr{S}' \cap \mathscr{D}'_+$ is identical to the space of all distributions in $\mathscr{S}'$ whose Fourier transforms have support on the negative half-axis. Thus for $f, g \in \mathscr{S}'_+$, the convolution product $\mathscr{F}f * \mathscr{F}g$ is well-defined. Therefore the product of f and g exists and may be explained [6], [29] as

$$(f \cdot g)(x) = \mathscr{F}^{-1}(\mathscr{F}f * \mathscr{F}g)(x).$$

In the light of Theorem 4.1 and Lemma 4.2, this definition has an obvious generalization to distributions in $\mathscr{D}' \cap Z'$ whose Fourier transforms are again in $\mathscr{D}' \cap Z'$. However, there does not appear to be known an analogous result for arbitrary distributions in $\mathscr{D}'_+$. Therefore we restrict ourselves here to the discussion of particular products that are useful in the solution of distributional Hilbert problems.

LEMMA 5.1. *Let $f \in \mathscr{D}'$. Let $\hat{h}(z)$ be an analytic representation of an infinitely differentiable function h. Then the distributional limit*

$$\lim_{y \to 0^+} \hat{h}(x + iy)\hat{f}(x + iy) = \hat{h}_+(x)\hat{f}_+(x)$$

exists and agrees with the usual product, for every analytic representation $\hat{f}(z)$ of f.

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R})$ have support K. Define $\alpha \in \mathscr{D}(\mathbb{R})$ to be such that $\alpha(x) \equiv 1$ on K. Then $(\alpha f)(x)$ has compact support. Thus there exists a continuously differentiable function g of compact support such that

$$(\alpha f)(x) = \frac{d^k g}{dx^k}(x) \quad \text{for } x \in K.$$

By [22, p. 38], the Cauchy integral $\hat{g}(z) = 1/(2\pi i)\langle g(t), 1/(t - z)\rangle$ converges uniformly to $\hat{g}_+(x)$ as $\operatorname{Im} z \to 0^+$. If $f_2(x) = [1 - \alpha(x)]f(x)$, then $f(x) = (\alpha f)(x) + f_2(x)$. Pick an analytic representation of f of the form

$$\hat{f}(z) = \frac{d^k \hat{g}}{dz^k}(z) + \hat{f}_2(z)$$

for some analytic representation $\hat{f}_2(z)$ of f_2. Then $\hat{f}_2(z)$ agrees with an analytic function on K and thus $\hat{h}(x + iy)\hat{f}_2(x + iy)$ converges uniformly to $\hat{h}_+(x)\hat{f}_{2+}(x)$ as $y \to 0^+$. Therefore we obtain

$$\begin{aligned}
&\lim_{y \to 0^+} \int_{-\infty}^{\infty} \hat{h}(x + iy)\hat{f}(x + iy)\varphi(x)\,dx \\
&= \lim_{y \to 0^+} \int_{-\infty}^{\infty} \hat{h}(x + iy)\hat{g}^{(k)}(x + iy)\varphi(x)\,dx \\
&\quad + \lim_{y \to 0^+} \int_{-\infty}^{\infty} \hat{h}(x + iy)\hat{f}_2(x + iy)\varphi(x)\,dx \\
&= \lim_{y \to 0^+} \int_{-\infty}^{\infty} (-1)^k \hat{g}(x + iy)\frac{d^k}{dx^k}(\hat{h}(x + iy)\varphi(x))\,dx \\
&\quad + \lim_{y \to 0^+} \int_{-\infty}^{\infty} \hat{h}(x + iy)\hat{f}_2(x + iy)\varphi(x)\,dx \\
&= \int_{-\infty}^{\infty} (-1)^k \hat{g}_+(x)\frac{d^k}{dx^k}(\hat{h}_+(x)\varphi(x))\,dx + \langle \hat{h}_+(x)\hat{f}_{2+}(x), \varphi(x)\rangle
\end{aligned}$$

$$= \langle (\alpha \hat{f})_+(x)\hat{h}_+(x), \varphi(x)\rangle + \langle \hat{h}_+(x)\hat{f}_{2_+}(x), \varphi(x)\rangle$$
$$= \langle \hat{h}_+(x)\hat{f}_+(x), \varphi(x)\rangle .$$

Thus

$$\lim_{y\to 0^+} \hat{h}(x + iy)\hat{f}(x + iy) = \hat{h}_+(x)\hat{f}_+(x).$$

Other analytic representations of f differ from $\hat{f}(z)$ by an entire function, so that the corresponding limits still exist.

LEMMA 5.2. *Let $h \in \mathscr{D}'$. Suppose h agrees with an infinitely differentiable function on $\mathbb{R}$ except possibly at $x = 0$. Then the distributional limit*

$$\lim_{y\to 0^+} \hat{h}(x + iy)\hat{f}(x + iy)$$

exists for every $f \in \mathscr{D}'$ and all analytic representations $\hat{f}(z)$ of f and $\hat{h}(z)$ of h, and represents a distribution in $\mathscr{D}'_+$.

Proof. Let U_0 be a neighborhood of zero. Let α be such that $\alpha(x) = 0$ for all $x \in U_0$ and $(1 - \alpha) \in \mathscr{D}$. Let $h_1 = \alpha h$, and $h_2 = (1 - \alpha)h$. Let β be such that $\beta(x) = 0$ in a neighborhood of the support of $(1 - \alpha)$, and $(1 - \beta) \in \mathscr{D}$. Define $f_1 = \beta f$ and $f_2 = (1 - \beta)f$.

$f_1(x)$ vanishes in a neighborhood of the support of h_2, so that every analytic representation $\hat{f}_1(z)$ of f_1 is analytic in a neighborhood (in the z-plane) of the support of h_2. Hence the distributional limit

$$\lim_{y\to 0^+} h_2(x + iy)\hat{f}_1(x + iy) = \hat{h}_{2_+}(x)\hat{f}_{2_+}(x)$$

exists by Lemma 5.1 and is a distribution in $\mathscr{D}'_+$.

Lemma 5.1 also applies to

$$\lim_{y\to 0+} \hat{h}_1(x + iy)[\hat{f}_1(x + iy) + \hat{f}_2(x + iy)] = \hat{h}_{1+}(x)[\hat{f}_{1+}(x) + \hat{f}_{2+}(x)]$$

since h_1 is infinitely differentiable.

Since f_2 and h_2 are both distributions of compact support, they may be considered distributions in $\mathscr{S}'$, so that we may choose analytic representations $\hat{f}_2(z)$ and $\hat{h}_2(z)$ of f_2 and h_2 such that $\hat{f}_{1+}$ and $\hat{h}_{2+}$ are in $\mathscr{S}'_+$. Then

$$\lim_{y\to 0^+} \hat{h}_2(x + iy)\hat{f}_2(x + iy) = h_{2_+}(x)\hat{f}_{2_+}(x)$$

is a distribution in $\mathscr{S}'_+ \subset \mathscr{D}'_+$. Thus for the particular analytic representations of f and h given in terms of $\hat{f}_1(z)$, $\hat{f}_2(z)$, and $\hat{h}_1(z)$, $\hat{h}_2(z)$ as above, by $\hat{f}(z) = \hat{f}_1(z) + \hat{f}_2(z)$ and $\hat{h}(z) = \hat{h}_1(z) + \hat{h}_2(z)$, the distributional limit $\lim_{y\to 0^+} f(x + iy)\hat{h}(x + iy)$ exists and represents a distribution in $\mathscr{D}'_+$. Any other analytic representations of f and h differ from $\hat{f}(z)$ and $\hat{h}(z)$ by an entire function. From Lemma 5.1 it follows that the corresponding limits exist and are in $\mathscr{D}'_+$.

CORROLLARY 5.3. *Let $h \in \mathscr{D}'$ agree with an infinitely differentiable function except at a finite number of points. Then*

$$\lim_{y\to 0^+} \hat{h}(x + iy)\hat{f}(x + iy)$$

exists and is in $\mathscr{D}'_+$ for every analytic representation $\hat{h}(z)$ and for all analytic representations $\hat{f}(z)$ of any distribution $f \in \mathscr{D}'$.

REFERENCES

[1] E. J. BELTRAMI AND M. R. WOHLERS, *Distributional boundary value theorems and Hilbert transforms*, Arch. Rational Mech. Anal., 18 (1965), pp. 304–309.

[2] ———, *Distributional boundary values of functions holomorphic in a half-plane*, J. Math. Mech., 15 (1966), pp. 137–146.

[3] ———, *Distributions and the Boundary Values of Analytic Functions*, Academic Press, New York, 1966.

[4] ———, *The Cauchy integral of tempered distributions and some theorems on analytical continuation*, SIAM J. Appl. Math., 15 (1967), pp. 1077–1087.

[5] H. J. BREMERMANN, *Distributions, Complex Variables, and Fourier Transforms*, Addison-Wesley, Reading, Mass., 1965.

[6] ———, *Some remarks on analytic representations and products of distributions*, SIAM J. Appl. Math., 15 (1967), pp. 929–943.

[7] H. J. BREMERMANN AND L. DURAND, *On analytic continuation, multiplication, and Fourier transformations of Schwartz distributions*, J. Math. and Phys., 2 (1961), pp. 240–258.

[8] P. L. BUTZER AND W. TREBELS, *Hilbert Transformation, gebrochene Integration und Differentiation*, Forschungsberichte des Landes Nordrhein-Westfalen, Nr. 1889, Westdeutscher Verlag, Köln, Germany, 1968.

[9] F. D. GAKHOV, *Boundary Value Problems*, Addison-Wesley, Reading, Mass., 1966.

[10] I. M. GEL'FAND AND G. E. SHILOV, *Generalized Functions*, Academic Press, New York, vol. I, 1964, vol. II, 1968.

[11] W. GÜTTINGER, *Generalized functions and dispersion relations in physics*, Fortschr. Physik, 14 (1966), pp. 483–602.

[12] ———, *Generalized functions in elementary particle physics and passive systems theory*, SIAM J. Appl. Math., 15 (1967), pp. 964–1000.

[13] J. HORVATH, *Sur l'iteration de la transformée de Hilbert d'une distribution complexe*, C. R. Acad. Sci. Paris Sér. A-B, 237 (1953), pp. 1480–1482.

[14] ———, *Hilbert transforms of distributions in R^n*, Proc. International Congress of Mathematics, Amsterdam, 1954, pp. 122–123.

[15] ———, *Singular integral operators and spherical harmonics*, Trans. Amer. Math. Soc., 82 (1956), pp. 52–63.

[16] D. S. JONES, *Some remarks on Hilbert transforms*, J. Inst. Math. Appl., 1 (1965), pp. 226–240.

[17] G. KÖTHE, *Die Randverteilungen analytischer Funktionen*, Math. Z., 57 (1952), pp. 13–33.

[18] H. A. LAUWERIER, *The Hilbert problem for generalized functions*, Arch. Rational Mech. Anal., 13 (1963), pp. 157–166.

[19] D. MITROVIĆ, *Plemelj formulas and analytic representations of distributions*, Glasnik Mat. Ser. III, 3 (1968), no. 23, pp. 231–239.

[20] ———, *Analytic representations of distributions in O'_α*, Ibid., 5 (1970), no. 25, pp. 43–49.

[21] ———, *The Plemelj distributional formulas*, Bull. Amer. Math. Soc., 77 (1971), pp. 562–563.

[22] N. I. MUSKHELISHVILI, *Singular Integral Equations*, P. Noordhoff, Groningen, the Netherlands, 1958.

[23] R. W. NEWCOMB, *Hilbert transforms and positive real functions*, Proc. IRE, 50 (1962), pp. 2516–2517.

[24] H. G. TILLMANN, *Randverteilungen analytischer Funktionen und Distributionen*, Math. Z., 59 (1953), pp. 61–83.

[25] ———, *Die Fortsetzung analytischer Funktionale*, Abh. Math. Sem. Univ. Hamburg, 21 (1957), pp. 139–154.

[26] ———, *Distributionen als Randverteilungen analytischer Funktionen II*, Math. Z., 76 (1961), pp. 5–21.

[27] ———, *Darstellung der Schwartzschen Distributionen durch analytische Funktionen*, Ibid., 77 (1961), pp. 106–124.

[28] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London, 1948.

[29] N. VLADIMIROV, *Construction of envelopes of holomorphy for regions of a special type and their application*, Trudy Mat. Inst. Steklov, 60 (1961), pp.101–104.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

A-STABLE METHODS AND PADÉ APPROXIMATIONS TO THE EXPONENTIAL*

BYRON L. EHLE†

Abstract. The set of Padé approximations to the exponential function is studied. It is shown that all entries on the first and second subdiagonal of the Padé table are analytic and bounded by 1 in the entire left half-plane. These results are then applied to the problem of producing A-stable numerical methods for solving initial value problems. It is shown that they easily permit one to generate several classes of methods of arbitrarily high order which are A-stable.

1. Introduction. There is currently considerable interest in numerical methods for solving systems of ordinary differential equations which exhibit the property of stiffness. A number of numerical methods have been proposed to solve such problems [8], [9], [12], [13]. Nearly all are designed to produce an approximation to the exponential function whose modulus is bounded by 1 when solving the initial value problem

$$y' = qy, \quad y(0) = 1, \tag{1}$$

with an arbitrary step size h, when q is any complex number with negative real part. Methods satisfying this condition are called A-stable [5]. A-stable methods generally permit the use of significantly larger step sizes than is possible with the classical fourth order Runge–Kutta or Adams' methods, for example, once the initial transient region is passed. This is because the A-stability condition guarantees that rapidly decaying terms will continue to decrease for any step size used. One difficulty in developing such methods has been a lack of suitable approximations to the exponential function which had moduli bounded by 1 in the entire left half-plane. Such approximations will be called A-acceptable in the remainder of this paper.

It has been shown by Varga [16] that the set of diagonal Padé approximations to the exponential are A-acceptable. Unfortunately, the moduli of all of these approximations approach 1 as $|z| \to \infty$, $\mathrm{Re}(z) < 0$ and this is not consistent with the behavior of e^z. A more satisfactory approximation to the exponential would be one that was not only A-acceptable but also satisfied the property that as $|z| \to \infty$, with $\mathrm{Re}(z) < 0$, its modulus approached zero. Such an approximation will be called L-acceptable. In a recent paper, Wright [19] has shown that the first eleven entries on the first subdiagonal of Padé approximations are L-acceptable.

In this paper it is shown, by an entirely different technique, that the set of all first and second subdiagonal Padé approximations to the exponential function are L-acceptable. Furthermore, evidence is given to suggest that these are the only L-acceptable Padé approximations to the exponential.

Finally, several classes of arbitrarily high order A-stable methods which produce L-acceptable approximations to the exponential are given.

* Received by the editors November 11, 1971, and in revised form June 2, 1972.

† Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

2. Preliminary theorems and definitions. A great deal is known about the Padé approximations to the exponential. In particular, if we denote by $P_{j,k}(z)$ the unique Padé approximation to the exponential with numerator $N_{j,k}(z)$ of degree k and denominator $D_{j,k}(z)$ of degree j, then it is known [11], [15] that

$$(2)\qquad \begin{aligned} N_{j,k}(z) &= \sum_{m=0}^{k} \frac{(j+k-m)!k!}{(j+k)!m!(k-m)!} z^m, \\ D_{j,k}(z) &= \sum_{m=0}^{j} \frac{(j+k-m)!j!}{(j+k)!m!(j-m)!}(-z)^m. \end{aligned}$$

In order to establish the L-acceptability of $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ we must establish three results:

(I) $P_{j,k}(z) - e^z = O(z^{m+1})$, $m \geqq 0$,
(II) $|P_{j,k}(z)| \to 0$ as $\operatorname{Re}(z) \to -\infty$,
(III) $|P_{j,k}(z)| \leqq 1$ for $\operatorname{Re}(z) \leqq 0$,

for $j = n+1$ or $n+2$, $k = n$, and $n = 0, 1, 2, \cdots$. Results (I) and (II) follow immediately from the fact [17, p. 394] that

$$P_{j,k}(z) - e^z = O(z^{j+k+1}),$$

and that $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ have denominators of higher degree than their numerators.

The proof that (III) is also satisfied is the subject of this paper. First it is established that $|P_{n+1,n}(z)|$ and $|P_{n+2,n}(z)|$ are bounded by 1 along the imaginary axis. Then it is shown that there are no zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$ in the left half-plane. Consequently, $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ are analytic for $\operatorname{Re}(z) \leqq 0$, $n = 0, 1, 2, \cdots$, and the maximum modulus theorem may be applied to establish boundedness.

In order to effect the above proof, a number of relationships which hold between various $N_{j,k}(z)$ and $D_{j,k}(z)$ as given by (2) will be needed. In particular it is easily verified that:

(A) $D_{j,k}(z) = N_{k,j}(-z)$ for $j, k \geqq 0$ and all z.
(B) $\overline{N_{n,n}(iy)} = D_{n,n}(iy)$ and $\overline{D_{n,n}(iy)} = N_{n,n}(iy)$, y real.
(C) For all $j, k \geqq 1$ and all z,
  (i) $N_{j,k}(z) = N_{j,k-1}(z) + AzN_{j-1,k-1}(z)$,
  (ii) $D_{j,k}(z) = D_{j,k-1}(z) + AzD_{j-1,k-1}(z)$,
  (iii) $N_{j,k}(z) = N_{j-1,k}(z) + BzN_{j-1,k-1}(z)$,
  where

$$A = j/[(j+k)(j+k-1)] \quad \text{and} \quad B = -k/[(j+k)(j+k-1)].$$

(D) The polynomial $F_n(z) = N_{n,n}(z) \cdot D_{n,n}(z)$ has no odd terms.
(E) For all $n \geqq 2$,

$$N_{n,n}(z) = N_{n-1,n-1}(z) + Az^2N_{n-2,n-2}(z)$$

and

$$D_{n,n}(z) = D_{n-1,n-1}(z) + Az^2D_{n-2,n-2}(z),$$

where

$$A = 1/[4(2n-1)(2n-3)].$$

LEMMA 1. *For all $n \geqq 1$, the only term with an odd power of z in the product $D_{n,n}(z)N_{n-1,n-1}(z)$ is the term of highest power, namely,*

$$\frac{(-1)^n n!(n-1)!z^{2n-1}}{(2n)!(2n-2)!}.$$

Proof. The proof is by induction. For $n = 1$, we have

$$D_{1,1}(z)N_{0,0}(z) = (1 - z/2)(1).$$

Now by property (E),

$$D_{n,n}(z)N_{n-1,n-1}(z) = D_{n-1,n-1}N_{n-1,n-1} + \frac{z^2}{4(2n-1)(2n-3)}D_{n-2,n-2}N_{n-1,n-1}.$$

Assume the only odd term in $D_{n-1,n-1}(z)\,N_{n-2,n-2}(z)$ is of the form given by the theorem. By property (D), the first term on the right has no odd terms, and by property (A), the product has only the stated odd term. As an immediate corollary we have the following.

COROLLARY 1. *For all $n \geqq 1$ and all z,*

$$\frac{z}{(2n-1)}[D_{n,n}(z)N_{n-1,n-1}(z) - N_{n,n}(z)D_{n-1,n-1}(z)] = (-1)^n\left[\frac{(n-1)!z^n}{(2n-1)!}\right]^2.$$

3. Bounds on the imaginary axis. To establish that $|P_{n+1,n}(z)|$ and $|P_{n+2,n}(z)|$ are bounded by 1 on the imaginary axis, it is sufficient to show that $|D_{n+1,n}(iy)| \geqq |N_{n+1,n}(iy)|$ and that $|D_{n+2,n}(iy)| \geqq |N_{n+2,n}(iy)|$ for y real. To establish the first inequality we prove the following theorem.

THEOREM 1. *For all $n \geqq 1$, if $z = iy$, y real, then*

$$|D_{n,n-1}(z)|^2 - |N_{n,n-1}(z)|^2 = \left[\frac{(n-1)!}{(2n-1)!}\right]^2 y^{2n} \geqq 0,$$

and hence $|P_{n+1,n}(iy)| \leqq 1$ for y real and $n \geqq 0$.

Proof. Employing property (C) and observing that $|N_{n,n}(iy)| = |D_{n,n}(iy)|$ for y real, $n \geqq 0$, we have

$$\begin{aligned}|D_{n,n-1}(iy)|^2 &- |N_{n,n-1}(iy)|^2 \\ &= \frac{iy}{2(2n-1)}[\overline{N_{n,n}(iy)}N_{n-1,n-1}(iy) - N_{n,n}(iy)\overline{N_{n-1,n-1}(iy)} \\ &\quad - \overline{D_{n,n}(iy)}D_{n-1,n-1}(iy) + D_{n,n}(iy)\,\overline{D_{n-1,n-1}(iy)}].\end{aligned}$$

We obtain the required result by applying property (B) to remove all conjugates and then applying Corollary 1.

In order to establish a similar result for $P_{n+2,n}(z)$, it is first necessary to observe the following.

LEMMA 2. *For all $n \geqq 2$ and all z,*

$$N_{n,n-2}(z) = \frac{1}{2(n-1)}[(4n-2)N_{n,n}(z) - (2n+z)N_{n-1,n-1}(z)]$$

and

$$D_{n,n-2}(z) = \frac{1}{2(n-1)}[(4n-2)D_{n,n}(z) - (2n+z)D_{n-1,n-1}(z)].$$

Proof. The two results follow from equations (2).

THEOREM 2. *For all $n \geqq 2$, if $z = iy$, y real, then*

$$|D_{n,n-2}(z)|^2 - |N_{n,n-2}(z)|^2 = \left[\frac{(n-2)!}{(2n-2)!}\right]^2 y^{2n} \geqq 0$$

and hence $|P_{n+2,n}(iy)| \leqq 1$ for y real and $n \geqq 0$.

Proof. Using Lemma 2 with $z = iy$, y real, we obtain

$$\begin{aligned} &|D_{n,n-2}(iy)|^2 - |N_{n,n-2}(iy)|^2 \\ &\quad = \frac{2(2n-1)}{4(n-1)^2}[-(2n-iy)D_{n,n}\overline{D_{n-1,n-1}} - (2n+iy)\overline{D_{n,n}}D_{n-1,n-1} \\ &\qquad + (2n-iy)N_{n,n}\overline{N_{n-1,n-1}} + (2n+iy)\overline{N_{n,n}}N_{n-1,n-1}]. \end{aligned}$$

Applying property (B) and Corollary 1 completes the proof.

Before proceeding to the proof that $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ are analytic for $\operatorname{Re}(z) \leqq 0$, it seems appropriate to note that the following theorem can be established for $P_{n+3,n}(z)$.

THEOREM 3. *For all $n \geqq 3$, y real,*

$$|D_{n,n-3}(iy)|^2 - |N_{n,n-3}(iy)|^2 = (y^2 - n^2 + 2n)\left[\frac{(n-3)!y^{n-1}}{(2n-3)!}\right]^2$$

and hence $|P_{n,n-3}(iy)|$ is not bounded by one over the interval

$$-\sqrt{n^2-2n} < y < \sqrt{n^2-2n} \quad \text{for } n \geqq 3.$$

The proof of this theorem proceeds in a fashion similar to Theorems 1 and 2 after establishing that

$$N_{n,n-3}(z) = \frac{(n-z-2)}{(n-2)}N_{n-1,n-1}(z) + \frac{z(n+z)}{2(n-2)(2n-3)}N_{n-2,n-2}(z)$$

and that a similar result for $D_{n,n-3}(z)$ is true.

Since it is easily verified that $|P_{4,0}(z)|$ is not bounded by 1 on the imaginary axis, and since it is also known that $|P_{n,0}(z)|$ is not bounded by 1 in the left half-plane for $n \geqq 5$ [7, p. 25], there seems to be little likelihood that $P_{n+j,n}(z)$ can be L-acceptable for any $n \geqq 0$ with $j \geqq 3$. For this reason we direct our attention to establishing the L-acceptability of only $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ in the remainder of this paper.

4. The zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$. As was noted in § 2, the proof of the L-acceptability of $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ would be complete if we could establish that no zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$ were in the left half-plane. By property (A), this is equivalent to showing that all the zeros of $N_{n,n+1}(z)$ and $N_{n,n+2}(z)$ are in the left half-plane. We choose to work with the numerators because all the coefficients are positive while those of the denominator alternate in sign.

Wimp [18] has shown that all the zeros of the Bessel polynomials

$$P_n^{(\delta)}(z) = \sum_{k=0}^{n} \binom{n}{k} (n + \delta)_k z^{n-k}, \qquad \delta \geqq 0, \quad n \geqq 1,$$

where

$$(n + \delta)_k = (n + \delta)(n + \delta + 1) \cdots (n + \delta + k - 1),$$
$$(n + \delta)_0 = 1,$$

are in the left half-plane. It is easily verified that $N_{n,n+1}(z) = [n!/(2n + 1)!]P_{n+1}^{(0)}(z)$ and hence it follows that the set of first subdiagonal Padé approximations to the exponential are L-acceptable. Unfortunately, $N_{n,n+2}(z) = [n!/(2n + 2)!]P_{n+2}^{(-1)}(z)$, and Wimp's result does not apply to the second subdiagonal.

In trying to verify and extend Wimp's result, an attempt was made to prove that all the zeros of $N_{n,n+1}(z)$ and $N_{n,n+2}(z)$ were in the left half-plane in a manner similar to Varga's proof [16] for $P_{n,n}(z)$. This was unsuccessful because the resulting continued fraction expansions were not easily related to one another. Instead, the proof is based on establishing the following two theorems.

THEOREM 4. *If for some $j, k \geqq 0$, $N_{j,k}(z)$ has all of its zeros in the open left half-plane, then for all $m \geqq j$, $N_{m,k}(z)$ has all of its zeros in the open left half-plane also.*

THEOREM 5. *For any $n \geqq 0$, if $N_{n+1,n+1}(z)$ has all of its zeros in the left half-plane, then $N_{n,n+2}(z)$ also has all of its zeros in the left half-plane.*

Assuming for the moment that the required proofs have been given, we note that the only zero of $N_{1,1}(z) = 1 + z/2$ is in the left half-plane. It then follows by repeated application of Theorems 5 and 4 as indicated by Fig. 1, that all the zeros of $N_{n,n+1}(z)$ and $N_{n,n+2}(z)$ are in the left half-plane. As noted above, it follows that none of the zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$ are in the left half-plane. Thus we would have established the following result.

THEOREM 6. *For all $n \geqq 0$, $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$, the first and second subdiagonal Padé approximations to the exponential function, are analytic in the entire left half-plane. Furthermore, they are bounded in absolute value by 1 in the entire left half-plane and hence are L-acceptable approximations to the exponential function.*

Clearly we could have stated Theorem 5 in a more positive way since one of Varga's results in establishing the A-acceptability of the diagonal Padé approximations was that all the zeros of $N_{n+1,n+1}(z)$ were in the left half-plane. By stating the theorem without including this result, however, an alternative proof of the location of the zeros of $N_{n+1,n+1}(z)$ as well as $N_{n,n+1}(z)$ is provided.

5. Proof of Theorems 4 and 5. We begin by recalling a result given in Marden [14, p. 69].

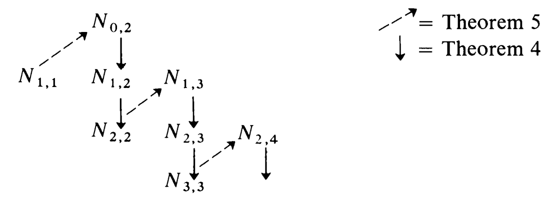

FIG. 1. *Basis of proof for Theorem* 6

THEOREM. *If* $f(z) = \sum_{k=0}^{n} a_k z^k$, $\beta_1 \neq \eta$ *and all the zeros of* $f(z)$ *lie in a circular region* C, *then every zero* Z *of the polynomial*

$$f_1(z) = \beta_1 f(z) - zf'(z)$$

may be written in the form $Z = \xi$ *or in the form*

$$Z = [\beta_1/(\beta_1 - \eta)]\xi,$$

where ξ *is a point of* C.

Observing that

$$N_{j,k}(z) = \frac{(j+k+1)}{(k+1)} N'_{j,k+1}(z) \quad \text{for all } j, k \geqq 0$$

follows immediately from (2), we can easily establish that

$$(j+k+1)N_{j+1,k} = (j+k+1)N_{j,k} - zN'_{j,k}$$

follows from property (C(iii)). With $\beta_1 = j + k + 1$ and $\eta = k$, it follows from the theorem [14] just given that all the zeros of $N_{j+1,k}$ are in the left half-plane provided all the zeros of $N_{j,k}$ are also. Thus Theorem 4 is established.

In order to prove Theorem 5, it is necessary to establish a relationship between $N_{n+1,n+1}(z)$ and $N_{n,n+2}(z)$. The following lemma will prove useful in this regard.

LEMMA 3. *For all* $j \geqq 0$, $k \geqq 0$,

$$N_{j,k+1}(z) = [1 + z/(j+1)]N_{j+1,k}(z) - [z/(j+1)]N'_{j+1,k}(z).$$

Proof. The proof follows directly from equation (2).

We now define $\theta_n(z)$ to be

$$\theta_n(z) = \frac{N_{n,n+2}(z)}{(-z/(n+1))N_{n+1,n+1}(z)} = \left[\frac{-(n+1)}{z} - 1\right] + \frac{N'_{n+1,n+1}(z)}{N_{n+1,n+1}(z)}.$$

The second equality is obtained using Lemma 3 with $j = n$ and $k = n + 1$.

Now, clearly all the zeros of $N_{n,n+2}(z)$ which are not zeros of $\theta_n(z)$ are zeros of $N_{n+1,n+1}(z)$. Conversely, all the zeros of $\theta_n(z)$ are zeros of $N_{n,n+2}(z)$. Thus, if $N_{n+1,n+1}(z)$ has zeros only in the left half-plane and we can show that all the zeros of $\theta_n(z)$ are in the left half-plane, then we will have shown that all the zeros of $N_{n,n+2}(z)$ are also in the left half-plane.

In order to study the zeros of $\theta_n(z)$ we consider the region S, bounded by the curve C, which is shown in Fig. 2. The boundary curve C is composed of the semicircle $|z| = R$, $\operatorname{Re}(z) \leqq 0$, R chosen so that all the zeros of $N_{n+1,n+1}(z)$ are inside $|z| = R$ together with the semicircle $|z| = r$, $r > 0$, $\operatorname{Re}(z) \geqq 0$, r chosen so that all the zeros of $N_{n,n+2}(z)$ are outside the circle $|z| = r$ together with the imaginary

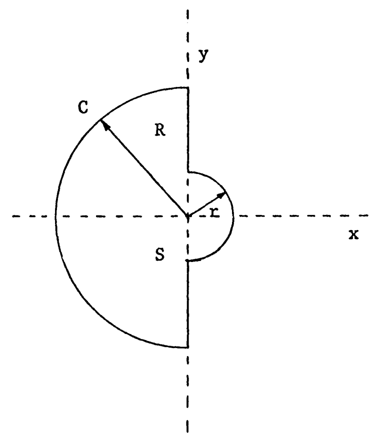

FIG. 2. *Region containing zeros of* $\theta_n(z)$

axis from $-R \leqq y \leqq -r$ and $r \leqq y \leqq R$. That values of $R < \infty$ and $r > 0$ can be found which satisfy these conditions follows at once from the known form of the polynomials $N_{n+1,n+1}(z)$ and $N_{n,n+2}(z)$ and several well-known results of Cauchy [14, p. 123–126].

The following well-known theorem [1, p. 123] from complex analysis will now be useful.

THEOREM. *Let* $f(z)$ *be meromorphic inside and on a simple closed curve C which does not pass through any of the zeros or poles of* $f(z)$. *Then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_C(f) - P_C(f),$$

where $N_C(f)$ *and* $P_C(f)$ *are, respectively, the numbers of zeros and poles of* $f(z)$ *inside C.*

Applying this theorem to $\theta_n(z)$, we obtain the following.

LEMMA 4. *If all the zeros of* $N_{n+1,n+1}(z)$ *are in the left half-plane, then*

$$N_C(N_{n,n+2}(z)) = (n + 2) + \frac{1}{2\pi i}\int_C \frac{\theta_n'(z)}{\theta_n(z)}\,dz,$$

where C is the curve in Fig. 2.

Turning our attention to the evaluation of $\int_C (\theta_n'/\theta_n)\,dz$, we observe that this can be done by determining the index of $\theta_n(C)$ with respect to the origin, that is, the index of the curve into which C is mapped by $\theta_n(z)$ taken relative to the origin.[1] For convenience, we call this new curve C^*. We shall now show that as C^* is traversed, its real part is always negative and hence its index with respect to the origin is zero.

LEMMA 5. *For* $|z| = R$, *R sufficiently large,* $\mathrm{Re}\,(\theta_n(z)) < 0$.

Proof. For $|z|$ large enough, $\theta_n(z) = -1 + O(1/z)$; hence for sufficiently large R, the result follows.

[1] Dieudonné has used this technique in considering a problem which is similar in spirit to the one we are considering ([14, p. 87], [6]).

LEMMA 6. *For* $|z| = r$, $r > 0$, *r sufficiently small, and* $\operatorname{Re}(z) > 0$, $\operatorname{Re}(\theta_n(z)) < 0$.

Proof. Since as $|z| \to 0$, $(N'_{n+1,n+1}(z)/N_{n+1,n+1}(z)) \to \frac{1}{2}$, we have that for r sufficiently small, $\operatorname{Re}(N'_{n+1,n+1}(z)/N_{n+1,n+1}(z)) \leqq \frac{3}{4}$. Thus for r sufficiently small, we have

$$\operatorname{Re}(\theta_n(z)) \leqq \left[\frac{-(n+1)\operatorname{Re}(z)}{r^2} - 1\right] + \tfrac{3}{4} < 0.$$

LEMMA 7. *For y real and* $n \geqq 0$, *if* $N_{n+1,n+1}(z)$ *has all of its zeros in the left half-plane, then*

$$\operatorname{Re}\left(\frac{N'_{n+1,n+1}(iy)}{N_{n+1,n+1}(iy)}\right) < \tfrac{1}{2}$$

and hence $\operatorname{Re}(\theta_n(iy)) \leqq -\frac{1}{2}$ *for* $0 < r \leqq |y| \leqq R < \infty$.

Proof. With $j = k = n + 1$, property (C(i)) can be written

$$N_{n+1,n} = N_{n+1,n+1} - \frac{z}{2(2n+1)} N_{n,n} = 2N'_{n+1,n+1}.$$

It follows by Lemma 1 and property (B) that

$$\operatorname{Re}(N'_{n+1,n+1}(iy)\overline{N_{n+1,n+1}(iy)}) = \frac{1}{2}\left[N_{n+1,n+1}(iy)\overline{N_{n+1,n+1}(iy)} - \frac{[(n+1)!]^2 y^{2n+2}}{[(2n+2)!]^2}\right]$$

and thus

$$\operatorname{Re}(N'_{n+1,n+1}(iy)/N_{n+1,n+1}(iy)) = (1-t)/2,$$

where

$$t = \frac{[(n+1)!]^2 y^{2n+2}}{[(2n+2)!]^2 |N_{n+1,n+1}(iy)|^2} \geqq 0.$$

Thus the first inequality is established. The second inequality follows directly from the first using the definition of $\theta_n(z)$ with $z = iy$.

Lemmas 5, 6 and 7 establish the next lemma.

LEMMA 8. *If* $N_{n+1,n+1}(z)$ *has all of its zeros in the left half-plane, then the index* C^* *with respect to the origin is zero, and hence*

$$\frac{1}{2\pi i}\int_C \frac{\theta'_n(z)}{\theta_n(z)}\,dz = 0.$$

Lemmas 4 and 8 now establish that all the zeros of $N_{n,n+2}(z)$ are inside the region S of Fig. 2 provided all the zeros of $N_{n+1,n+1}(z)$ are in the left half-plane. But since all the zeros of $N_{n,n+2}(z)$ are outside the circle of radius r, we are also able to conclude that all the zeros of $N_{n,n+2}(z)$ are in the region S' given in Fig. 3. Noting that S' is entirely in the left half-plane, we have provided the necessary proof of Theorem 5.

6. Conclusions. Based on the results given above, it is possible to construct many new methods which are A-stable. We mention only two possibilities here, both being generalizations of ideas given in [8]. The first is to choose the coefficients

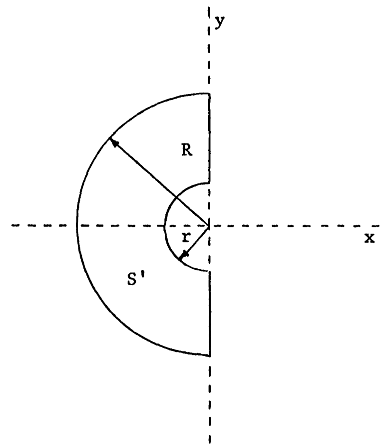

FIG. 3. *Region containing zeros of* $N_{n,n+2}(z)$

in the generalized one-step method

$$y_{n+1} = y_n + \sum_{i=1}^{n} h^i(\alpha_i y_n^{(i)} + \beta_i y_{n+1}^{(i)}), \qquad n = 1, 2, 3, \cdots, \tag{3}$$

so that it reduces to a first or second subdiagonal Padé approximation to the exponential when solving the initial problem given by (1).

The appropriate choice for a first subdiagonal Padé approximation $P_{n,n-1}(z)$ is clearly

$$\alpha_i = (2n-1-i)!(n-1)!/[(2n-1)!i!(n-1-i)!], \quad i = 1, 2, \cdots, n-1,$$

$$\alpha_n = 0,$$

and

$$\beta_i = (2n-1-i)!n!/[(2n-1)!i!(n-i)!], \quad i = 1, 2, \cdots, n,$$

and for a second subdiagonal Padé approximation $P_{n,n-2}(z)$ the choice is

$$\alpha_i = (2n-2-i)!(n-2)!/[(2n-2)!i!(n-2-i)!], \quad i = 1, 2, \cdots, n-2,$$

$$\alpha_n = \alpha_{n-1} = 0,$$

and

$$\beta_i = (2n-2-i)!n!/[(2n-2)!i!(n-i)!].$$

Hermite [10], Hummel and Seebeck [11] and others have studied (3) and have observed that the coefficients given above are also those needed to make (3) a general method of order $2n - 1$ and $2n - 2$, respectively. Thus we have two sets of methods which are of arbitrarily high order and produce L-acceptable approximations to the exponential.

The second class of L-acceptable methods we consider is based on the implicit Runge–Kutta processes studied by Butcher [2], [3]. The obvious choice is to look at his methods based on Radau and Lobatto quadrature. Unfortunately, it is easily shown using the method given in [8] that none of the methods he proposes using these two quadratures are A-stable, since they reduce to above diagonal Padé approximations to the exponential when solving (1). Ehle [7], however, has shown that it is possible to construct implicit Runge–Kutta methods from these quadrature rules which reduce to subdiagonal Padé approximations $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ for small n. Chipman [4] has recently been able to show that the rules developed in [7] produce implicit Runge–Kutta processes which result in first and second subdiagonal Padé approximations for all n. This combined with Butcher's result about the order of such methods gives a second class of L-acceptable methods of arbitrarily high order.

REFERENCES

[1] L. V. AHLFORS, *Complex Analysis*, McGraw-Hill, New York, 1953.

[2] J. C. BUTCHER, *Implicit Runge–Kutta processes*, Math. Comp., 18 (1964), pp. 50–64.

[3] ———, *Integration processes based on Radau quadrature formulas*, Ibid., 18 (1964), pp. 233–244.

[4] F. H. CHIPMAN, *Numerical Solution of initial value problems using A-stable Runge–Kutta processes*, Research Rep. CSRR 2042, Dept. of AACS, Univ. of Waterloo, Ontario, Canada, 1971.

[5] G. DAHLQUIST, *A special stability problem for linear multistep methods*, BIT, 3 (1963), pp. 27–43.

[6] M. J. DIEUDONNÉ, *Sur un problème de la théorie des polynômes*, C. R. Acad. Sci. Paris, 199 (1934), pp. 999–1001.

[7] B. L. EHLE, *On Padé approximations to the exponential function and A-stable methods for the numerical solution of initial value problems*, Research Rep. CSRR 2010, Dept. of AACS, Univ. of Waterloo, Ontario, Canada, 1969.

[8] ———, *High order A-stable methods for the numerical solution of systems of D. E.'s*, BIT, 8 (1968), pp. 276–278.

[9] C. W. GEAR, *The automatic integration of ordinary differential equations*, Comm. ACM, 14 (1971), pp. 176–179.

[10] C. HERMITE, *Sur la formule d'interpolation de Lagrange*, J. Reine Angew. Math., 84 (1878), pp. 70–79; *Oeuvres*, vol. 3, pp. 432–443.

[11] P. M. HUMMEL AND C. L. SEEBECK, *A generalization of Taylor's theorem*, Amer. Math. Monthly, 56 (1949), pp. 243–247.

[12] R. K. JAIN, *A-stable multi-step methods for stiff ordinary differential equations*, Proc. of the Manitoba Conference on Numerical Mathematics, Univ. of Manitoba, Winnipeg, Canada, 1971.

[13] J. D. LAWSON, *Generalized Runge–Kutta processes for stable systems with large Lipshitz constants*, SIAM J. Numer. Anal., 4 (1967), pp. 372–380.

[14] M. MARDEN, *Geometry of Polynomials*, American Mathematical Society, Providence, R.I., 1966.

[15] H. PADÉ, *Sur la représentation approchée d'une fonction par des fractions rationnelles*, thesis, Ann. de l'Ec. Nor., 9 (1892).

[16] R. S. VARGA, *On higher order stable implicit methods for solving parabolic partial differential equations*, J. Mathematical Phys., 40 (1961), pp. 220–231.

[17] H. S. WALL, *Analytic Theory of Continued Fractions*, Van Nostrand, Princeton, N.J., 1948.

[18] J. WIMP, *On zeros of a confluent hypergeometric function*, Proc. Amer. Math. Soc., 16 (1965), pp. 281–283.

[19] K. WRIGHT, *Some relationships between implicit Runge–Kutta, collocation, and Lanczos τ methods, and their stability properties*, BIT, 10 (1970), pp. 217–227.

SIAM J. MATH. ANAL.
Vol. 4, No. 4, November 1973

SPHERICAL SUMMABILITY OF CONJUGATE MULTIPLE FOURIER SERIES AND INTEGRALS AT THE CRITICAL INDEX*

Dedicated to the Memory of Michael Hanna

G. E. LIPPMAN†

Abstract. In Euclidean k-space, $k \geqq 2$, for Bochner–Riesz summability at the critical index $\gamma = (k - 1)/2$, we obtain a localization theorem for Fourier integrals conjugate with respect to spherical harmonic kernels. It is also shown that this result is best possible with respect to the index of summability and that localization does not hold at the critical index for conjugate multiple Fourier series.

1. Introduction. Let T_k be the k-dimensional torus, $k \geqq 2$, f be a function integrable on T_k and periodic in each coordinate of E_k, and x a point in T_k. Bochner [1] proved that the limit behavior at x of the Bochner–Riesz partial sums of order α for the Fourier series of f depends only on the values of f in any neighborhood (no matter how small) of x, as long as α remains greater than $(k - 1)/2$. In the same paper Bochner proved an analogous result for Bochner–Riesz means of order α, for α greater than or equal to $(k - 1)/2$, for Fourier transforms of functions integrable on E_k, and that localization fails in the Fourier integral case for an index of summability less than the critical index $(k - 1)/2$. In the same paper Bochner gave an ingenious proof of the existence of a function f periodic in E_k, integrable on T_k, identically zero in an open ball centered at the origin, with the property that the Bochner–Riesz sums of order $(k - 1)/2$ for its Fourier series diverge at $x = 0$, thus proving the failure of localization at the critical index of summability for Bochner–Riesz sums of Fourier series.

Calderón and Zygmund [2], using their singular integral theory, defined the notion of conjugate multiple Fourier series, and Shapiro [6] proved localization theorems for Bochner–Riesz summability, of order greater than the critical index, for conjugate Fourier–Stieltjes series, under very general conditions on the conjugate kernel. In this paper we prove a localization theorem for Bochner–Riesz means of order $(k - 1)/2$ of Fourier integrals conjugate with respect to spherical harmonic kernels, and we prove that localization fails below the critical index of summability for such conjugate Fourier integrals. We also prove that localization fails at the critical index for conjugate Fourier series.

2. Definitions and notation. We shall work in k-dimensional Euclidean space, E_k, $k \geqq 2$, using the following notation:

$x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k), (x, y) = x_1y_1 + \cdots + x_ky_k, |x| = (x, x)^{1/2}$. $B(x, r)$ will be the open k-ball with center x and positive radius r. $S(x, r)$ will represent its boundary, $S_j = \{x \in E_k : j < |x| \leqq j + 1\}$, T_k will designate the k-dimensional torus $T_k = \{x | -\pi < x_j \leqq \pi : j = 1, \cdots, k\}$, m will be an integral lattice point in E_k and Ω will be used to represent the surface of the unit $(k - 1)$-sphere in E_k.

In this paper we shall work with a special class of Calderón–Zygmund kernels in defining conjugate series and integrals, namely the spherical harmonic kernels.

* Received by the editors August 22, 1972.

† Department of Mathematics, California State University, Hayward, California 94542.

Letting $x' = x/|x|$, we shall call $K(x)$ a spherical harmonic kernel if

$$K(x) = P(x')/|x|^k, \tag{1}$$

where $P(x)$ is a surface spherical harmonic of degree n, $n \geqq 1$; thus P is a homogeneous harmonic polynomial. For such a kernel, the principal-valued Fourier transform

$$\hat{K}(y) = \lim_{\lambda\to\infty} \lim_{\varepsilon\to 0} (2\pi)^{-k} \int_{B(0,\lambda)-B(0,\varepsilon)} e^{-i(x,y)} K(x)\, dx \tag{2}$$

exists and is finite for every y (see [6, p. 69]).

For a multiple trigonometric series $S = \sum a_m e^{i(m,x)}$, the conjugate series with respect to the kernel K is defined to be

$$\tilde{S}_k = \sum_{m\neq 0} a_m \hat{K}(m)\, e^{i(m,x)}, \tag{3}$$

where $\hat{K}$ is as defined in (2). Throughout this paper, γ will be used to denote the critical index

$$\gamma = (k-1)/2. \tag{4}$$

3. Statement of the main results.

THEOREM 1. *If f is integrable on E_k, $k \geqq 2$, and $K(x) = P(x')/|x|^k$ is a spherical harmonic kernel P, of degree $n \geqq 1$, then at every point x_0 in E_k for which*

$$\int_0^\eta t^{-1} \left| \int_\Omega f(x_0 + t\xi) P(\xi)\, d\Omega(\xi) \right| dt < +\infty \tag{5}$$

for some $\eta > 0$,

$$\lim_{R\to\infty} \left\{ \int_{B(0,R)} \hat{K}(p) \hat{f}(p)\, e^{i(p,x_0)} (1 - |p|^2/R^2)^\gamma\, dp - (2\pi)^{-k} \int_{E_k - B(0,1/R)} f(x_0 - x) K(x)\, dx \right\} = 0. \tag{6}$$

Note. If f satisfies the restricted "Dini" condition at x_0, that is, if for some $\eta > 0$,

$$\int_0^\eta t^{-1} \int_\Omega |f(x_0 + t\xi) - f(x_0)|\, d\Omega(\xi)\, dt < +\infty, \tag{7}$$

then, since $P(\xi)$ is bounded on Ω and its integral over the sphere Ω is zero (due to harmonicity), the inequality (5) holds.

THEOREM 2. *If f is integrable on E_k, $k \geqq 2$, and satisfies condition* (5) *at x_0, K is a spherical harmonic kernel of degree $n \geqq 1$, and if $\alpha = \gamma - \beta$, $0 \leqq \beta < 1$, then*

$$\lim_{R\to\infty} \left\{ \int_{B(0,R)} \hat{K}(p) \hat{f}(p)\, e^{i(p,x_0)} (1 - |p|^2/R^2)^\alpha\, dp - (2\pi)^{-k} \int_{E_k - B(0,1/R)} f(x_0 - x) K(x)\, dx \right\} = 0 \tag{8}$$

if and only if

$$\lim_{R\to\infty} R^{\beta} \int_{E_k - B(0,1/R)} |x|^{\beta}\{f(x) - f(x_0)\} \cdot K(-x)\cos\left(R|x| - \frac{\pi}{2}(k + n - \beta)\right) dx = 0. \tag{9}$$

For $\beta > 0$, condition (9) need not hold, even for an integrable function $f(x)$ which vanishes outside a finite ball centered at the origin. However, for $\beta = 0$, condition (5) implies condition (9), so that with respect to the index of summability, Theorem 1 is the best possible result we can obtain.

For example, we may construct such a function as follows.
Define

$$f(x) = \begin{cases} |x|^{1-\beta}\Psi_{\beta}(|x|)\bar{P}(-x), & 0 \leqq |x| \leqq 2\pi, \\ 0, & |x| > 2\pi, \end{cases}$$

where $\Psi_{\beta}(|x|)$ is the periodic function given in Theorem 13.7, Chapter 6, of [9],

$$\Psi_{\beta}(r) = \lim_{n\to\infty} \frac{2\pi}{\Gamma(\beta)}\left\{r^{\beta-1} + (r + 2\pi)^{\beta-1} + \cdots + (r + 2n\pi)^{\beta-1} - \frac{(2\pi)^{\beta-1}}{\beta}n^{\beta}\right\}.$$

$\Psi_{\beta}(r)$ has Fourier series

$$\sum_{j=-\infty}^{\infty}{}' |j|^{-\beta} \exp(-\pi i\beta \operatorname{sgn} j/2)\, e^{ijr}.$$

Then for integer values of j,

$$\begin{aligned} &\lim_{j\to\infty} j^{\beta} \int_{E_k - B(0,1/j)} |x|^{\beta} f(x)K(-x)\cos j|x|\, dx \\ &= \lim_{j\to\infty} j^{\beta} \int_{1/j}^{2\pi} \Psi_{\beta}(r)\cos jr\left\{\int_{\Omega} |P(-\xi)|^2\, d\Omega(\xi)\right\} dr \\ &= \text{const.} \lim_{j\to\infty} j^{\beta} \int_0^{2\pi} \Psi_{\beta}(r)\cos jr\, dr \\ &= \text{const.} \neq 0 \quad \text{as } j \to \infty \end{aligned}$$

since the Fourier coefficients of $\Psi_{\beta}(r)$ are $O|j|^{-\beta}$. Therefore (9) does not hold for f as defined.

THEOREM 3. *Let* $0 < \delta < 1$ *be given. There exists a function* f, *periodic in* E_k, *integrable on* T, *and identically zero in* $B(0, \delta)$, *with the property that*

$$\limsup_{R\to\infty} |\tilde{\sigma}_R(f, x)| = +\infty, \tag{10}$$

where

$$\tilde{\sigma}_R(f, x) = \sum_{0<|m|\leqq R} \hat{f}(m)\hat{K}(m)\, e^{i(m,x)}(1 - |m|^2/R^2)^{\gamma}. \tag{11}$$

Remark. Since f is identically zero in a neighborhood of the origin, $\int_{E_k} f(-x)K(x)\, dx$ is finite (see Shapiro [6, p. 45]). Theorem 3 shows that the difference between the Bochner–Riesz means of order γ of the conjugate Fourier series and the principal-valued Hilbert transform of f is divergent as R increases to plus infinity.

4. Lemmas necessary to prove main results. Stein [7] proved important theorems relating the Dirichlet kernel for multiple Fourier series to the corresponding kernel for Fourier integrals in $E_k, k \geqq 2$. Chang [3] extended the ideas of Stein to obtain similar results for conjugate multiple Fourier series and integrals. In obtaining his results Chang established the following lemmas.

LEMMA 1. *If $P(x)$ is a spherical harmonic of degree $n \geqq 1$, the dimension k is greater than or equal to one, and s is complex with* $\mathrm{Re}(s) > -(k+1)/2$, *then for every $y \in E_k$,*

$$\int_{|x| \leqq R} P(x/|x|)(1 - |x|^2/R^2)^{\gamma+s} e^{i(y,x)}\, dx = i^n R^{n+k} P(y) E(k, s, n, R|y|), \tag{12}$$

where

$$E(k, s, n, u) = \frac{\pi^{k/2}}{2^n} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j + (n+k)/2)\Gamma(\gamma + s + 1)}{j!\Gamma(j + k/2 + n)\Gamma(\gamma + s + 1 + j + (n+k)/2)} \left(\frac{u}{2}\right)^{2j} \tag{13}$$

for u complex. The radius of convergence of this power series is $+\infty$.

LEMMA 2. *Let $k \geqq 2$, $n \geqq 1$ be integers and suppose that $s = \sigma + i\tau$ has its real part σ confined to $1 \geqq \sigma \geqq -1$. Then, for all $u \geqq 1$, we have*

$$\begin{aligned} E(k, s, n, u) &= c(k, n)\frac{1}{u^{n+k}} + \frac{G_1(s, u)}{u^{k+n+1}} \\ &\quad + \frac{G_2(s, u)}{u^{k+n+s+1}} + a(k, s)\left\{\cos\left[u - \frac{\pi}{2}(k + s + n)\right]\right\}\frac{1}{\pi u^{n+k+s}}, \end{aligned} \tag{14}$$

where

$$c(k, n) = \pi^{k/2} 2^k \Gamma\left(\frac{n+k}{2}\right) \bigg/ \Gamma(n/2), \tag{15}$$

$$a(k, s) = \pi^{(k-1)/2} 2^{s+k} \Gamma\left(\frac{k+1}{2} + s\right), \tag{16}$$

$$|G_1(s, u)| \leqq A_{k,n} \exp\left(\frac{3\pi}{2}|\tau|\right), \tag{17}$$

$$|G_2(s, u)| \leqq A_{k,n} \exp\left(\frac{3\pi}{2}|\tau|\right), \tag{18}$$

where $A_{k,n}$ is a constant depending only on k and n.

LEMMA 3. The Poisson summation formula. *Suppose $R \geqq 1/\pi$, $s = \sigma + i\tau$ with $1 \geqq \sigma > 0$. Then the function $\varphi(x)$ defined by*

$$\varphi(x) = \begin{cases} P(x/|x|)(1 - |x|^2/R^2)^{\gamma+s} & \text{if } 0 < |x| < R, \\ 0 & \text{if } x = 0, \\ 0 & \text{if } |x| \geqq R \end{cases} \tag{19}$$

satisfies the Poisson summation formula, for all $x \in T$, in the following sense:

$$\sum_m \varphi(m)\, e^{i(m,x)} = \sum_{j=0}^{\infty} \sum_{m \in S_j} \Phi(x + 2\pi m), \tag{20}$$

where

$$\Phi(y) = \int_{E_k} \varphi(x)\, e^{i(x,y)}\, dx. \tag{21}$$

5. Proof of Theorem 1. Without loss of generality we may assume x_0 is the origin, for if not, set $g(x) = f(x + x_0)$. Then $\hat{g}(p) = \hat{f}(p)\, e^{i(p,x_0)}$ and $f(x_0 - x) = g(-x)$. The function g is integrable on E_k and satisfies the hypotheses of Theorem 1 at the origin. We also note that since

$$\lim_{\lambda \to \infty} \int_{B(0,\lambda) - B(0,1/R)} K(x)\, dx = 0, \tag{22}$$

we may assume $f(0) = 0$, for

$$\int_{E_k - B(0,1/R)} f(-x) K(x)\, dx = \lim_{\lambda \to \infty} \int_{B(0,\lambda) - B(0,1/R)} \{f(-x) - f(0)\} K(x)\, dx. \tag{23}$$

Therefore we must show that, as R goes to $+\infty$, the difference

$$\int_{B(0,R)} \hat{K}(p) \hat{f}(p)(1 - |p|^2/R^2)^\gamma\, dp - (2\pi)^{-k} \int_{E_k - B(0,1/R)} f(-x) K(x)\, dx$$

goes to zero.

By Remark I, p. 69 of [6], we have that $\hat{K}(y)$ exists for all y in E_k and for $y \neq 0$,

$$\hat{K}(y) = \frac{(-i)^n P(y/|y|) \Gamma(n/2)}{2^k \pi^{k/2} \Gamma((n + k)/2)}. \tag{24}$$

Define

$$I(R) = \int_{B(0,R)} \hat{f}(p) \hat{K}(p)(1 - |p|^2/R^2)^\gamma\, dp. \tag{25}$$

By the definition of the Fourier transform of f, Fubini's theorem, and the expression for the principal-valued Fourier transform of K,

$$I(R) = (-i)^n b(n,k) \int_{E_k} f(x)\, dx \int_{B(0,R)} e^{-i(p,x)} (1 - |p|^2/R^2)^\gamma P(p/|p|)\, dp, \tag{26}$$

where

$$b(n,k) = \Gamma(n/2)/\Gamma\left(\frac{n+k}{2}\right)(2\pi)^{3k/2} 2^{k/2}. \tag{27}$$

By Lemma 1, $I(R)$ is equal to

$$b(n,k) R^{n+k} \int_{E_k} f(x) P(-x) E(k, 0, n, R|x|)\, dx. \tag{28}$$

We now break $I(R)$ up into two integrals

$$I(R) = I' + I'',$$

where, from this point on in the proof, integrals I with superscripts and subscripts will be understood to depend on R.

By (13),

$$I' = \text{const.}\, R^{n+k} \int_{B(0,1/R)} f(x)P(-x) \cdot \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j + (k+n)/2)\Gamma(\gamma + 1)}{j!\Gamma(j + k/2 + n)\Gamma(\gamma + 1 + j + (k+n)/2)} (R|x|/2)^{2j} \right\} dx, \tag{29}$$

where the constant depends only on n and k.

By Lemma 2,

$$I'' = b(n,k)R^{n+k} \int_{E_k - B(0,1/R)} f(x)P(-x) \cdot \left[c(k,n)(R|x|)^{-k-n} + G(R|x|)(R|x|)^{-k-n-1} + a(k,0)\left\{\cos\left(R|x| - \frac{\pi}{2}(n+k)\right)\right\}(R|x|)^{-n-k}\right] dx, \tag{30}$$

where $c(k,n)$ and $a(k,0)$ are the constants given in (15) and (16) respectively, and $G(R|x|)$ is bounded by a constant depending only on n and k.

We first show that I' goes to zero as R increases to infinity. I' is dominated by

$$\text{const.} \left| \sum_{j=0}^{\infty} \frac{\Gamma(\gamma + 1)\Gamma(j + (n+k)/2)}{j!\Gamma(j + k/2 + n)\Gamma(\gamma + 1 + j + (k+n)/2)} \right| \cdot \int_0^{1/R} \frac{1}{t} \left| \int_{\Omega} f(t\xi)P(\xi)\, d\Omega(\xi) \right| dt,$$

which goes to zero as R goes to infinity by hypothesis (5), since the series is uniformly bounded by a constant independent of R.

We now deal with I''. Using Lemma 2 we have shown in (30) that I'' can be written as $I'' = I_1 + I_2 + I_3$, where

$$I_1 = (2\pi)^{-k} \int_{E_k - B(0,1/R)} f(x)K(-x)\, dx, \tag{31}$$

since $K(x) = |x|^{-n-k}P(x)$ and by (15) and (27) $c(n,k)b(n,k) = (2\pi)^{-k}$,

$$I_2 = \text{const.}\, (1/R) \int_{E_k - B(0,1/R)} f(x)K(-x)|x|^{-1}G(R|x|)\, dx \tag{32}$$

and

$$I_3 = \text{const.} \int_{E_k - B(0,1/R)} f(x)K(-x) \cos\left[R|x| - \frac{\pi}{2}(n+k)\right] dx, \tag{33}$$

where the constants depend only on n and k. In order to complete the proof we need only show

(34) $$\lim_{R\to\infty} I_2 = 0$$

and

(35) $$\lim_{R\to\infty} I_3 = 0,$$

since expression (31) is equal to the second term in (6) with x_0 equal to zero.

For the moment let δ be a fixed positive number, independent of R. In order to prove (34) we must show that

(36) $$\lim_{R\to\infty} (1/R)\left\{\int_{B(0,\delta)-B(0,1/R)} + \int_{E_k - B(0,\delta)}\right\} f(x)K(-x)|x|^{-1}G(R|x|)\,dx = 0.$$

The function $f(x)K(-x)|x|^{-1}G(R|x|)$ is integrable over $E_k - B(0,\delta)$ since it is dominated by $\delta^{-n+1}A_{k,n}\sup|P(\xi)|\cdot|f(x)|$, where the sup is taken over all $\xi\in\Omega$. Therefore, as R goes to infinity, the second integral, when multiplied by $1/R$, goes to zero.

The first term of (36) can be written in the form

(37) $$\lim_{R\to\infty} (1/R)\int_{1/R}^{\delta} \frac{G(Rt)t^{k-1}}{t^{k+1}} \int_{\Omega} f(t\xi)P(-\xi)\,d\Omega(\xi)\,dt,$$

which, since $1/(Rt) \leqq 1$, is dominated by a constant times

(38) $$\int_0^{\delta} \frac{1}{t}\left|\int_{\Omega} f(t\xi)P(\xi)\,d\Omega(\xi)\right| dt.$$

By hypothesis (5), for a given positive ε, we choose δ so small that (38) is less than ε. Thus $\lim_{R\to\infty} I_2 = 0$, proving (34).

Proof of (35). We need only show

(39) $$\lim_{R\to\infty}\int_{E_k - B(0,1/R)} f(x)K(-x) \cdot\left\{\cos R|x|\cos\frac{\pi}{2}(n+k) + \sin R|x|\sin\frac{\pi}{2}(n+k)\right\} dx = 0.$$

We shall show

(40) $$\lim_{R\to\infty}\int_{E_k - B(0,1/R)} f(x)K(-x)\cos R|x|\,dx = 0.$$

The proof for the term with $\sin R|x|$ in it follows in similar fashion.

Let $\varepsilon > 0$. Choose $\delta > 0$ from hypothesis (5) so that

(41) $$\left|\int_{B(0,\delta)-B(0,1/R)} f(x)K(-x)\,dx\right| \leqq \int_{1/R}^{\delta} r^{-1}\left|\int_{\Omega} f(r\xi)P(\xi)\,d\Omega(\xi)\right| dr < \varepsilon,$$

as was done in the proof that $\lim_{R\to\infty} I' = 0$. We must show

$$\lim_{R\to\infty} \int_{E_k - B(0,\delta)} f(x)K(-x)\cos R|x|\,dx = 0. \tag{42}$$

However, using spherical coordinates, this limit is equal to $\lim_{R\to\infty}\int_\delta^\infty r^{-1}\cos Rr \cdot \{\int_\Omega P(-\xi)f(r\xi)\,d\Omega(\xi)\}\,dr$. The function $r^{-1}\chi_{(\delta,\infty)}(r)\{\int_\Omega P(-\xi)f(r\xi)\,d\Omega(\xi)\}$, where χ is the indicator function, is integrable in E_1, since it is dominated by

$$\sup_{\xi\in\Omega}|P(\xi)|\delta^{-1}\int_\Omega |f(r\xi)|\,d\Omega(\xi),$$

which is integrable with respect to r. Therefore by the one-dimensional Riemann–Lebesgue theorem for Fourier integrals, (42) holds. This proves (35) and the theorem.

6. Proof of Theorem 2. As before, we can assume $x_0 = 0$ and $f(0) = 0$. We define

$$I_R^\alpha = \int_{B(0,R)} \hat{K}(p)\hat{f}(p)(1 - |p|^2/R^2)^\alpha\,dp. \tag{43}$$

By the techniques used in (25), (26), and (27) in the proof of Theorem 1, this is equal to

$$b(n,k)R^{n+k}\int_{E_k} f(x)P(-x)E(k, -\beta, n, R|x|)\,dx,$$

where $b(n,k)$ is the constant, depending only on n and k, given in (27).

As in the proof of Theorem 1, we break up the integral I_R^α into the sum of the integral over $B(0,1/R)$ and the integral over $E_k - B(0,1/R)$. The integral over $B(0,1/R)$ goes to zero by the same argument as in the previous proof, and using Lemma 2, we break up the second integral, which we shall call $I_R^{\alpha''}$, into four terms,

$$\begin{aligned} I_R^{\alpha''} = b(n,k)R^{n+k}\int_{E_k - B(0,1/R)} & f(x)P(-x)\,[c(k,n)(R|x|)^{-n-k} \\ & + G_1(-\beta, R|x|)(R|x|)^{-k-n-1} + G_2(-\beta, R|x|)(R|x|)^{\beta-(k+n+1)} \\ & + a(k,-\beta)\{\cos(R|x| - (\pi/2)(k - \beta + n))\}(R|x|)^{\beta-(n+k)}]\,dx, \end{aligned} \tag{44}$$

where G_1 and G_2 are bounded by constants depending only on k and n. The first term of this expression plays the same role as in Theorem 1, that is, it is exactly equal to the second term in (8) with $x_0 = 0$. The second term, by the same reasoning as in the previous theorem, goes to zero as R goes to infinity as a consequence of hypothesis (5). Up to a constant factor, the third term is equal to

$$R^{\beta-1}\int_{E_k - B(0,1/R)} f(x)P(-x)\frac{G_2(R|x|)}{|x|^{n+k+1-\beta}}\,dx,$$

which equals

$$(45) \qquad R^{\beta-1}\int_{E_k - B(0,1/R)} f(x)K(-x)\frac{G_2(R|x|)}{|x|^{1-\beta}}\,dx$$

since $K(y) = P(y/|y|)/|y|^k = P(y)/|y|^{n+k}$ for all $y \in E_k$.

We break up (45) into the sum of two integrals

$$(46) \qquad \begin{aligned} &R^{\beta-1}\int_{B(0,\delta)-B(0,1/R)} f(x)K(-x)\frac{G_2(R|x|)}{|x|^{1-\beta}}\,dx \\ &+R^{\beta-1}\int_{E_k - B(0,\delta)} f(x)K(-x)\frac{G_2(R|x|)}{|x|^{1-\beta}}\,dx, \end{aligned}$$

where δ is a fixed positive number, independent of R. In the first term of (46), $(R|x|)^{\beta-1} \leqq 1$. Thus this term is bounded by

$$(47) \qquad \int_{1/R}^{\delta} |G_2(Rr)|r^{-1}\left|\int_{\Omega} f(r\xi)P(\xi)\,d\Omega(\xi)\right|\,dr.$$

As G_2 is bounded by a constant depending only on n and k (see (17) and (18)), given $\varepsilon > 0$, we can choose $\delta > 0$ so that by hypothesis (5) expression (47) is less than ε.

Since for δ fixed independent of R, the function $f(x)K(-x)G(R|x|)|x|^{\beta-1}$ is integrable over $E_k - B(0,\delta)$ and $0 \leqq \beta < 1$, the second term of (46) goes to zero as $R \to \infty$.

We are thus left with the term

$$b(n,k)A(k,-\beta)R^{\beta}\int_{E_k - B(0,1/R)} |x|^{\beta} f(x)K(-x)\cos\left(R|x| - \frac{\pi}{2}(k+n-\beta)\right)dx,$$

which goes to 0 as $R \to \infty$ if and only if (9) holds. This completes the proof of Theorem 2.

7. Proof of Theorem 3. To prove the theorem, for $\beta \geqq 0$ we set

$$(48) \qquad \Phi_R^{\beta}(x) = R^{\beta}\sum_{|m|\leqq R} \hat{K}(m)\,e^{i(m,x)}(1 - |m|^2/R^2)^{\gamma+\beta},$$

where we define

$$(49) \qquad \Phi_R(x) \equiv \Phi_R^0(x).$$

Using Banach space techniques, the theorem will follow once we establish the following.

(*) There exists x_0 in $T_k - B(0,\delta)$ and an increasing sequence $\{R_j\}_{j=1}^{\infty}$ such that

$$(50) \qquad \lim_{j\to\infty} |\Phi_{R_j}(x_0)| = +\infty.$$

For consider the Banach space B consisting of all real-valued functions integrable on T which vanish almost everywhere in $B(0,\delta)$. Then

$$(51) \qquad \sigma_{R_j}^{\gamma}(f,0) = (2\pi)^{-k}\int_{T_k} f(x)\Phi_{R_j}(x)\,dx$$

gives rise to a sequence of bounded linear functionals on B,

$$F_j(f) = \sigma^{\gamma}_{R_j}(f, 0). \tag{52}$$

If the conclusion to the theorem is false, then $\sup_j |F_j(f)| < +\infty$ for each $f \in B$. But then by the Banach–Steinhaus theorem, $\sup_j \|F_j\|$ is finite. However,

$$\|F_j\| = (2\pi)^{-k} \sup |\Phi_{R_j}(x)|, \tag{53}$$

where the sup is taken over all $x \in T_k - B(0, \delta)$. Therefore,

$$\|F_j\| \geqq (2\pi)^{-k} |\Phi_{R_j}(x_0)|, \tag{54}$$

and consequently

$$\sup_j |\Phi_{R_j}(x_0)| < +\infty. \tag{55}$$

This contradicts (*), therefore the theorem holds, pending the proof of (*). We shall show more than (*), namely that

$$\limsup_{R \to \infty} |\Phi_R(x)| = +\infty \tag{56}$$

for almost all $x \in T_k$.

The following lemma is due to Bochner [1, p. 192].

LEMMA 4. *Let $f(t)$ be a bounded measurable function in $0 < t < \infty$ for which*

$$a(s) = \lim_{\tau \to \infty} \tau^{-1} \int_0^{\tau} f(t)\, e^{-ist}\, dt \tag{57}$$

exists for every s in $0 \leqq s < \infty$. Then $a(s)$ is different from zero for at most a countable set of numbers which we denote by $S = \{s_1, s_2, \cdots\}$. If the numbers in S are linearly independent with respect to integer coefficients, then $\sum_{j=1}^{\infty} |a(s)| < +\infty$.

The next step in establishing (*) is to show that $R^{-1} \int_0^R \Phi_r(x)\, e^{-\lambda r}\, dr$ tends to a finite limit for $0 \leqq \lambda < \infty$. We do this in the following lemma.

LEMMA 5. *If x is not of the form $2\pi m$, where m is a lattice point in E_k, then for each λ, $0 \leqq \lambda < \infty$,*

$$\lim_{R \to \infty} R^{-1} \int_0^R \Phi_r(x)\, e^{-i\lambda r}\, dr \tag{58}$$

exists. If $\lambda \neq \lambda_m$ for every m, the above limit is equal to 0, with the exception that when $\lambda = 0$, the limit equals

$$c(k, n) \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{\lambda_m(x)^{n+k}}, \tag{59}$$

$c(k, n)$ being given by (15) and the series being convergent by Lemma 9, in [3]. If $\lambda = \lambda_m(x)$ for some lattice point m, where $\lambda_m(x) = |x - 2\pi m|$, then the expression (58) is equal to $c_{n,k} q_m(x)$, where

$$q_m(x) = P\left(\frac{x - 2\pi m}{|x - 2\pi m|}\right) \Big/ |x - 2\pi m|^k \tag{60}$$

and $c_{n,k}$ is a nonzero constant depending only on the dimension k and on n, the degree of P.

Since every x in $T_k - \{0\}$ is not of the form $2\pi m$, and since, furthermore, by continuity of $P(\xi)$, there exists an $\varepsilon > 0$ such that $|P(\xi)| > \varepsilon$ for all ξ in some spherical cap on Ω, by the integral test

$$\sum_m |q_m(x)| = +\infty, \tag{61}$$

we see from Lemmas 4 and 5 that we cannot have simultaneously $\sup_{0<R<\infty}|\Phi_R(x)| < +\infty$ and $\{q_m(x)\}_m$ linearly independent with respect to integer coefficients. Thus in order to complete the proof of the theorem we need only prove Lemma 5.

In order to prove Lemma 5 for $\Phi_r(x)$ as defined in (48) and (49), we will use the Poisson summation formula (20) given in Lemma 3. For $r > 1/\pi$ and $1 > \beta > 0$,

$$\Phi_r^\beta(x) = (i)^n r^{k+n+\beta} \sum_{j=0}^{\infty} \sum_{m\in S_j} P(x + 2\pi m)E(k, \beta, n, r|x + 2\pi m|), \tag{62}$$

where S_j is the sphere of radius j centered at the origin. We can also take the inner sum over $m' = -m$ in S_j, in which case we can write

$$\Phi_r^\beta(x) = i^n r^{n+k+\beta} \sum_{j=0}^{\infty} \sum_{m\in S_j} P(x - 2\pi m)E(k, \beta, n, r|x - 2\pi m|). \tag{63}$$

We now use the decomposition (14) from Lemma 2, so that for $r \geqq R_0$, where R_0 is chosen such that $R_0 \geqq \pi^{-1}$ and

$$R_0|x - 2\pi m| \geqq 1 \quad \text{for all } m \in E_k, \tag{64}$$

$$\begin{aligned}\Phi_r^\beta(x) = i^n r^{k+n+\beta} &\sum_{j=0}^{\infty} \sum_{m\in S_j} P(x - 2\pi m) \\ &\cdot\Bigg[\frac{c(k,n)}{r^{n+k}|x - 2\pi m|^{n+k}} + \frac{G_1(\beta, r|x - 2\pi m|)}{r^{n+k+1}|x - 2\pi m|^{n+k+1}} \\ &+ \frac{G_2(\beta, r|x - 2\pi m|)}{r^{n+k+\beta+1}|x - 2\pi m|^{n+k+\beta+1}} \\ &+ \frac{a(k,\beta)\cos\{r|x - 2\pi m| - (\pi/2)(k + n + \beta)\}}{r^{n+k+\beta}|x - 2\pi m|^{n+k+\beta}}\Bigg],\end{aligned} \tag{65}$$

which is equal to

$$\begin{aligned}&r^\beta \sum_{j=0}^{\infty} \sum_{m\in S_j} \frac{i^n P(x - 2\pi m)c(k,n)}{|x - 2\pi m|^{k+n}} \\ &+ r^{\beta-1} \sum_{j=0}^{\infty} \sum_{m\in S_j} \frac{i^n P(x - 2\pi m)G_1(\beta, r|x - 2\pi m|)}{|x - 2\pi m|^{n+k+1}} \\ &+ r^{-1} \sum_{j=\infty}^{\infty} \sum_{m\in S_j} \frac{i^n P(x - 2\pi m)G_2(\beta, r|x - 2\pi m|)}{|x - 2\pi m|^{k+n+\beta+1}} \\ &+ \sum_{j=0}^{\infty} \sum_{m\in S_j} \frac{i^n a(k,\beta)\cos\{r|x - 2\pi m| - (\pi/2)(k + \beta + n)\}P(x - 2\pi m)}{|x - 2\pi m|^{n+k+\beta}}\end{aligned} \tag{66}$$

for $r \geqq R_0$. As a result of this decomposition, for $0 < \beta < 1/4$ and $r \geqq R_0$,

$$
\begin{aligned}
&\Bigg| \Phi_r^\beta(x) - c(k,n) r^\beta \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k}} \\
&\quad - a(k,\beta) \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{k+n+\beta}} \cos\left\{ r|x - 2\pi m| - \frac{\pi}{2}(k + n + \beta) \right\} \Bigg| \\
&\leqq \frac{1}{\sqrt{r}} \Bigg| \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k+1}} G_1(\beta, r|x - 2\pi m|) \\
&\quad + \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k+\beta+1}} G_2(\beta, r|x - 2\pi m|) \Bigg| .
\end{aligned}
\tag{67}
$$

Due to [3, Lemma 9, p. 60] the series

$$
\sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k}}
\tag{68}
$$

is uniformly convergent in x.

Since G_1 and G_2 are both bounded by $A_{k,n}$, a constant depending only on k and n (see (17) and (18)) and since P is bounded on the unit $(k - 1)$-sphere, we have each of the two series on the right-hand side of the inequality (67) dominated by

$$
r^{-1/2} \sum_m |x - 2\pi m|^{-k-1} = o(1)
$$

as r goes to plus infinity.

Notice, for $0 < \beta < 1/4$,

$$
\begin{aligned}
\left| R^{-1} \int_0^{R_0} \Phi_r^\beta(x)\, e^{-i\lambda r}\, dr \right| &\leqq (R_0/\pi) \sup_m |\hat{K}(m)| R^{\beta - 1} \\
&\leqq \frac{c'(k, n, x)}{R^{1/2}} ,
\end{aligned}
\tag{69}
$$

by the definition (48) of $\Phi_r^\beta(x)$, where $c'(k, n, x)$ is a constant depending on k, n, and x, since R_0 depends on x. Therefore for $R > R_0$,

$$
\begin{aligned}
&\Bigg| R^{-1} \int_0^R \Phi_R^\beta(x)\, e^{-i\lambda r}\, dr - c(k,n) R^{-1} \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k}} \int_{R_0}^R r^\beta e^{-i\lambda r}\, dr \\
&\quad - a(k,\beta) R^{-1} \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+\beta+k}} \\
&\quad \cdot \int_{R_0}^R \cos\left\{ r|x - 2\pi m| - \frac{\pi}{2}(k + \beta + n) \right\} e^{-i\lambda r}\, dr \Bigg| \\
&\leqq \text{const.}\, R^{-1} \int_{R_0}^R r^{-1/2}\, dr.
\end{aligned}
\tag{70}
$$

Thus the left-hand side of inequality (69) is dominated by $O(R^{-1/2})$ as $R \to \infty$.

We now assume $\lambda \neq \lambda_m$ for every $m \in E_k$. Consider the third integral on the left-hand side of (70),

$$\int \cos\left\{ r\lambda_m - \frac{\pi}{2}(k + \beta + n)\right\} e^{-i\lambda r}\, dr$$

$$= \tfrac{1}{2} \exp\left\{-\frac{\pi i}{2}(k + \beta + n)\right\} \int e^{ir(\lambda_m - \lambda)}\, dr \qquad (71)$$

$$+ \tfrac{1}{2} \exp\left\{\frac{\pi i}{2}(k + \beta + n)\right\} \int e^{-ir(\lambda_m + \lambda)}\, dr.$$

Putting the limits on the integrals, this is equal to

$$\tfrac{1}{2} \exp\left\{-\frac{\pi i}{2}(k + \beta + n)\right\} \frac{1}{i(\lambda_m - \lambda)} [\exp\{iR(\lambda_m - \lambda)\} - \exp\{iR_0(\lambda_m - \lambda)\}]$$

$$(72)$$

$$-\tfrac{1}{2} \exp\left\{\frac{\pi i}{2}(k + \beta + n)\right\} \frac{1}{i(\lambda_m + \lambda)} [\exp\{-iR(\lambda_m + \lambda)\} - \exp\{-iR_0(\lambda_m + \lambda)\}].$$

Substituting this expression into (70), we see that the power of λ_m in the denominator is now large enough to insure uniform absolute convergence of the series in the third term of (70) for all β, $0 \leqq \beta < 1/4$. Since the integrand in the second term is finite and integrable for all β, $0 < \beta < 1/4$, and independent of m, and since the series in this term is uniformly convergent, we may pass to the limit as $\beta \to 0$, holding R fixed, under the integral sign in this term. Therefore, passing to the limit as $\beta \to 0$ on the left-hand side of (70) we obtain, for $\lambda \neq 0$,

$$\Bigg| R^{-1} \int_0^R \Phi_R(x)\, e^{-i\lambda r}\, dr - c(k,n) R^{-1} \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{m+k}} \int_{R_0}^{R} e^{-i\lambda r}\, dr$$

$$- a(k,0) R^{-1} \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k}}$$

$$\cdot \left\{ \tfrac{1}{2} \exp\left(\frac{\pi i}{2}(n + k)\right) \frac{1}{i(\lambda_m - \lambda)} [\exp(iR(\lambda_m - \lambda)) - \exp(iR_0(\lambda_m - \lambda))] \right. \qquad (73)$$

$$\left. - \tfrac{1}{2} \exp\left(\frac{\pi i}{2}(n + k)\right) \frac{1}{i(\lambda_m + \lambda)} [\exp(-iR(\lambda_m + \lambda)) - \exp(-iR_0(\lambda_m + \lambda))] \right\} \Bigg|$$

$$= O(R^{-1/2}) \quad \text{as } R \to \infty.$$

Integrating the second term with respect to r, we see that both the second and third terms on the left-hand side of inequality (73) are $O(R^{-1})$ as $R \to \infty$. Hence,

$$\lim_{R \to \infty} \left| R^{-1} \int_0^R \Phi_R(x)\, e^{-i\lambda r}\, dr \right| = 0 \qquad (74)$$

for $\lambda \neq \lambda_m$ for every m, and $\lambda \neq 0$.

If $\lambda = 0$, the second term on the left-hand side of (73) goes to

$$c(k,n) Q(x) = c(k,n) \sum_{j=0}^{\infty} \sum_{m \in S_j} \frac{i^n P(x - 2\pi m)}{|x - 2\pi m|^{n+k}} \qquad (75)$$

as R goes to plus infinity. The third term is $O(R^{-1})$. Hence,

$$\lim_{R\to\infty} R^{-1} \int_0^R \Phi_R(x)\, e^{-i\lambda r}\, dr = c(k,n)Q(x) \tag{76}$$

if $\lambda = 0$.

We now suppose $\lambda = \lambda_{m'}$ for some m' in E_k. We substitute $\lambda_{m'}$ for λ in inequality (70), obtaining

$$\begin{aligned} &\Bigg| R^{-1} \int_0^R \Phi_R^\beta(x) \exp(-i\lambda_{m'} r)\, dr - c(k,n) R^{-1} \sum_{j=0}^{\infty} \sum_{m\in S_j} \frac{i^n P(x-2\pi m)}{|x-2\pi m|^{n+k}} \\ &\cdot \int_{R_0}^R r^\beta \exp(-i\lambda_{m'} r)\, dr - a(k,\beta) R^{-1} \frac{i^n P(x-2\pi m')}{\lambda_{m'}^{n+\beta+k}} \\ &\cdot \int_{R_0}^R \cos\left\{ r\lambda_{m'} - \frac{\pi}{2}(k+\beta+n)\right\} \exp(-i\lambda_{m'} r)\, dr + O(R^{-1}) \Bigg| \\ &= O(R^{-1/2}). \end{aligned} \tag{77}$$

Due to uniform convergence of the series (68), we again may pass to the limit as $\beta \to 0$ in the second term, obtaining a term which is $O(R^{-1})$ as $R \to \infty$. In the third term of (77), the integral from R_0 to R is equal to

$$\begin{aligned} &\exp\left\{-\frac{\pi}{2}(k+\beta+n)\right\} \int_{R_0}^R dr + \exp\left\{\frac{\pi}{2}(k+\beta+n)\right\} \int_{R_0}^R \exp(-2i\lambda_{m'} r)\, dr \\ &= (R-R_0) \exp\left\{-\frac{\pi}{2}(k+\beta+n)\right\} + \exp\left\{\frac{\pi}{2}(k+\beta+n)\right\} \frac{i}{2\lambda_m} \\ &\cdot \int_{R_0}^R \exp(-2i\lambda_{m'} r). \end{aligned} \tag{78}$$

Substituting this back into (77), passing to the limit as β goes to zero, we see that

$$\begin{aligned} &\Bigg| R^{-1} \int_0^R \Phi_R(x) \exp(-i\lambda_{m'} r)\, dr - a(k,0) \exp\left\{-\frac{\pi}{2}(k+n)\right\} \\ &\cdot \frac{i^n P(x-2\pi m')}{\lambda_{m'}^{n+k}} + O(R^{-1}) \Bigg| = O(R^{-1/2}). \end{aligned} \tag{79}$$

Defining $c_{n,k} = a(k,0) \exp\{-(\pi/2)(k+n)\} i^n$ and recalling (60), we have thus established Lemma 5 and therefore Theorem 3.

REFERENCES

[1] S. BOCHNER, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc., 40 (1936), pp. 175–207.

[2] A. P. CALDERÓN AND A. ZYGMUND, *Singular integrals and periodic functions*, Studia Math., 14 (1954), pp. 249–271.

[3] C. P. CHANG, *On certain exponential sums arising in conjugate multiple Fourier series*, Ph.D. dissertation, University of Chicago, Chicago, 1964.

[4] V. L. SHAPIRO, *The conjugate Fourier–Stieltjes integral in the plane*, Bull. Amer. Math. Soc., 65 (1959), pp. 12–15.

[5] ———, *Fourier series in several variables*, Ibid., 70 (1964), pp. 48–93.
[6] ———, *Topics in Fourier and geometric analysis*, Memoirs Amer. Math. Soc., No. 39, 1961.
[7] E. M. STEIN, *On certain exponential sums arising in multiple Fourier series*, Ann. of Math., 73 (1961), pp. 87–109.
[8] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1966.
[9] A. ZYGMUND, *Trigonometric Series*, vol. I, Cambridge University Press, Cambridge, 1959.